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Research Article

On the Boundary Functional of a Semi-Markov Process

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Abstract: In this paper, we consider the semi-Markov random walk process with negative drift, positive jumps. An integral equation for the Laplace transform of the conditional distribution of the boundary functional is obtained. In this work, we define the residence time of the system by generalized exponential distributions with different parameters via fractional order integral equation. The purpose of this paper is to reduce an integral equation for the Laplace transform of the conditional distribution of a boundary functional of the semi-Markov random walk processes to fractional order differential equation with constant coefficients.

Keywords: Laplace transform, random variable, semi-Markov random walk process, Riemann-Liouville fractional derivative.

1. Introduction

A semi-Markov processes are investigated in different directions. In recent years, a semi-Markov random walk with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. It is well known that the semi-Markov processes have been introduced by Levy [14], Smith [22] and Takàcs [23] in order to reduce the limitation induced by the exponential distribution of the corresponding time intervals. This is the immediate generalization of Markov chains since the Markov property is the typical consequence of the lack of memory of the exponential distribution. The semi-Markov process is constructed by the so-called Markov renewal process. The Markov renewal process is defined by the transition probabilities matrix, called the renewal kernel, and by an initial distribution. The essential developments of semi-Markov processes theory were proposed by Pyke [21], Feller [7], Cinlar [6], Gikhman and Skorokhod [9], Limnios and Oprisan [15] and Grabski [8]. In the work of Abdel-Rehim, Hassan and El-Sayed [3] a simulation of the short and long memory of ergodic Markov and non-Markov genetic diffusion processes on the long run was

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investigated. Using asymptotic methods and factorization methods similar problems were studied by Lebowitz and Percus [13] and Lotov and Orlova [16]. In many cases asymptotic analysis of the factorization representations of dual transforms leads to the complete asymptotic expansions of the distributions under consideration [4]. But in particular case of semi-Markov random walk process we can obtain the explicit form for probability characteristics. The authors of [10, 12, 19, 20] have found the Laplace transform of the distribution of the first moment reaching level zero of the semi-Markov random walk processes. It should be noted that finding the Laplace transform of the semi-Markov random walk processes is a powerful tool in applied mathematics and engineering. It is well known the connection between semigroup theory and the Markov processes. In the semigroup theory of Markov processes, a particular process is usually represented as a semigroup of contraction operators in some concrete Banach space, and the properties of the particular process are deduced from the properties of the associated semigroup of operators. From this point of view, by Atangana in [1] it was shown that the Atangana Baleanu fractional derivative possesses the Markovian and non-Markovian properties. We also refer to [2] for more results on fractional modeling of probabilistic processes. We recall that, in [5] the authors studied a stepwise semi-Markov processes. Then authors used the fractional Riemann-Liouville derivative. Moreover, the obtained solution for the fractional differential equation was in the form of a threefold sum. But in the presented paper, we obtained a mathematical model of a semi-Markov process with negative drift, positive jumps for a certain general class of probability distributions, and in the class of gamma distributions we managed to reduce the study of a mathematical model through a fractional differential equation with a fractional Riemann-Liouville derivative. In conclusion, we were able to find solution for the fractional differential equation.

In this paper, jump processes with a waiting time between jumps that is not necessarily given by an exponential random variable is consider. In the present paper, we study the semi-Markov random walk process with negative drift, positive jumps and delaying barrier.

An integral equation for the Laplace transform of the conditional distribution of the boundary functional is constructed. In particular, constructed integral equation is reduce to the fractional order differential equation in the class of gamma distributions. Finally, solution of the fractional order differential equation is obtained. The paper is organized in five section. In Section 2, we introduce analytic expression of a stochastic process and some notation. Section 3 is devoted to construct an integral equation for the Laplace transform of the conditional distribution of the boundary functional, also it is shown that the obtained integral equation is reduce a fractional differential equation in the class of gamma distributions. The main result is obtained in Section 4. Finally, the conclusion is given in Section 5.

2. Problem Statement and Preliminaries

Let's assume that sequences of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k=1}^{\infty}$,be given on the any probability space (Ω, F, P), where the random variables ξ_k and ζ_k are independent, positive. Now, we can construct the stochastic process $X_1(t)$ as follows

$$X_1(t) = z - t + \sum_{i=0}^{k-1} \zeta_i, \text{ if } \sum_{i=0}^{k-1} \xi_i \leq t < \sum_{i=0}^{k} \xi_i,$$

where $\xi_0 = \zeta_0 = 0$. This stochastic model is called "a semi-Markov random walk process with negative drift, positive jumps". Let this process is delayed by a barrier zero:

$$X(t) = X_1(t) - \inf_{0 \leq s \leq t} \{0, X_1(s)\}.$$

Now, we introduce the following random variable

$$\tau_0 = \inf\{t : X(t) = 0\}.$$

We set $\tau_0 = \infty$ if $X(t) > 0$ for every t. Notice that the random variable τ_0 is the time of the first crossing of the process $X(t)$ into the delaying barrier at zero level. τ_0 is called boundary functional of the semi-Markov random walk process with negative drift, positive jumps.

