Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (2019), 1 – 10
DOI : 10.15672/HJMS.xx

Research Article

Log-Harmonic mappings associated with the sine function

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Abstract

In this paper, we define new subclasses $\mathcal{ST}_{lh}(s)$ and $\mathcal{CST}_{lh}(s)$ of sine starlike log-harmonic mappings and sine close-to-starlike log-harmonic mappings, respectively, defined in the open unit disc $\mathbb{D}$. We investigate representation theorem and integral representation theorem for functions in the class $\mathcal{ST}_{lh}(s)$. Further, we determine radius of starlikeness for functions in the classes $\mathcal{ST}_{lh}(s)$ and $\mathcal{CST}_{lh}(s)$.

Mathematics Subject Classification (2020). 30C35, 30C45, 35Q30

Keywords. starlike function, sine starlike log-harmonic mapping, radius of starlikeness, sine close-to-starlike log-harmonic mapping

1. Introduction

Let $L(\mathbb{D})$ be the linear space of all analytic functions defined in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$, and let $\mathcal{A}$ be a subclass of $L(\mathbb{D})$ consisting of functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Also, let $\mathcal{B}$ be the set of all bounded analytic functions $\mu \in L(\mathbb{D})$ satisfying $|\mu(z)| < 1$ for each $z \in \mathbb{D}$. For $z = x + iy$, the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

satisfy the Laplacian

$$\triangle = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Thus a C^2-function f defined in the unit disc $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if $\triangle f = 0$. Analogously, a log-harmonic mapping defined in the disc $\mathbb{D}$ is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}(z)}}{\overline{f(z)}} = \mu(z)\left(\frac{f_z(z)}{f(z)}\right), \tag{1.1}$$

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Received: 04.11.2023; Accepted: 12.03.2024

for some $\mu \in \mathcal{B}$, where μ is the second complex-dilatation of the function f. Hence, the Jacobian

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |f_z(z)|^2(1 - |\mu(z)|^2)$$

is positive, and all non-constant log-harmonic mappings are sense-preserving in $\mathbb{D}$.

Abdulhadi and Bshouty [3] observed that if f is a non vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are analytic in $\mathbb{D}$. On the other hand, if f is a non-constant log-harmonic mapping that vanishes only at $z = 0$, then f admits the representation given by

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)},$$

where m is a non-negative integer, $\Re(\beta) > -1/2$, h and g are analytic functions in $\mathbb{D}$ with $h(0) \neq 0$ and $g(0) = 1$. The exponent β depends only on $\mu(0)$, and can be expressed by

$$\beta = \overline{\mu(0)} \frac{1 + \mu(0)}{1 - |\mu(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$. A univalent log-harmonic mapping in $\mathbb{D}$ vanishes at the origin if and only if $m = 1$. Thus every univalent log-harmonic mapping in $\mathbb{D}$ which vanishes at the origin has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)},$$

where $\Re(\beta) > -1/2$ and $0 \notin hg(\mathbb{D})$. The class of log-harmonic mappings have been studied extensively in [1,5,6] and references therein.

In this paper, we focus on sense-preserving univalent log-harmonic mappings in $\mathbb{D}$ with the condition $\mu(0) = 0$ having the form

$$f(z) = zh(z)\overline{g(z)}, \tag{1.2}$$

where h and g are analytic in $\mathbb{D}$ such that

$$h(z) = \exp\left(\sum_{n=1}^{\infty} a_n z^n\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{n=1}^{\infty} b_n z^n\right).$$

Here, h and g are the analytic and the co-analytic parts of f, respectively. The class of such mappings is denoted by $\mathcal{S}_{lh}$. It follows from (1.2) that the functions h, g and the dilatation μ satisfy the relation

$$\mu(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = \frac{z(\log g)'(z)}{1 + z(\log h)'(z)}. \tag{1.3}$$

In [4], it is shown that the mapping $f(z) = zh(z)\overline{g(z)}$ is starlike log-harmonic mapping of order α if

