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Analysis of Fractional Advection Equation with Improved Homotopy Analysis Method

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ABSTRACT

The enhanced homotopy analysis method was employed to solve the nonlinear time-fractional advection equation, resulting in the derivation of a series of solutions. The objective of this study is to minimize the absolute error by identifying the ideal value for a certain parameter, indicated as h, by utilizing the residual error (RE) function associated with that parameter. The 3-dimensional graphs illustrating the absolute discrepancies between the solutions obtained from the exact approach and the modified homotopy analysis method have been made using the MAPLE software. Hence, the proposed approach demonstrates efficacy and appropriateness in tackling fractional partial differential equations. It has been discovered that all optimal homotopy-analysis methods significantly enhance the speed at which series solutions converge. It is highly recommended to use the most effective methods that involve four unknown convergence-control parameters. This efficient method possesses broad implications and can be employed to obtain rapidly converging series solutions for various types of equations exhibiting significant nonlinearity.

Geliştirilmiş Homotopi Analiz Metodu ile Kesirli Mertebeden Adveksiyon Denkleminin Analizi

Araştırma Makalesi

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ÖZ

Doğrusal olmayan zaman-kesirli mertebeden adveksiyon denklemini çözmek için geliştirilmiş homotopi analiz yöntemi kullanıldı ve bu da bir dizi çözümün türetilmesiyle sonuçlandı. Bu çalışmanın amacı, h ile gösterilen belirli bir parametre için o parametreye ilişkin artık hata fonksiyonundan yararlanılarak ideal değeri belirleyerek mutlak hatayı en aza indirmektir. Kesin yaklaşımla elde edilen çözümler ile değiştirilmiş homotopi analiz yöntemi arasındaki mutlak farklılıkları gösteren 3 boyutlu grafikler MAPLE yazılımı kullanılarak yapılmıştır. Dolayısıyla önerilen yaklaşım kesirli kısmi diferansiyel denklemlerin çözümünde etkinlik ve uygunluk göstermektedir. Tüm optimal homotopi analizi yöntemlerinin, seri çözümlerin yakınsama hızını önemli ölçüde arttırdığı keşfedilmiştir. Bilinmeyen dört yakınsama kontrol parametresini içeren en etkili yöntemlerin kullanılması şiddetle tavsiye edilir. Bu etkili yöntemin geniş sonuçları vardır ve önemli ölçüde doğrusal olmayanlık sergileyen çeşitli denklem türleri için hızla yakınsayan seri çözümler elde etmek için kullanılabilir.

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1. Introduction

The field of fractional calculus has been subject to substantial research and has been rigorously defined by a multitude of eminent scientists. The researchers have developed innovative conceptualizations of fractional calculus, which have later established the foundation for the discipline of fractional analysis. Fractional partial differential equations (FPDEs) are commonly utilized in the development of nonlinear models and the analysis of dynamical systems. The application of fractional calculus has been employed to examine and investigate several topics, including chaos theory, financial models, disordered environments, and optics. The identification and analysis of nonlinear phenomena in natural systems are predominantly dependent on the resolution of fractional differential equations. A wide array of analytical and numerical approaches are utilized to obtain precise solutions for fractional differential equations that encompass nonlinear phenomena, owing to their inherent intricacy (Liouville, 1832; Riemannn, 1896; Caputo, 1969; Miller and Ross, 1993; Podlubny, 1999; Caponetto et al., 2010; Baleanu et al., 2012; Liu et al., 2015; Povstenko, 2015; Baleanu et al., 2017; Sweilam et al., 2017; Esen et al., 2018; Veeresha et al., 2019).