The aim of this study is to find the Laplace transform of the conditional distribution of the random variable τ_0. Laplace transform of the conditional distribution of the random variable τ_0 by

$$L(\theta | z) = E\left[e^{-\theta \tau_0} | X(0) = z\right], \quad \theta > 0, \ z \geq 0.$$

Let us denote the conditional distribution of random variable of τ_0 and the Laplace transform of the conditional distribution are defined by

$$N(t | z) = P\left[\tau_0 > t | X(0) = z\right],$$

and

$$\tilde{N}(\theta | z) = \int_0^{\infty} e^{-\theta t} N(t | z) dt,$$

respectively.

Thus, it is easy to see that

$$\tilde{N}(\theta | z) = \frac{1 - L(\theta | z)}{\theta}$$

or, equivalently,

$$L(\theta | z) = 1 - \theta \tilde{N}(\theta | z).$$

3. The Construction of an Integral Equation for the $\tilde{N}(\theta|z)$ and Reduction to the Fractional Order Differential Equation

Theorem 3.1 *Function $\tilde{N}(\theta|z)$ satisfy the following integral equation:*

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt + \int_z^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}$$

$$+ \int_0^z \tilde{N}(\theta|y) \int_{z-y}^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}. \tag{1}$$

Proof Using by the law of total probability, we can get

$$N(t|z) = P\{\tau_0 > t;\ \xi_1 > t\,|X(0) = z\} + P\{\tau_0 > t;\ \xi_1 < t\,|X(0) = z\}$$

$$= P\{z - t > 0; \xi_1 > t\} + \int_0^t \int_0^\infty P\{\xi_1 \in ds;\ z - s > 0; z - s + \zeta_1 \in dy\} P\{\tau_0 > t - s|X(0) = z\}.$$

Then

$$N(t,\ |z) = P\{z - t > 0\}\ P\{\xi_1 > t\}$$

$$+ \int_0^t \int_0^\infty P\{\xi_1 \in ds; z - s > 0; z - s + \zeta_1 \in dy\}\ N(t - s\,|y). \tag{2}$$

By applying the Laplace transform with respect to t both sides of the equation (2) we get:

$$\tilde{N}(\theta|z) = \int_0^\infty e^{-\theta t} P\{z - t > 0; \xi_1 > t\}\, dt$$

$$+ \int_0^\infty e^{-\theta t} \int_0^t \int_0^\infty P\{\xi_1 \in ds; z - s > 0; z - s + \zeta_1 \in dy\}\ N(t - s\,|y),\ \ \theta > 0$$

or

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+ \int_0^\infty \tilde{N}(\theta|y) \int_0^\infty e^{-\theta t} P\{\xi_1 \in dt;\ z - t > 0\}\, d_y P\{\zeta_1 < y - z + t\},\ \ \theta > 0.$$

The following equation can be easily obtained from the last equation:

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+ \int_0^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}.$$

It is clear that it should be taken $y - z + t > 0$. From this condition, it follows that $t > max(0, z - y)$. Then the last equation can be rewritten as follows

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+ \int_0^\infty \tilde{N}(\theta|y) \int_{\max(0,z-y)}^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}. \tag{3}$$

Finally, from integral equation (3), (1) is obtained. □

This completes the proof.

Suppose that the distribution of the random variable ξ_1 has the density function $p_{\xi_1}(s)$, $s > 0$ and the distribution of the random variable ζ_1 has the density function $p_{\zeta_1}(s)$, $s > 0$. Then equation (1) has the form

$$\tilde{N}(\theta|z) = \frac{1}{\theta}\left[1 - e^{-\theta z}\right] + \frac{1}{\theta} e^{-\theta z} P\{\xi_1 < z\} - \frac{1}{\theta}\int_0^z e^{-\theta t} p_{\xi_1}(t)dt$$

$$+ \int_z^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} p_{\xi_1}(t) p_{\zeta_1}(y - z + t)\, dtdy$$

$$+ \int_0^z \tilde{N}(\theta|y) \int_{z-y}^z e^{-\theta t} p_{\xi_1}(t) p_{\zeta_1}(y - z + t)\, dtdy. \tag{4}$$

Let's assume that random variable ξ_1 has the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, while random variable ζ_1 has Erlang distribution of first order with the parameters μ:

$$p_{\xi_1}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0 \\ 0, & x \le 0, \end{cases} \qquad \rho_{\zeta_1}(x) = \begin{cases} \mu e^{-\mu x}, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