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) = \Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) > \alpha$$

for all $z = re^{i\theta} \in \mathbb{D}\backslash\{0\}$ and for some $0 \leq \alpha < 1$. The class of all starlike log-harmonic mappings of order α is denoted by $\mathcal{ST}_{lh}(\alpha)$. For $\alpha = 0$, we get the class $\mathcal{ST}_{lh}(0) = \mathcal{ST}_{lh}$ of starlike log-harmonic mappings. Also, denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α. For $\alpha = 0$, we get the class $\mathcal{S}^*(0) = \mathcal{S}^*$ of starlike functions.

The following theorem provides a link between the classes $\mathcal{ST}_{lh}(\alpha)$ and $\mathcal{S}^*(\alpha)$.

Theorem A (Theorem 2.1 [4]). *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping in $\mathbb{D}$ with $0 \notin (hg)(\mathbb{D})$, where h and g are analytic functions. Then $f \in \mathcal{ST}_{lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in \mathcal{S}^*(\alpha)$.*

Let $\mathcal{P}_{lh}$ be the set of all log-harmonic mappings R defined in $\mathbb{D}$ which are of the form $R(z) = H(z)\overline{G(z)}$, where H and G are in $L(\mathbb{D})$, $H(0) = G(0) = 1$ such that $\Re(R(z)) > 0$ for all $z \in \mathbb{D}$. In particular, the set $\mathcal{P}$ of all analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ is a subset of $\mathcal{P}_{lh}$. The next result describes the connection between the classes $\mathcal{P}_{lh}$ and $\mathcal{P}$.

Theorem B ([2]). *A function $R(z) = H(z)\overline{G}(z) \in \mathcal{P}_{lh}$ if and only if $p(z) = H(z)/G(z) \in \mathcal{P}$.*

Denote by Ω the class of Schwarz functions w which are analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. For analytic functions f_1 and f_2 in $\mathbb{D}$, we state that f_1 is subordinate to f_2, symbolized by $f_1 \prec f_2$, if there exists a function w in Ω satisfying $f_1(z) = f_2(w(z))$. The comprehensive details of subordination can be found in [8]. Ma and Minda [11] investigated the class of analytic functions ϕ with positive real part in $\mathbb{D}$ that map the disc $\mathbb{D}$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. These authors introduced the class of starlike functions

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{D} \right\}.$$

For the case $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$, the family of Janowski starlike functions $\mathcal{S}^*[A, B]$ is obtained ([9]). When $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$, we have the family $\mathcal{S}^*(\alpha)$ of starlike functions of order α. Particularly, $\alpha = 0$ yields the usual class $\mathcal{S}^*(0) =: \mathcal{S}^*$ of starlike functions. Recently, Cho *et al.* [7] defined the subclass $\mathcal{S}^*_s$ of Ma–Minda class $\mathcal{S}^*(\phi)$ which is endowed with the analytic function $\phi(z) = 1 + \sin z$. Then, the function $f \in \mathcal{S}^*_s$ if $zf'(z)/f(z) \prec 1 + \sin z$ for all $z \in \mathbb{D}$. The following lemma provides the largest disc and the smallest disc centered, respectively, at $(a, 0)$ and $(1, 0)$ such that the domain $\Omega_s : (1 + \sin z)(\mathbb{D})$ is contained in the smallest disc and contains the largest disc.

Lemma 1.1 ([7]). *Let $1 - \sin 1 \le a \le 1 + \sin 1$ and $r_a = \sin 1 - |a - 1|$. Then the following inclusions hold:*

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subset \Omega_s \subset \{w \in \mathbb{C} : |w - 1| < \sinh 1\}.$$

Motivated by the above discussed literature, we introduce the notion of sine starlike log-harmonic mappings. Due to Cho *et al.* [7], we first give Ma-Minda type sine starlike function class:

An analytic function $\varphi \in \mathcal{S}^*_s$ if $z\varphi'(z)/\varphi(z) \prec 1 + \sin z$ for all $z \in \mathbb{D}$. Since $\varphi \in \mathcal{S}^*_s$,

$$\frac{z\varphi'(z)}{\varphi(z)} \prec 1 + \sin z \quad \text{if and only if} \quad \frac{z\varphi'(z)}{\varphi(z)} = 1 + \sin w(z),$$

where w is a Schwarz function with $|w(z)| \le |z|$. Let $w(z) = r^* e^{it}$ with $r^* \le |z| = r,\ t \in [-\pi, \pi]$. Thus, easy calculations show that

$$|\sin w(z)| \le \sinh r^* \le \sinh r.$$

Therefore, we have

$$\Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \ge 1 - \sinh r.$$

Consider the function $\varphi(z) = zh(z)/g(z)$. Then taking logarithmic derivative, we observe that

$$\frac{z\varphi'(z)}{\varphi(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec 1 + \sin z.$$

Hence, taking into account the above relations, we define the following classes:

Definition 1.2. An analytic mapping $\varphi(z) = zh(z)/g(z)$ such that $\varphi(0) = 0$ and $h(0) = g(0) = 1$, is said to be sine starlike if

$$\Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right) = \Re\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) \geq 1 - \sin hr$$

for all $z \in \mathbb{D}$. The class of sine starlike functions is denoted by $\mathcal{S}_s^*$.

Definition 1.3. A log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ such that $f(0) = 0$ and $h(0) = g(0) = 1$, is said to be sine starlike log-harmonic mapping if

$$\Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) \geq 1 - \sin hr$$

for all $z \in \mathbb{D}$. The class of sine starlike log-harmonic mapping is denoted by $\mathcal{ST}_{lh}(s)$.

The main purpose of this paper is to show that a log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ is sine starlike log-harmonic in $\mathbb{D}$ if and only if the function $\varphi(z) = zh(z)/g(z)$ is in the class $\mathcal{S}_s^*$. In Section 2, we first investigate a representation theorem which gives a relation between the classes $\mathcal{ST}_{lh}(s)$ and $\mathcal{S}_s^*$. We next obtain integral representation theorem for functions in the class $\mathcal{ST}_{lh}(s)$. In Section 3, we investigate radius of starlikeness for the class $\mathcal{ST}_{lh}(s)$. Further, we define the concept of sine close-to-starlike log-harmonic mappings, denoted by $\mathcal{CST}_{lh}(s)$, and investigate the radius of starlikeness for such mappings.

2. Representation Theorems

In this section, we first establish a representation theorem, which provides a relation between the classes $\mathcal{ST}_{lh}(s)$ and $\mathcal{S}_s^*$.

Theorem 2.1. *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping in $\mathbb{D}$ with $0 \notin hg(\mathbb{D})$. Then f belongs to the class $\mathcal{ST}_{lh}(s)$ if and only if $\varphi(z) = zh(z)/g(z)$ belongs to the class $\mathcal{S}_s^*$.*

Proof. Let $f(z) = zh(z)\overline{g(z)}$ be in the class $\mathcal{ST}_{lh}(s)$. Then

$$\begin{aligned}\frac{\partial}{\partial\theta}\big(\arg f(re^{i\theta})\big) &= \Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right)\\ &= \Re\left(1 + \frac{zh'(z)}{h(z)} - \frac{\overline{\overline{z}g'(z)}}{\overline{g(z)}}\right)\\ &= \Re\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) \geq 1 - \sin hr. \end{aligned} \tag{2.1}$$

Consider the function $\varphi(z) = zh(z)/g(z)$, thus logarithmic differentiation gives

$$\frac{z\varphi'(z)}{\varphi(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}. \tag{2.2}$$

In view of (2.1) and (2.2), we arrive at

$$\Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) = \Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \geq 1 - \sinh r. \tag{2.3}$$

Since the function f is univalent, we have $0 \notin f_z(\mathbb{D})$. Also,

$$q_1(w) = \varphi \circ f^{-1}(w) = w|g \circ f^{-1}(w)|^{-2}$$

is locally univalent in $f(\mathbb{D})$. Thus, we have

$$\frac{z\varphi'(z)}{\varphi(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = \frac{zf_z(z)}{f(z)} - \mu(z)\frac{zf_z(z)}{f(z)} = (1 - \mu(z))\frac{zf_z(z)}{f(z)} \neq 0$$

for all $z \in \mathbb{D}$. Therefore, φ is univalent, and in view of (2.3) we conclude that $\varphi \in \mathcal{S}^*_s$.