The concept of the HAM was initially presented by Liao (1992) in his PhD dissertation. In subsequent years, the application of HAM was utilized to tackle various diverse issues. The application of the analytical series solution technique enabled the management of convergence control in combination with the HAM. By employing these techniques, a multitude of scholars have successfully addressed a wide range of physical and engineering challenges. Sun (2004) conducted a study that focused on the subject of nonlinear traveling waves. In this investigation, the HAM approach was employed. The HAM, initially proposed by Song and Zhang in 2007, was initially employed to investigate the fractional KdV-Burgers-Kuramoto problem. The HAM was effectively utilized to solve the generalized Benjamin-Bona-Mahony model. The application of the HAM has demonstrated efficacy in addressing several FPDEs, including the fractional wave equation, hyperbolic equation, and Fisher equation. The determination of the range of the convergence control parameter (CCP) h in the HAM is accomplished by utilizing the approach of plotting h curves. A study was conducted to ascertain the most effective parameter of the proposed methodology. A range of approaches were employed to determine the value of the h parameter (Liao, 1992; Liao, 1999; Liao, 2003; Liao, 2005; Abbasbandy, 2008; Abdulaziz et al., 2008; Dehghan et al., 2009; Liao, 2010; Niu et al., 2010; Elsaid, 2011; Arafa et al., 2012; Fan et al., 2013; Freihat et al., 2013; Lu, 2014; Lu and Liu, 2014; Aslanov, 2015; Jia et al., 2017; Hariharan, 2017; Alkan, 2022).

The utilization of the HAM has proven to be effective in the resolution of nonlinear functional partial differential equations (FPDEs). Yusufoglu and Selam (2010) conducted a study to examine the convergence range of the CCP h. The modified equal-width wave equation was investigated using the HAM. The authors further demonstrated the efficacy of the approach by determining the optimal value of h. The objective of Elsaid's (2011) work was to employ the HAM to address the space-time Riesz-Caputo FPDEs. The utilization of the HAM has been employed as a means to provide a numerical solution for the time-fractional Swift-Hohenberg problem. The nonlinear Fornberg-Whitham problem has been computationally solved using the HAM. The researchers also determined the ideal values for the CCP within the stated limits. Furthermore, the HAM has been employed as a computational technique to address the nonlinear fractional wave-like equation (Song and Zhang, 2007; Sakar and Erdoğan, 2013; Shaiq et al., 2013; Odibat and Bataineh, 2015; Pandey and Mishra, 2017; Sakar, 2017; Odibat, 2018).

The HAM has been utilized to computationally solve the Korteweg-de Vries Burger equation. The application of the HAM has been utilized as a strategy to acquire solutions for nonlinear wave-like equations. A unique methodology has been introduced for utilizing the HAM to address nonlinear issues. The proposed methodology effectively addresses the challenge associated with the computation of intricate integrals. An approach was devised to effectively determine the most suitable critical control points (CCPs) utilized in the analysis of the convergence zone inside hierarchical agglomerative clustering. The present study presents a recommended methodology for effectively addressing third-order fractional dispersive wave equations through the utilization of a hybrid approach. The methodology utilized in this research involved the integration of the HAM and the Sumudu transform, as outlined in the reference (Sun, 2004). The utilization of the HAM has demonstrated efficacy in the resolution of Partial Differential Equations (PDEs) frequently encountered within the realm of engineering. The Optimal Homotopy Analysis Method (OHAM) has been utilized to address optimal control problems. A novel methodology was developed to ascertain the most suitable choice of a linear operator and initial condition. A novel methodology has been suggested for the use of the Optimal Homotopy Asymptotic Method (OHAM) and error control in the field of nonlinear ordinary differential equations (ODEs) (Liao, 1992; Van Gorder and Vajravelu, 2009; Yusufoğlu and Selam, 2010; Vishal et al., 2012; Fan and You, 2013; Turkyilmazoğlu, 2016; Hariharan, 2017; Van Gorder, 2019).

