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Almost Hermitian Structures From Almost Contact Metric Manifolds and Their Curvature Properties

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Abstract

In this manuscript, we consider almost Hermitian manifolds and almost contact metric manifolds. We construct almost Hermitian manifolds from the product of almost contact metric manifolds with $\mathbb{R}$ by warped product. Depending on the function of warped product, we investigate the curvature properties of the almost Hermitian manifolds obtained in this way. In particular, we study Einstein almost Hermitian manifolds obtained from Einstein almost contact metric manifolds. In addition, we study the relationships between some classes of almost contact metric manifolds and almost Hermitian manifolds.

Keywords: *almost contact metric structure; almost Hermitian structure;curvature; Einstein manifold; product manifold; warped product.*
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1. Introduction

Almost contact metric structures are odd dimensional counterparts of almost Hermitian structures. Manifolds with almost Hermitian structures were classified by [11] according to properties of Levi-Civita covariant derivative of the Kaehler form. The classification of manifolds with almost contact metric structures was first done in [13] by considering the canonical almost Hermitian structure on $M \times \mathbb{R}$ and new classes of almost contact metric structures were obtained together with several examples. A complete classification of almost contact metric manifolds was given in [2, 8] by considering the symmetry of the Levi-Civita covariant derivative of the fundamental 2-form of the almost contact metric structure.

In the literature there are papers on constructing a structure from another on a product manifold, for example, see [14] and references therein for almost Hermitian structures obtained from two almost contact metric structures. In [5], Kaehlerian structures obtained from two trans-Sasakian structures were considered. In [4], $\mathscr{D}$-homothetic warping was introduced and studied. In [6], the idea of $\mathscr{D}$-homothetic warping was generalized and relations between some classes of almost contact structures and almost Hermitian structures was studied. For recent progress on relations between almost contact metric structures and almost Hermitian structures, also see [10, 7, 3, 19] and related references. In addition to the almost contact case, almost parahermitian manifolds were constructed by multiplying almost paracontact metric manifolds with $\mathbb{R}$ in [15]. A similar study has been carried out for almost contact (complex) B-metric structures in [17]. In this study, we focus on to construct an almost Hermitian structure on $M \times \mathbb{R}$ by an almost contact metric structure on M and a smooth function σ on $\mathbb{R}$ by warped product. The function σ enables one to obtain various almost Hermitian structures from a given almost contact metric structure.

We also obtain results about curvature properties on $M \times \mathbb{R}$ and state theorems about the existence of certain structures such as Einstein manifold, a structure with zero scalar curvature, by specifying the function σ. We also give examples supporting our results.

2. Preliminaries

An ordered triple (φ, ξ, η), where φ is an endomorphism, ξ is a vector field, η is a 1-form is called an almost contact structure on a smooth manifold M^{2n+1} if

$$\eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 = -I + \eta \otimes \xi. \tag{2.2}$$

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If there also exists a compatible Riemannian metric g with the property that

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.3}$$

where X,Y are smooth vector fields on M, then (M,φ,ξ,η,g) is called an almost contact metric manifold. Identities (2.1), (2.2), (2.3) imply

$$\eta\circ\varphi=0,\ \ \varphi\xi=0,\ \ \eta(X)=g(X,\xi),\ \ g(\varphi X,Y)=-g(X,\varphi Y).$$

The fundamental 2-form Φ is defined by $\Phi(X,Y)=g(\varphi X,Y)$.
We denote smooth vector fields and also tangent vectors by letters X,Y,Z.
There are 2^{12} classes of almost contact metric manifolds. Note that the non-existence of certain classes in dimensions higher than 3 is shown in [20]. Let α be the tensor

$$\alpha(X,Y,Z)=g((\nabla_X\varphi)(Y),Z),$$

for all $X,Y,Z\in T_pM$ where T_pM is the tangent space at p and ∇ denotes the covariant derivative of g. By (2.1), (2.2), (2.3), the tensor α satisfies the followings:

$$\alpha(X,Y,Z)=-\alpha(X,Z,Y) \tag{2.4}$$

$$\alpha(X,Y,Z)=-\alpha(X,\varphi Y,\varphi Z)+\eta(Y)\alpha(X,\xi,Z)+\eta(Z)\alpha(X,Y,\xi). \tag{2.5}$$