Conversely, let $\varphi \in \mathcal{S}^*_s$ and $\mu \in \mathcal{B}$ such that $|\mu(z)| < 1$ for each $z \in \mathbb{D}$. Since $\varphi(z) = zh(z)/g(z)$, we have the equation (2.2). Also, from (1.1), we get

$$\frac{zg'(z)}{g(z)} = \mu(z)\left(1 + \frac{zh'(z)}{h(z)}\right). \tag{2.4}$$

Combining (2.2) and (2.4), we observe that

$$g(z) = \exp\int_0^z \frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds, \tag{2.5}$$

and

$$zh(z) = \varphi(z)\exp\int_0^z \frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds, \tag{2.6}$$

where $\frac{z\varphi'(z)}{\varphi(z)} = p(z) \prec 1+\sin z$ such that $p(0)=1$ and $\Re(p(z)) > 0$. It follows that

$$\begin{aligned} f(z) = zh(z)\overline{g(z)} &= \varphi(z)\exp\int_0^z \frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds\ \overline{\exp\int_0^z \frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds} \\ &= \varphi(z)\exp\left(2\,\Re\int_0^z \frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right), \end{aligned}$$

and

$$f(z) = zh(z)\overline{g(z)} = \varphi(z)|g(z)|^2.$$

Therefore, h and g are non-vanishing analytic functions, normalized by $h(0) = g(0) = 1$, in $\mathbb{D}$ and f is a solution of (1.1) with respect to μ. Hence, we observe that

$$\Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}\right) = \Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \geq 1 - \sinh r.$$

Moreover,

$$q_2(w) = f\circ\varphi^{-1}(w) = w|g\circ\varphi^{-1}(w)|^2$$

is locally univalent in $\varphi(\mathbb{D})$, and therefore f is univalent. It follows that $f \in \mathcal{ST}_{lh}(s)$. $\square$

We now give an integral representation for $f \in \mathcal{ST}_{lh}(s)$ with the case $\mu(0) = 0$. Hence, we need the following lemma.

Lemma 2.2 ([10]). *If the function $\mu \in \mathcal{B}$ with $\mu(0) = 0$, then*

$$\frac{\mu(z)}{1-\mu(z)} = \int_{\partial\mathbb{D}} \frac{\xi z}{1-\xi z}d\kappa(\xi),\ \ (z \in \mathbb{D})$$

for some probability measure κ on $\partial\mathbb{D}$.

Theorem 2.3. *A log-harmonic mapping $f(z) = zh(z)\overline{g(z)} \in \mathcal{ST}_{lh}(s)$ if and only if there are two probability measures ν and κ on $\partial\mathbb{D}$ such that*

$$g(z) = \exp\left(\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} K_1(z,t,\xi)d\nu(t)d\kappa(\xi)\right), \tag{2.7}$$

where

$$K_1(z,t,\xi) = \sin\left(\frac{t}{\xi}\right)\left\{\mathrm{Ci}\left(-\frac{t}{\xi}\right) - \mathrm{Ci}\left(tz - \frac{t}{\xi}\right)\right\} + \cos\left(\frac{t}{\xi}\right)\left\{\mathrm{Si}\left(\frac{t}{\xi} - tz\right) - \mathrm{Si}\left(\frac{t}{\xi}\right)\right\} - \log(1-\xi z)$$