The fractional advection-diffusion equation is a significant FPDE. The resolution of this equation holds significance in enhancing comprehension of advection and diffusion phenomena within a fractional framework. Consequently, numerical and approximate analytical techniques are typically necessary for this endeavor. The finite element method was developed by Zheng et al. to solve the space fractional advection-diffusion problem. In this study, Wang and Wang (2019) proposed a rapid characteristic finite difference approach to solve the space fractional advection-diffusion problem. Shen et al. have established explicit and implicit difference approximations for space-time fractional advection-diffusion equations. Jiang et al. (2011) obtained analytical solutions for the Caputo-Riesz fractional advection-diffusion equations on a finite domain, incorporating multiple time-space terms were provided. The equations were subject to Dirichlet nonhomogeneous boundary conditions. The analytical solution was derived by utilizing the spectrum form of the fractional Laplacian operator, as demonstrated in reference. Liu et al. investigated a scheme utilizing the finite volume method for solving the space fractional diffusion problem. Bu et al. devised a finite element multigrid approach for solving multi-term time fractional advection-diffusion equations. In their study, Parvizi et al. (2014) introduced a Jacobi collocation technique to numerically solve the classical fractional advection-diffusion equation, incorporating a nonlinear source factor. Rubab et al. (2015) examined analytical solutions for the time fractional advection-diffusion equation, specifically focusing on cases where the boundary experiences time-dependent pulses. The analytical solutions of the fractional advection-diffusion equation with the time fractional Caputo-Fabrizio derivative were determined using the Laplace and Fourier transforms as reference. The solutions of the space-time fractional advection-diffusion equations have been derived using two methodologies. This study employed the Caputo time fractional derivative and the Riesz fractional Laplacian in its analysis. A fully implicit finite difference scheme has been used to solve the time fractional advection–diffusion equation (Zheng et al., 2010; Wang and Wang, 2011; Shen et al., 2011; Jiang et al., 2012; Liu et al., 2014; Bu et al., 2015; Parvizi et al., 2015; Rubbab et al., 2016; Povstenko and Kyrylych, 2017; Mohyud-din et al., 2018).

The main aim of this study is to obtain the numerical solutions for the nonlinear time-fractional advection equation, which includes an arbitrary parameter ħ, by the use of the HAM. Several solutions that have not been previously discussed in the existing body of literature are identified, and their graphical attributes are depicted comprehensively. Although the Adomian decomposition approach does not guarantee the convergence of its approximation series, the homotopy analysis method guarantees the convergence of its approximation series.

The current article is organized subsequent: The following section of the article provides a comprehensive analysis of HAM and its evolutionary path. The third chapter presents the application of the HAM to get numerical solutions for the nonlinear time-fractional advection equation. The concluding chapter of this study presents a comprehensive overview of the significant discoveries that have emerged from the examination.

2. Homotopy Analysis Method

In this section, we will present an introduction to the HAM and the OHAM.

2.1. Homotopy Analysis Method

The purpose of this inquiry is to analyze the underlying notion of the technique employed by examining the given nonlinear equation (Alkan, 2022**)**

$$\mathcal{N}[u(\rho,\sigma)] = 0, \tag{1}$$

where $\boldsymbol{\mathcal{N}}$ is a nonlinear operator, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are spatial and time variables, $\boldsymbol{u}(\boldsymbol{\rho},\boldsymbol{\sigma})$ is unknown function. Let q is an embedding parameter in the range $[0,1]$. Let h be a nonzero CCP. Assume that $H(\rho,\sigma)$ and M are the auxiliary function and linear operator, respectively. Let $u_0(\rho,\sigma)$ be an initial iteration of $u(\rho,\sigma)$.

Hence, the equation describing the deformation of order zero for the solution series $\psi(\rho,\sigma;q)$ can be expressed as follows:

$$(1-q)M[\psi(\rho,\sigma;q)-u_0(\rho,\sigma)] = qhH(\rho,\sigma)\mathcal{N}[\psi(\rho,\sigma;q)]. \quad (2)$$

When the value of q is set to 0 and 1 in Eq. (2), the resulting expressions are derived as follows, correspondingly:

$$\psi(\rho,\sigma;0) = u_0(\rho,\sigma), \psi(\rho,\sigma;1) = \mathrm{u}(\rho,\sigma). \quad (3)$$

As the homotopy parameter q varies from 0 to 1, the function $\psi(\rho,\sigma;q)$ exhibits continuous convergence from the initial iteration $u_0(\rho,\sigma)$ to the exact solution $u(\rho,\sigma)$. Within the field of homotopy theory, the process of continuous transformation is commonly denoted as deformation.