The following 1-forms are associated with α:

$$f(X)=\sum_i\alpha(e_i,e_i,X),\ \ f^*(X)=\sum_i\alpha(e_i,\varphi e_i,X),\ \ \omega(X)=\alpha(\xi,\xi,X),$$

where $X\in T_pM$, $\{e_i,\xi\}$ is an orthonormal basis for T_pM.
Let $\mathscr{F}$ be the set of all $(0,3)$ tensors over T_pM having properties (2.4) and (2.5). Then $\mathscr{F}$ is the direct sum of twelve subspaces $\mathscr{F}_i$, $i=1,\dots,12$. The defining conditions of the classes we consider are listed below [2, 8].

$$\mathscr{F}_1:\ \alpha(X,Y,Z)\ =\ \eta(X)\eta(Y)\omega(Z)-\eta(X)\eta(Z)\omega(Y) \tag{2.6}$$

$$\mathscr{F}_2:\qquad \alpha(X,Y,Z)=\frac{f(\xi)}{2n}\{\eta(Z)g(X,Y)-\eta(Y)g(X,Z)\}, \tag{2.7}$$

$\mathscr{F}_2$ is the class of γ-Sasakian manifolds, where $\gamma=\frac{f(\xi)}{2n}$. Note that γ-Sasakian manifolds are usually denoted by α-Sasakian. However we use γ since α is used for the $(0,3)$ tensor of the almost contact structure in this paper.

$$\mathscr{F}_3:\ \alpha(X,Y,Z)=-\frac{f^*(\xi)}{2n}\{\eta(Z)g(X,\varphi Y)-\eta(Y)g(X,\varphi Z)\},$$

$\mathscr{F}_3$ is the class of γ-Kenmotsu manifolds, where $\gamma=-\frac{f^*(\xi)}{2n}$.

$$\mathscr{F}_8:\ \alpha(hX,hY,hZ)\ =\ \alpha(X,Y,\xi)=0,\ \ hX=-\varphi^2X$$

$$\mathscr{F}_{11}:\ \alpha(\xi,Y,Z)=\alpha(X,Y,\xi)=0,\ \ \alpha(X,X,Z)=0$$

$\mathscr{F}_{11}$ is the class of nearly-K-cosymplectic manifolds.
The class of cosymplectic manifolds is characterized by $\alpha=0$ and is contained in all $\mathscr{F}_i$, $i=1,\dots,12$.
An almost contact metric manifold is said to be in the class $\mathscr{F}_i\oplus\mathscr{F}_j$, etc if the tensor α is in the class $\mathscr{F}_i\oplus\mathscr{F}_j$ over T_pM for all $p\in M$.
Note that the classes $\mathscr{F}_1,\dots,\mathscr{F}_{12}$ correspond to classes $\mathscr{C}_{12}$, $\mathscr{C}_6$, $\mathscr{C}_5$, $\mathscr{C}_7$, $\mathscr{C}_8$, $\mathscr{C}_9$, $\mathscr{C}_{10}$, $\mathscr{C}_{11}$, $\mathscr{C}_4$, $\mathscr{C}_3$, $\mathscr{C}_1$, $\mathscr{C}_2$ in the classification of [8], respectively. We use the classification of [2] and classes $\mathscr{W}_i$ in [2] are denoted by $\mathscr{F}_i$.
An almost Hermitian manifold is an even dimensional Riemannian manifold (N,h) together with an almost complex structure J (i.e. $J^2=-I$) such that

$$h(J(X),J(Y))=h(X,Y),$$

for all vector fields X,Y in N. The Kaehler form of an almost Hermitian manifold (N,h,J) is defined by

$$F(X,Y)=h(J(X),Y).$$

The fundamental $(0,3)$ tensor β of the structure is

$$\beta(X,Y,Z)=(\nabla_XF)(Y,Z)=h((\nabla_XJ)(Y),Z).$$

In [11], almost Hermitian manifolds were classified into four $U(n)-$irreducible invariant subspaces $\mathscr{W}_1,\mathscr{W}_2,\mathscr{W}_3,\mathscr{W}_4$, depending on the vector space of the fundamental tensors of almost Hermitian structures. Thus there are 16 invariant subspaces, each corresponding to a different class of almost Hermitian manifolds. The defining condition of the classes we study are:

$$\mathscr{W}_4 : \beta(X,Y,Z) = \frac{-1}{2(n-1)}\{h(X,Y)\delta F(Z) - h(X,Z)\delta F(Y) - h(X,JY)\delta F(JZ) + h(X,JZ)\delta F(JY)\} \tag{2.8}$$

$$\mathscr{W}_1 \oplus \mathscr{W}_4 : \beta(X,X,Y) = \frac{-1}{2(n-1)}\{h(X,X)\delta F(Y) - h(X,Y)\delta F(X) - h(JX,Y)\delta F(JX)\} \tag{2.9}$$