and

$$h(z) = \exp\left(\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} K_2(z,t,\xi)d\nu(t)d\kappa(\xi)\right), \tag{2.8}$$

where

$$K_2(z,t,\xi) = \mathrm{Si}(tz) + \sin\left(\frac{t}{\xi}\right)\left\{\mathrm{Ci}\left(-\frac{t}{\xi}\right) - \mathrm{Ci}\left(tz - \frac{t}{\xi}\right)\right\} + \cos\left(\frac{t}{\xi}\right)\left\{\mathrm{Si}\left(\frac{t}{\xi} - tz\right) - \mathrm{Si}\left(\frac{t}{\xi}\right)\right\} - \log(1-\xi z)$$

if $|\xi| = |t| = 1,\ \xi \neq t$.

Proof. By Theorem 2.1, we know that $f(z) = zh(z)\overline{g(z)} \in \mathcal{ST}_{lh}(s)$ if and only if $\varphi(z) = zh(z)/g(z) \in \mathcal{S}^*_s$, thus

$$\frac{z\varphi'(z)}{\varphi(z)} = p(z) \prec 1 + \sin z,$$

where $p \in \mathcal{P}$ such that $p(0) = 1$ and $\Re(p(z)) > 0$. Hence, for $p(z) = 1 + \sin z$, there exists a probability measure ν defined on the Borel $\sigma-$algebra of $\partial\mathbb{D}$ such that

$$\frac{z\varphi'(z)}{\varphi(z)} = \int_{\partial\mathbb{D}} (1 + \sin tz) d\nu(t) \Rightarrow \varphi(z) = z \exp\left(\int_{\partial\mathbb{D}} \int_0^z \frac{\sin ts}{ts} ds d\nu(t) \right). \tag{2.9}$$

Setting (1.3), (2.9) and Lemma 2.2 into (2.5), we get

$$g(z) = \exp\left(\int_0^z \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \frac{\xi}{1 - \xi s} (1 + \sin ts) d\nu(t) d\kappa(\xi) ds \right)$$

for probability measures ν and κ on $\partial\mathbb{D}$. Integrating above function, we arrive at

$$\begin{aligned} g(z) &= \exp\left(\int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \int_0^z \frac{\xi}{1 - \xi s} (1 + \sin ts) ds d\nu(t) d\kappa(\xi) \right) \\ &= \exp\left(\int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} K_1(z, t, \xi) d\nu(t) d\kappa(\xi) \right), \end{aligned} \tag{2.10}$$

where

$$K_1(z, t, \xi) = \sin\left(\frac{t}{\xi}\right) \left\{ \mathrm{Ci}\left(-\frac{t}{\xi}\right) - \mathrm{Ci}\left(tz - \frac{t}{\xi}\right) \right\} + \cos\left(\frac{t}{\xi}\right) \left\{ \mathrm{Si}\left(\frac{t}{\xi} - tz\right) - \mathrm{Si}\left(\frac{t}{\xi}\right) \right\} - \log(1 - \xi z).$$

Here, $\mathrm{Ci}(z)$ is the cosine integral and $\mathrm{Si}(z)$ is the sine integral given, respectively, by

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos s}{s} ds \quad \text{and} \quad \mathrm{Si}(z) = \int_0^z \frac{\sin s}{s} ds.$$

Moreover, in similar way, by plugging (2.9) and (2.10) into $h(z) = (\varphi(z)/z)g(z)$, we get the integral representation for h given by (2.8). This completes the proof. $\square$

3. Radii of Starlikeness

The first result gives radius of starlikeness for sine starlike log-harmonic mappings f, which satisfy the condition $\Re(\frac{f(z)}{z}) > 0$.