The derivatives for the $\boldsymbol{m}$-th deformation equation are described as:

$$u_0^{(m)}(x,t) = \left.\frac{\partial^m \psi(x,t;q)}{\partial q^m}\right|_{q=0}. \quad (4)$$

By utilizing Taylor's theorem, we can derive the power series expansion of $\boldsymbol{\psi(\rho,\sigma;q)}$ with respect to $\boldsymbol{q}$.

$$\psi(\rho,\sigma;q) = u_0(\rho,\sigma) + \sum_{m=1}^{+\infty} u_m(\rho,\sigma)q^m. \quad (5)$$

If the auxiliary linear operator M, initial iteration $u_0(\rho,\sigma)$, CCP h, and auxiliary function $H(\rho,\sigma)$ are suitably selected, the power series $\psi(\rho,\sigma;q)$ exhibits convergence at $q=1$ and can be derived as

$$u(\rho,\sigma) = u_0(\rho,\sigma) + \sum_{m=1}^{\infty} u_m(\rho,\sigma). \quad (8)$$

According to Liao's (2005) findings in the literature, it has been demonstrated that when one of the solutions to the Eq. (1) is $\hbar = -1$ and $H(x,t) = 1$, the Eq. (2) may be converted to a specific form

$$(1-q)M[\psi(\rho,\sigma;q)-u_0(\rho,\sigma)] + q\mathcal{N}[\psi(\rho,\sigma;q)] = 0. \quad (9)$$

that is employed in the homotopy perturbation method (Liao, 2005).

The vector $\vec{\boldsymbol{u}}_{\boldsymbol{m}}$ is described by

$$\vec{u}_m = \{u_0(\rho,\sigma), u_1(\rho,\sigma), \dots, u_m(\rho,\sigma)\}. \quad (10)$$

The equation for $u_m(\rho,\sigma)$ can be derived from the zeroth-order deformation equation, as stated in Eq. (6).

By employing the $\boldsymbol{k_m}$ function described by

$$k_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (11)$$

Then we derive the equation

$$M[u_m(\rho,\sigma) - k_m u_{m-1}(\rho,\sigma)] = hH(\rho,\sigma)R_m(\vec{u}_{m-1},\rho,\sigma), \quad (12)$$

where $\boldsymbol{R_m(\vec{u}_{m-1},\rho,\sigma)}$ is expressed in the following.

$$R_m(\vec{u}_{m-1},\rho,\sigma) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{N}[\psi(\rho,\sigma;q)]}{\partial q^{m-1}}\right|_{q=0}. \quad (13)$$

The m-th order deformation equation can be derived by utilizing Equation (13). By utilizing equations (5) through (13), $R_m(\vec{u}_{m-1},\rho,\sigma)$ has been determined as

$$R_m(\vec{u}_{m-1},\rho,\sigma) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\left[\sum_{n=0}^{+\infty} u_n(\rho,\sigma)q^n\right]\Bigg|_{q=0}. \tag{14}$$

The way to obtaining the $\boldsymbol{m-th}$ order solution is as follows:

$$u(\rho,\sigma) = \sum_{k=0}^{m} u_k(\rho,\sigma), \tag{15}$$

where the solution $u(\rho,\sigma)$ incorporates the CCP ħ.