In the class of these distributions the integral equation (4) can be written as follows

$$\tilde{N}(\theta|z) = \frac{1}{\theta}\left[1 - e^{-\theta z}\right] + \frac{\beta^\alpha e^{-\theta z}}{\theta\Gamma(\alpha)} \int_0^z e^{-\beta y} y^{\alpha-1} dy - \frac{\beta^\alpha}{\theta\Gamma(\alpha)} \int_0^z e^{-(\theta+\beta)t} t^{\alpha-1} dt$$

$$+ \frac{\mu\beta^\alpha e^{\mu z}}{\Gamma(\alpha)} \int_z^\infty e^{-\mu y} \tilde{N}(\theta|y) \int_0^z e^{-(\mu+\theta+\beta)t} t^{\alpha-1}\, dtdy$$

$$+ \frac{\mu\beta^\alpha e^{\mu z}}{\Gamma(\alpha)} \int_0^z e^{-\mu y} \tilde{N}(\theta|y) \int_{z-y}^z e^{-(\mu+\theta+\beta)t} t^{\alpha-1}\, dtdy. \tag{5}$$

Multiplying both sides of equation (5) by $e^{-\mu z}$ and differentiating both sides with respect to z, we obtain

$$e^{-\mu z} \tilde{N}'(\theta|z) - \mu e^{-\mu z} \tilde{N}(\theta|z) = \frac{1}{\theta}\left[(\mu + \theta) e^{-(\mu+\theta)z} - \mu e^{-\mu z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)e^{-(\mu+\theta)z}}{\theta\Gamma(\alpha)}\int_0^z e^{-\beta y}y^{\alpha-1}dy+\frac{\mu\beta^{\alpha}e^{-\mu z}}{\theta\Gamma(\alpha)}\int_0^z e^{-(\theta+\beta)t}t^{\alpha-1}dt$$

$$+\frac{\mu\beta^{\alpha}e^{-(\mu+\theta+\beta)z}}{\Gamma(\alpha)}z^{\alpha-1}\int_0^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy$$

$$-\frac{\mu\beta^{\alpha}e^{-(\mu+\theta+\beta)z}}{\Gamma(\alpha)}\int_0^z e^{(\theta+\beta)y}\tilde{N}(\theta|y)(z-y)^{\alpha-1}dy. \tag{6}$$

It is easy to see that, $\int\limits_0^z e^{-\beta y}y^{\alpha-1}dy=e^{-\beta z}z^{\alpha}\sum\limits_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$.

Multiplying both sides of last equation (6) by $e^{-(\mu+\theta+\beta)z}$, we obtain

$$e^{(\theta+\beta)z}\tilde{N}'(\theta|z)-\mu e^{(\theta+\beta)z}\tilde{N}(\theta|z)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int\limits_0^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy$$

$$-\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}\int\limits_0^z e^{(\theta+\beta)y}\tilde{N}(\theta|y)(z-y)^{\alpha-1}dy. \tag{7}$$

We denote

$$Q(\theta|z)=e^{(\theta+\beta)z}\tilde{N}(\theta|z). \tag{8}$$

Then equation (7) can be rewritten as follows

$$Q'(\theta|z)-(\mu+\theta+\beta)Q(\theta|z)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int\limits_0^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy-\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}\int_0^z Q(\theta|y)(z-y)^{\alpha-1}dy. \tag{9}$$

It is known that the Riemann-Liouville integral can be expressed by (see, [17, 18])

$$D_z^{-\alpha}(Q(\theta|z))=\frac{1}{\Gamma(\alpha)}\int_0^z Q(\theta|y)(z-y)^{\alpha-1}dy,\ \ 0<\alpha\le 1.$$

Taking into account the last equality in (9), we obtain

$$Q'(\theta|z)-(\mu+\theta+\beta)Q(\theta|z)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\beta^{n}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^{n}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int_{0}^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy-\mu\beta^{\alpha}D_{z}^{-\alpha}Q(\theta|z)\,. \tag{10}$$

By applying Riemann-Liouville fractional derivative of order α to both sides equation (10), we obtain

$$D_{z}^{\alpha+1}Q(\theta|z)-(\mu+\theta+\beta)D_{z}^{\alpha}Q(\theta|z)+\mu\beta^{\alpha}Q(\theta|z)$$

$$=\frac{1}{\theta}\left[(\mu+\theta)D_{z}^{\alpha}e^{\beta z}-\mu D_{z}^{\alpha}e^{(\theta+\beta)z}\right]-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\beta^{n}D_{z}^{\alpha}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}D_{z}^{\alpha}z^{\alpha-1}\int_{0}^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^{n}D_{z}^{\alpha}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}. \tag{11}$$