Theorem 3.1. *Suppose that $f(z) = zh(z)\overline{g(z)} \in \mathcal{ST}_{lh}(s)$ in $\mathbb{D}$ with $h(0) = g(0) = 1$, and $\varphi(z) = \frac{zh(z)}{g(z)} \in \mathcal{S}^*_s$ in $\mathbb{D}$. If $\Re(\frac{f(z)}{z}) > 0$ for $z \in \mathbb{D}$, then f is univalent and starlike in*

$$|z| \leq r = \frac{\sinh 1}{\sqrt{1 + (\sinh 1)^2} + 1} \approx 0.462117.$$

Proof. Since $f \in \mathcal{ST}_{lh}(s)$, it follows that

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta}) \right) = \Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} \right) = \Re\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \right).$$

Taking logarithmic derivative of $\varphi(z) = zh(z)/g(z)$, and using the above relation, we get

$$\Re\left(\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} \right) = \Re\left(\frac{z\varphi'(z)}{\varphi(z)} \right). \tag{3.1}$$

Let $p(z) = \varphi(z)/z$, then we observe that

$$\frac{zp'(z)}{p(z)} = \frac{z\varphi'(z)}{\varphi(z)} - 1. \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\Re\left(\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}\right)=\Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right)=1+\Re\left(\frac{zp'(z)}{p(z)}\right). \tag{3.3}$$

We will show that the function f in (3.3) is univalent and starlike. Since

$$\Re\left(\frac{f(z)}{z}\right)=\Re\left(\frac{zh(z)\overline{g(z)}}{z}\right)=|g(z)|^2\Re(p(z))>0,$$

it follows that $\Re(p(z))>0$. Thus we conclude that $p\in\mathcal{P}$, which satisfying

$$\left|\frac{zp'(z)}{p(z)}\right|\leq\frac{2r}{1-r^2}.$$

Hence, from (3.2) and the above relation, we obtain

$$\left|\frac{z\varphi'(z)}{\varphi(z)}-1\right|=\left|\frac{zp'(z)}{p(z)}\right|\leq\frac{2r}{1-r^2}.$$

Since the above disc centered at 1, by Lemma 1.1, it follows that $|w-1|\leq 2r/(1-r^2)$ contains the disc Ω_s if

$$\frac{2r}{1-r^2}\leq\sinh 1$$

or $(\sinh 1)r^2+2r-\sinh 1\leq 0$. Thus, the radius of $\mathcal{ST}_{lh}(s)$ is the smallest positive root of the equation $(\sinh 1)r^2+2r-\sinh 1=0$ in $(0,1)$, and this implies that $|z|\leq r=\frac{\sinh 1}{\sqrt{1+(\sinh 1)^2}+1}$.

Moreover, the function

$$f(z)=zh(z)\overline{g(z)}=\varphi(z)\exp\left(2\,\Re\int_0^z\frac{\mu(s)}{1-\mu(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right),$$

where $\varphi(z)=z(1+z)/(1-z)$ and $\mu(z)=z$, holds $\Re(\frac{f(z)}{z})>0$ for $z\in\mathbb{D}$, and is univalent in $|z|\leq r=\frac{\sinh 1}{\sqrt{1+(\sinh 1)^2}+1}$.

Sharpness is satisfied for the function

$$\frac{z\varphi'(z)}{\varphi(z)}-1=\frac{2z}{1-z^2}=\sinh 1.$$

This completes the proof. □

Now, we define the class of sine close-to-starlike log-harmonic mappings: Let $F(z)=zh(z)\overline{g(z)}$ be a log-harmonic mapping with respect to $\mu\in\mathcal{B}$. We say that F is sine close-to-starlike log-harmonic mapping denoted by $\mathcal{CST}_{lh}(s)$ if there exists a log-harmonic mapping $f(z)=zh_1(z)\overline{g_1(z)}\in\mathcal{ST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$ such that

$$\Re\left(\frac{F(z)}{f(z)}\right)>0,$$

or equivalently

$$F(z)=f(z)R(z),$$

where $R(z)=H(z)\overline{G(z)}\in\mathcal{P}_{lh}$ with $H(0)=G(0)=1$.

The next theorem gives the radius of starlikeness for functions $F(z)=zh(z)\overline{g(z)}$ in the class $\mathcal{CST}_{lh}(s)$.