2.2. Improved Homotopy Analysis Method

The exact calculation of the square RE in ODEs for the approximation of order $\boldsymbol{m}$ is formally established as (Liao, 2010; Alkan, 2022**)**

$$\Delta_m = \int_0^{+\infty} \left(\mathcal{N}\left(\sum_{i=0}^{m} s_i\,(\mathrm{r})\right)\right)^2 dr, \tag{16}$$

where the formula $\Delta_{\boldsymbol{m}}$ incorporates a CCP, $\boldsymbol{\mathcal{N}}$ is nonlinear operator, denoted as $\boldsymbol{h}$, which is currently unknown. The CCP $\boldsymbol{h}$ for the approximation of order $\boldsymbol{m}$ is determined by finding a minimum value of $\Delta_{\boldsymbol{m}}$, which represents the optimal value. Therefore, we have the equality

$$\frac{d\Delta_m}{dh} = 0. \tag{17}$$

Nevertheless, previous research has demonstrated that the computation of Δ_m, as defined by Liao using formula (16), necessitates a substantial amount of CPU time, even when employing a low approximation. To decrease CPU time, the average quadratic RE ($\sqrt{E_m}$) is formulated in the following [55]:

$$E_m = \frac{1}{n+1}\sum_{j=0}^{n}\left(N\left(\sum_{i=0}^{m} s_i\left(\frac{j}{n},h\right)\right)\right)^2. \tag{18}$$

The nonlinear time-fractional advection equation (NTFAE), as explained in this work, employed the appropriate form of Eq. (18).

2.3. Convergence Analysis

Theorem 2.1. If the homotopy series presented in Equation (8) exhibits convergence, the outcome is described as (Liao, 2010; Alkan, 2022**)**

$$\sum_{n=1}^{\infty} R_n(\vec{u}_{n-1},\rho,\sigma) = 0. \tag{19}$$

Proof. The source of this information can be located in (Liao, 2010).

Theorem 2.2. In the event that the homotopy series shown in Equation (8) exhibits convergence, this series must serve as a solution to the initial nonlinear equation denoted as Equation (1) (Liao, 2010; Alkan, 2022**).**

Proof. It can be seen in (Liao, 2010; Alkan, 2022**).**

3. The Time-Fractional Advection Equation

In this section, we will examine the nonlinear cases of the NTFAE.

Example 3.1.

Consider the NTFAE with the initial condition

$$D_\sigma^\alpha u(\rho,\sigma)+u(\rho,\sigma)u_\rho(\rho,\sigma)=f(\rho,\sigma), 0\le\rho,\sigma\le 1\,,\ 0<\alpha\le 1,$$

$$u(\rho,0)=0, \tag{20}$$

where, D_σ^α is the Caputo fractional derivative operator, $f(\rho,\sigma)$ is an unknown function.

If $f(\rho,\sigma)$ in Eq. (20) is taken as

$$f(\rho,\sigma)=\frac{\rho(2\sigma^{(2-\alpha)}+\sigma^4\Gamma(3-\alpha))}{\Gamma(3-\alpha)}, \tag{21}$$

then Eq. (20) becomes

$$D_\sigma^\alpha u(\rho,\sigma)+u(\rho,\sigma)u_\rho(\rho,\sigma)=\frac{\rho(2\sigma^{(2-\alpha)}+\sigma^4\Gamma(3-\alpha))}{\Gamma(3-\alpha)}, \tag{22}$$

$$u(\rho,0)=0. \tag{23}$$

With initial condition in Eq. (23), the exact solution of Eq. (22) is $u(\rho,\sigma)=\rho\sigma^2$. In Eq. (22), the expression with fractional derivative

$$M[u(\rho,\sigma;q)]=D_\sigma^\alpha u(\rho,\sigma;q), \tag{24}$$

is chosen as the linear operator. In addition, the nonlinear operator from Eq. (22) is chosen as

$$N[u(\rho,\sigma;q)]=D_\sigma^\alpha u(\rho,\sigma)+u(\rho,\sigma)u_\rho(\rho,\sigma)-\frac{\rho\left(2\sigma^{(2-\alpha)}+\sigma^4\Gamma(3-\alpha)\right)}{\Gamma(3-\alpha)}. \tag{25}$$