It is well known that the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{\mathrm{z}^{\mathrm{k}}}{(\alpha\mathrm{k}+\beta)},\quad \alpha,\beta>0,\ \ \alpha,\beta\in R.$$

Obviously, $E_{1,1}(\beta z)=e^{\beta z}$. The Riemann-Liouville fractional derivative of the power and the exponential functions are given, respectively, by

$$D_{z}^{\alpha}z^{\alpha-1}=0,\quad D_{z}^{\alpha}z^{n+\alpha}=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\,z^{n},\ \ D_{z}^{\alpha}e^{\beta z}=z^{-\alpha}E_{1,1-\alpha}(\beta z).$$

Therefore, the (11) can be rewritten as

$$D_{z}^{\alpha+1}Q(\theta|z)-(\mu+\theta+\beta)D_{z}^{\alpha}Q(\theta|z)+\mu\beta^{\alpha}Q(\theta|z)$$

$$=\frac{1}{\theta}\left[(\mu+\theta)z^{-\alpha}E_{1,1-\alpha}(\beta z)-\mu z^{-\alpha}E_{1,1-\alpha}((\theta+\beta)z)\right]-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}E_{1,1}(\beta z)$$

$$+\frac{\mu\beta^{\alpha}}{\theta}E_{1,1}((\theta+\beta)z). \tag{12}$$

4. Solution of Fractional Differential Equation (12)

Theorem 4.1 *Let $s>\theta+\beta$ and $\left|s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}\right|>|\mu\beta^{\alpha}|$. Then, a solution of the fractional order differential equation (12) has the form*

$$Q(\theta|z)=\frac{(\mu+\theta)}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,(\alpha+1)n+\ell+2}(\beta z)$$

$$-\frac{\mu}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left((\theta+\beta)z\right)$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left(\beta z\right)$$

$$-\frac{\beta^{\alpha}\mu}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left((\theta+\beta)z\right)$$

$$+\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}D_{z}^{\alpha}Q\left(\theta\,|\,0\right). \tag{13}$$

Proof Applying to the equation (12) Laplace transform by z and taking into account $D_{z}^{\alpha-1}(Q\left(\theta|0\right))=0$, we can write

$$L\left[Q\left(\theta\,|\,z\right)\right]=\frac{(\mu+\theta)}{\theta}\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\mu}{\theta}\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}\frac{1}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\beta^{\alpha}\mu}{\theta}\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$+\frac{D_{z}^{\alpha}Q\left(\theta|\,0\right)}{s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}}, \tag{14}$$

where $s>\theta+\beta$.

Now, applying to the equation (14) inverse Laplace transform by s, we obtain

$$Q\left(\theta\,|\,z\right)=\frac{(\mu+\theta)}{\theta}L^{-1}\left[\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\mu}{\theta}L^{-1}\left[\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}L^{-1}\left[\frac{1}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\beta^{\alpha}\mu}{\theta}L^{-1}\left[\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$+D_{z}^{\alpha}Q\left(\theta\,|\,0\right)L^{-1}\left[\frac{1}{s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}}\right]. \tag{15}$$

It is known that ([11], Lemma 5), for $s > \theta + \beta$ and $\left|s^{\alpha+1} - (\mu + \theta + \beta)s^{\alpha}\right| > |\mu\beta^{\alpha}|$, we obtain

$$L^{-1}\left[\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}(\beta z),$$

$$L^{-1}\left[\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}((\theta+\beta)z),$$

$$L^{-1}\left[\frac{1}{(s-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}(\beta z),$$

$$L^{-1}\left[\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}((\theta+\beta)z)$$

and

$$L^{-1}\left[\frac{1}{s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}}\right] = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}.$$

This concludes the proof of theorem. □

Taking into account (13) in (8), expression of the function $\tilde{N}(\theta|z)$ can be given as follows

$$\tilde{N}(\theta|z) = \frac{(\mu+\theta)}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}(\beta z)$$

$$-\frac{\mu}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}((\theta+\beta)z)$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}(\beta z)$$

$$-\frac{\beta^{\alpha}\mu}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\, z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}((\theta+\beta)z)$$

$$+e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}D_{z}^{\alpha}Q(\theta|0).$$

5. Conclusion

The main purpose of this study is to investigate the semi-Markov random walk process with negative drift, positive jumps. In general case, we construct an integral equation for the Laplace transform of the conditional distribution of the random variable. In particular, the fractional order differential equation is obtained from constructed integral equation in the class of gamma distributions. The fractional derivatives are described in the Riemann-Liouville sense. In conclusion, we find solution of the fractional order differential equation.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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