Theorem 3.2. *Let $F(z)=zh(z)\overline{g(z)}\in\mathcal{CST}_{lh}(s)$. Then F maps the disc $|z|<\rho\approx 0.309757$ onto a starlike domain, where ρ is the smallest positive root of the equation*

$$(1-\sinh\rho)(1-\rho^2)-2\rho=0. \tag{3.4}$$

Proof. Since $F(z)=zh(z)\overline{g(z)}\in \mathcal{CST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$, there exist a function $f(z)=zh_1(z)\overline{g_1(z)}\in \mathcal{ST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$, and a log-harmonic mapping with positive real part $R(z)=H(z)\overline{G(z)}\in\mathcal{P}_{lh}$ with respect to $\mu\in\mathcal{B}$ such that

$$F(z)=f(z)R(z). \tag{3.5}$$

Since $R\in\mathcal{P}_{lh}$, we have

$$\Re\left(\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\right)=\Re\left(\frac{zp'(z)}{p(z)}\right), \tag{3.6}$$

where $\Re(p(z))=\Re(\frac{H(z)}{G(z)})>0$ by Theorem B, and

$$\Re\left(\frac{zp'(z)}{p(z)}\right)\geq -\frac{2r}{1-r^2}. \tag{3.7}$$

From (2.3), (3.5), (3.6) and (3.7), we get

$$\begin{aligned}\Re\left(\frac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)}\right)&=\Re\left(\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}\right)+\Re\left(\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\right)\\&=\Re\left(\frac{z\varphi'(z)}{\varphi(z)}\right)+\Re\left(\frac{zp'(z)}{p(z)}\right)\\&\geq 1-\sinh r-\frac{2r}{1-r^2}.\end{aligned}$$

Thus,

$$\Re\left(\frac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)}\right)>0$$

if $1-\sinh r-\frac{2r}{1-r^2}>0$. Therefore, the radius of starlikeness ρ is the smallest positive root of the equation $(1-\sinh\rho)(1-\rho^2)-2\rho=0$ in $(0,1)$. The function $F(z)=\frac{z(1+z)}{(1-z)^3}$ belongs to the class $\mathcal{CST}_{lh}(s)$. □

Next, we prove the following radius of starlikeness for functions $F\in\mathcal{CST}_{lh}(s)$.

Theorem 3.3. *Let $K(z)=zh(z)\overline{g(z)}$ be a log-harmonic mapping with respect to $\mu\in\mathcal{B}$, and let $F(z)=zh_1(z)\overline{g_1(z)}\in\mathcal{CST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$ such that $\Re(\frac{K(z)}{F(z)})>0$. Then F maps the disc $|z|<\rho_1\approx 0.193715$ onto a starlike domain, where ρ_1 is the smallest positive root of the equation*

$$(1-\sinh\rho_1)(1-\rho_1^2)-4\rho_1=0. \tag{3.8}$$

Proof. Since $K(z)=zh(z)\overline{g(z)}$ is a log-harmonic mapping with respect to $\mu\in\mathcal{B}$, and $F(z)=zh_1(z)\overline{g_1(z)}\in\mathcal{CST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$, there exist a function $f(z)=zh_2(z)\overline{g_2(z)}\in\mathcal{ST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$ and log-harmonic mappings with positive real parts R and R^* in $\mathcal{P}_{lh}$ with respect to $\mu\in\mathcal{B}$ such that

$$K(z)=f(z)R(z)R^*(z). \tag{3.9}$$

From (3.9), we get

$$\Re\left(\frac{zK_z(z)-\overline{z}K_{\overline{z}}(z)}{K(z)}\right)=\Re\left(\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}\right)+\Re\left(\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\right)+\Re\left(\frac{zR^*_z(z)-\overline{z}R^*_{\overline{z}}(z)}{R^*(z)}\right). \tag{3.10}$$