Using the homotopy definition, the equation for zeroth-order deformation is constructed in the form of

$$(1-q)M[u(\rho,\sigma;q)-u_0(\rho,\sigma)]=q\hbar H(\rho,\sigma)N[u(\rho,\sigma;q)], \tag{26}$$

where ħ is the convergence-control parameter. When the values of q are taken to 0 and 1 in Eq. (26), the resulting outcomes are derived as

$$u(\rho,\sigma;0)=u_0(\rho,\sigma)=u(x,0),\qquad u(\rho,\sigma;1)=u(\rho,\sigma). \tag{26}$$

m-th order deformation equation is also in the form of

$$L[u_m(\rho,\sigma)-k_m u_{m-1}(\rho,\sigma)]=\hbar R_m(\vec{u}_{m-1}(\rho,\sigma)), \tag{27}$$

where,

$$R_m(\vec{u}_{m-1}(\rho,\sigma))=D_\sigma^\alpha u_{m-1}(\rho,\sigma)+\sum_{k=0}^{m-1}\left[u_k(\rho,\sigma)\frac{\partial u_{m-1-k}(\rho,\sigma)}{\partial\rho}\right]-(1-k_m)$$

$$\times \frac{\rho(2\sigma^{(2-\alpha)} + \sigma^4\Gamma(3-\alpha))}{\Gamma(3-\alpha)}. \tag{28}$$

In Eq. (27), for $m \geq 1$ using the inverse of the operator, the equation for m-th order deformation is derived as

$$u_m(\rho,\sigma) = k_m u_{m-1}(\rho,\sigma) + \hbar M^{-1}\left(R_m\left(\vec{u}_{m-1}(\rho,\sigma)\right)\right). \tag{29}$$

For $m = 0,1,2,\ldots$ in Eq. (29), the iterations are derived sequentially as follows.

$$u_0(\rho,\sigma) = 0, \tag{30}$$

$$u_1(\rho,\sigma) = -\rho\hbar\sigma^2 - \frac{24\rho\hbar\sigma^{\alpha+4}}{\Gamma(\alpha+5)}, \tag{31}$$

$$\vdots$$

The CCP in the iteration is denoted by the symbol ħ. Also, the approach of squared RE is employed to determine the optimal value of this parameter. $(\rho,\sigma) \in [0,1] \times [0,1]$ will be taken as the region. The Residual function will be defined in the following.

$$r_4(\rho,\sigma,\hbar) = D_\sigma^\alpha w_4(\rho,\sigma,\hbar) + w_4(\rho,\sigma,\hbar)\frac{\partial w_4(\rho,\sigma,\hbar)}{\partial\rho} - \frac{\rho(2\sigma^{(2-\alpha)} + \sigma^4\Gamma(3-\alpha))}{\Gamma(3-\alpha)}. \tag{32}$$

The second norm of this residual function has the form of

$$e_4(\hbar) = \left(\int_0^1\int_0^1 |r_4(\rho,\sigma,\hbar)|\, d\sigma d\rho\right)^{\frac{1}{2}}. \tag{33}$$

When specifying the optimal values of parameter ħ, the minimum of $e_4(\hbar)$ based on norm 2 will be selected. Table 1 presents the optimal ħ parameters for several values of α, specifically $\alpha = 1, 0.9, 0.8,$ and 0.7. The numerical outcomes are depicted as tables and graphs, specifically Tables 2-5 and Figures 1-4.