Since $R,R^*\in\mathcal{P}_{lh}$, we have

$$\Re\left(\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\right)=\Re\left(\frac{zp'(z)}{p(z)}\right)\geq -\frac{2r}{1-r^2}, \tag{3.11}$$

$$\Re\left(\frac{zR^*_z(z)-\overline{z}R^*_{\overline{z}}(z)}{R^*(z)}\right)=\Re\left(\frac{zp'(z)}{p(z)}\right)\geq -\frac{2r}{1-r^2}. \tag{3.12}$$

Substituting (2.3), (3.11) and (3.12) into (3.10), we get

$$\Re\left(\frac{zK_z(z)-\overline{z}K_{\overline{z}}(z)}{K(z)}\right) \geq 1-\sinh r-\frac{4r}{1-r^2}.$$

Hence,

$$\Re\left(\frac{zK_z(z)-\overline{z}K_{\overline{z}}(z)}{K(z)}\right) > 0$$

if $1-\sinh r-\frac{4r}{1-r^2}>0$. Therefore, the radius ρ_1 is the smallest positive root of the equation $(1-\sinh\rho_1)(1-\rho_1^2)-4\rho_1=0$ in $(0,1)$. The function $F(z)=\frac{z(1+z)}{(1-z)^4}$ belongs to the class $\mathcal{CST}_{lh}(s)$. □

Finally, we prove the following radius of starlikeness for functions $F\in\mathcal{CST}_{lh}(s)$.

Theorem 3.4. *Let $F(z)=zh(z)\overline{g(z)}\in\mathcal{CST}_{lh}(s)$ be a log-harmonic mapping with respect to $\mu\in\mathcal{B}$, and let $f^*(z)=zh^*(z)\overline{g^*(z)}\in\mathcal{ST}_{lh}(s)$ with respect to $\mu\in\mathcal{B}$. Then $S(z)=F(z)^{\lambda}f^*(z)^{1-\lambda}$, $\lambda\in(0,1)$ is univalent and starlike in $|z|<\rho_2$, where ρ_2 is the smallest positive root of the equation*

$$(1-\sinh\rho_2)(1-\rho_2^2)-2\lambda\rho_2=0. \tag{3.13}$$

Proof. Let $S(z)=F(z)^{\lambda}f^*(z)^{1-\lambda}$, $\lambda\in(0,1)$, where $F(z)=f(z)R(z)$ such that $f\in\mathcal{ST}_{lh}(s)$, $R\in\mathcal{P}_{lh}$, and where $f^*\in\mathcal{ST}_{lh}(s)$ are log-harmonic mappings with respect to $\mu\in\mathcal{B}$, then $S(z)$ is log-harmonic with respect to the same $\mu\in\mathcal{B}$ such that

$$S(z)=F(z)^{\lambda}f^*(z)^{1-\lambda}=(f(z)R(z))^{\lambda}(f^*(z))^{1-\lambda}. \tag{3.14}$$

From (2.3), (3.6), (3.7) and (3.14), we get

$$\begin{aligned}\Re\left(\frac{zS_z(z)-\overline{z}S_{\overline{z}}(z)}{S(z)}\right) &= \lambda\Re\left(\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}\right)+\lambda\Re\left(\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\right)\\ &\quad+(1-\lambda)\Re\left(\frac{zf^*_z(z)-\overline{z}f^*_{\overline{z}}(z)}{f^*(z)}\right)\\ &\geq \lambda\left(1-\sinh r-\frac{2r}{1-r^2}\right)+(1-\lambda)(1-\sinh r)\\ &= 1-\sinh r-\frac{2\lambda r}{1-r^2}.\end{aligned}$$

Hence,

$$\Re\left(\frac{zS_z(z)-\overline{z}S_{\overline{z}}(z)}{S(z)}\right) > 0$$

if $1-\sinh r-\frac{2\lambda r}{1-r^2}>0$. Therefore, the radius ρ_2 is the smallest positive root of the equation $(1-\sinh\rho_2)(1-\rho_2^2)-2\lambda\rho_2=0$ in $(0,1)$. □

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