Table 1. The optimal ħ parameters for $\alpha = 0.7,0.8,0.9,1$ for Example 3.1.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
4	-0.8740320877	-0.8888546878	-0.8825292730	-0.8605335608

Table 2. Values of solutions and errors with $\alpha = 1$ for Example 3.1

ρ	σ	HAM	Exact Solution	Error
0.20	0.10	0.2011817673	0.200	1.18E-3
	0.20	0.0080008508	0.008	8.50E-7
	0.40	0.0320795815	0.032	7.95E-5
	0.60	0.0725622976	0.072	5.62E-4
	0.80	0.1296835214	0.128	1.68E-3
0.40	0.10	0.4023635349	0.400	2.36E-3
	0.20	0.0160017017	0.016	1.70E-6
	0.40	0.0641591631	0.064	1.59E-4
	0.60	0.1451245952	0.144	1.12E-3
	0.80	0.2593670425	0.256	3.36E-3

Table 3. Values of solutions and errors with $\alpha = 0.9$ for Example 3.1

ρ	σ	HAM	Exact Solution	Error
0.20	0.10	0.1970572236	0.200	2.94E-3
	0.20	0.0080023240	0.008	2.32E-6
	0.40	0.0320935092	0.032	9.35E-5
	0.60	0.0725621051	0.072	5.62E-4
	0.80	0.1291435433	0.128	1.14E-3
0.40	0.10	0.3941144472	0.400	5.88E-3
	0.20	0.0160046481	0.016	4.64E-6
	0.40	0.0641870184	0.064	1.87E-4
	0.60	0.1451242103	0.144	1.12E-3
	0.80	0.2582870866	0.256	2.28E-3

Table 4. Values of solutions and errors with $\alpha = 0.8$ for Example 3.1

ρ	σ	HAM	Exact Solution	Error
0.20	0.10	0.1931274324	0.200	6.87E-3
	0.20	0.0080036345	0.008	3.63E-6
	0.40	0.0321247177	0.032	1.24E-4
	0.60	0.0726722204	0.072	6.72E-4
	0.80	0.1289050112	0.128	9.05E-4
0.40	0.10	0.3862548647	0.400	1.37E-2
	0.20	0.0160072690	0.016	7.26E-7
	0.40	0.0642494354	0.064	2.49E-4
	0.60	0.1453444408	0.144	1.34E-3
	0.80	0.2578100224	0.256	1.81E-3

Table 5. Values of solutions and errors with $\alpha = 0.7$ for Example 3.1

ρ	σ	HAM	Exact Solution	Error
0.20	0.10	0.1895745570	0.200	1.04E-2
	0.20	0.0080052958	0.008	5.29E-6
	0.40	0.0321829967	0.032	1.82E-4
	0.60	0.0729356757	0.072	9.35E-4
	0.80	0.1290735321	0.128	1.07E-3
0.40	0.10	0.3791491139	0.400	2.08E-2
	0.20	0.0160105915	0.016	1.05E-5
	0.40	0.0643659935	0.064	3.65E-4
	0.60	0.1458713515	0.144	1.87E-3
	0.80	0.2581470643	0.256	2.14E-3

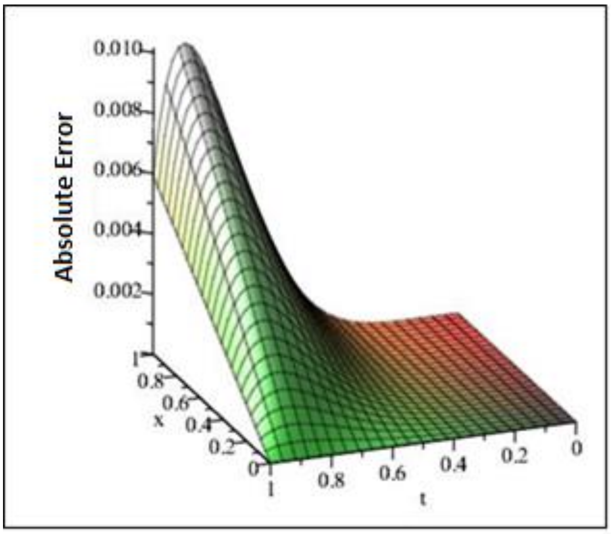

Figure 1. Absolute error graph for $\alpha = 1$ for Example 3.1.

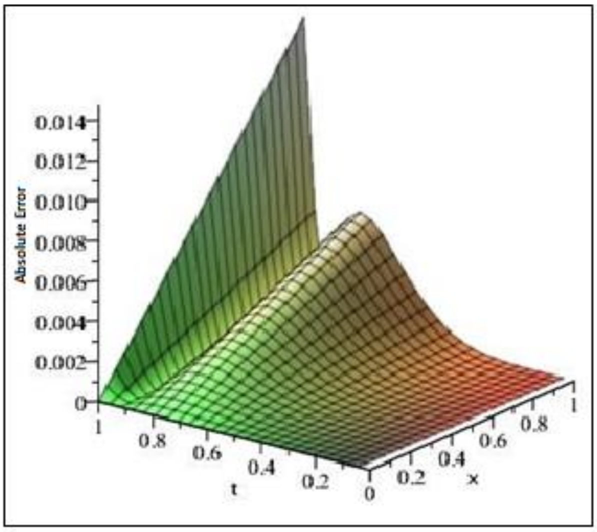

Figure 2. Absolute error graph for $\alpha = 0.9$ for Example 3.1.

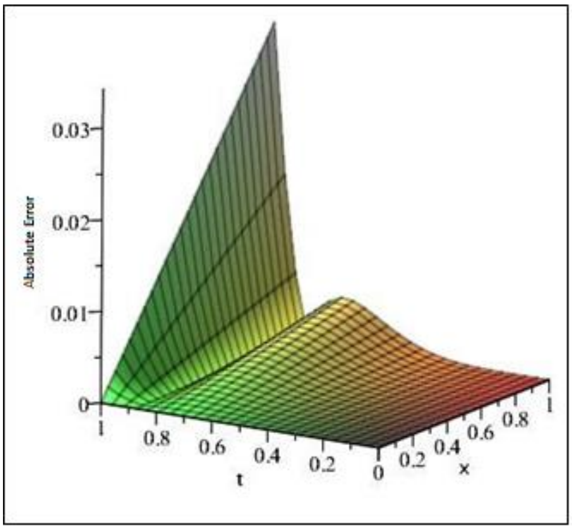

Figure 3. Absolute error graph for $\alpha = 0.8$ for Example 3.1.

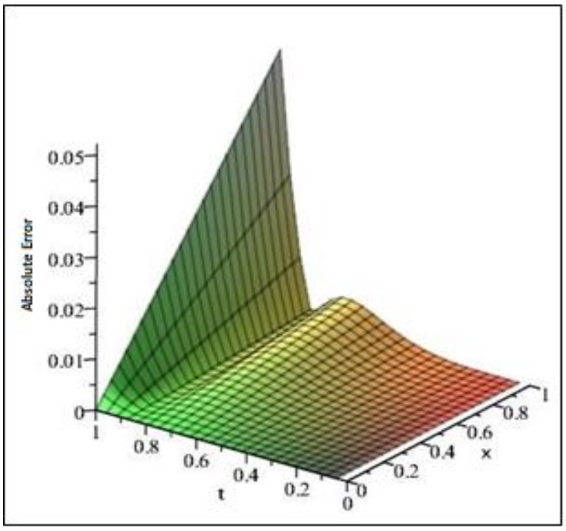

Figure 4. Absolute error graph for $\alpha = 0.7$ for Example 3.1.

4. Conclusion

The present study successfully implemented the optimal parameter for partial differential equations to solve the time-fractional advection equation, yielding a series of solutions. By implementing a more focused examination area or augmenting the number of iterations, it becomes feasible to reduce the absolute error. Based on the solutions derived for the time-fractional advection equation as presented in this study, it is believed that this approach has the potential to be applied in the resolution of both linear and nonlinear FPDEs.

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Conflict of Interest Statement

The author of the article declares that she has no conflict of interest.

Authors' Contributions

The author declares that she has contributed 100% to the article.

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