$\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3 = \mathscr{S}\mathscr{K}$: $\delta F = 0$

$$\begin{aligned}\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_4 \quad : \quad & \beta(X,Y,Z) + \beta(JX,JY,Z) \\ &= \frac{-1}{n-1}\{h(X,Y)\delta F(Z) - h(X,Z)\delta F(Y) - h(X,JY)\delta F(JZ) + h(X,JZ)\delta F(JY)\}\end{aligned}$$

$$\mathscr{W}_1 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_1 : \beta(X,X,Y) - \beta(JX,JX,Y) = 0 \tag{2.10}$$

$\mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_2 : \mathfrak{S}_{XYZ}\{\beta(X,Y,Z) - \beta(JX,JY,Z)\} = 0$

$\mathscr{W}$: No condition

It is well known that if (M,φ,ξ,η) is an almost contact manifold, then $M \times \mathbb{R}$ is canonically an almost complex manifold with the almost complex structure

$$J(X,a\frac{d}{dt}) = \left(\varphi(X) - a\xi, \eta(X)\frac{d}{dt}\right)$$

defined by [16]. If g is the associated Riemannian metric of an almost contact structure, then two Riemannian metrics are defined on $M \times \mathbb{R}$ by

$$h\left((X,a\frac{d}{dt}),(Y,b\frac{d}{dt})\right) = g(X,Y) + ab$$

and

$$h^0 = e^{2f}h,$$

where $f : M \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(X,t) = t$. Then h and h^0 are Hermitian metrics on $(M \times \mathbb{R}, J)$ and

$$(M \times \mathbb{R}, J, h), \tag{2.11}$$

$$(M \times \mathbb{R}, J, h^0) \tag{2.12}$$

are almost Hermitian manifolds [13]. In [13], the relation between the almost contact structure on M and the almost Hermitian structure on $M \times \mathbb{R}$ is studied. There are several other ways of obtaining an almost Hermitian structure on a product manifold [4, 6, 3, 19]. In the next section, we will use warped product to construct an almost Hermitian structure.

3. Construction of Almost Hermitian Structures

Let (M,φ,ξ,η,g) be an almost contact metric metric manifold and consider the product manifold $M \times \mathbb{R}$. We consider the almost complex structure J on $M \times \mathbb{R}$ obtained in the following way.

$$J(X,a\frac{d}{dt}) = \left(\varphi(X) - e^{-\sigma}a\xi, e^{\sigma}\eta(X)\frac{d}{dt}\right), \tag{3.1}$$

where $(X,a\frac{d}{dt})$ is a vector field on $M \times \mathbb{R}$, $X \in \chi(M)$, t is the coordinate of $\mathbb{R}$, $a \in C^{\infty}(M \times \mathbb{R})$ and σ is a function of t. The warped Riemannian metric h on $M \times \mathbb{R}$ given by

$$h\left((X,a\frac{d}{dt}),(Y,b\frac{d}{dt})\right) = e^{2\sigma}g(X,Y) + ab$$

is a Hermitian metric on $(M \times \mathbb{R}, J)$, that is,

$$h(J(X,a\frac{d}{dt}),J(Y,b\frac{d}{dt})) = h((X,a\frac{d}{dt}),(Y,b\frac{d}{dt})).$$

The Kaehler form F of $(M \times \mathbb{R}, J, h)$ is

$$F((X,a\frac{d}{dt}),(Y,b\frac{d}{dt})) = h(J(X,a\frac{d}{dt}),(Y,b\frac{d}{dt})).$$

The Levi-Civita covariant derivative $\tilde{\nabla}$ of the metric h is evaluated from the Kozsul's formula as follows:

$$\tilde{\nabla}_{(X,a\frac{d}{dt})}(Y,b\frac{d}{dt}) \quad = \quad \left(\nabla_X Y + \frac{d\sigma}{dt}(aY+bX)\left\{X[b]+a\frac{db}{dt} - e^{2\sigma}\frac{d\sigma}{dt}g(X,Y)\right\}\frac{d}{dt}\right). \tag{3.2}$$

Unless otherwise stated, throughout the paper, we will use the notation $\tilde{X},\tilde{Y},\tilde{Z},...$ for the vector fields $(X,a\frac{d}{dt}),(Y,b\frac{d}{dt}),(Z,c\frac{d}{dt}),...$ on the product manifold, respectively.
By choosing $(Y,b\frac{d}{dt}) = (\xi,0\frac{d}{dt})$ in (3.2), one can see that, if ξ is a parallel vector field, then $(\xi,0)$ need not be parallel.
In [6], it is proved that for a $\mathscr{D}$-homothetic bi-warping, if the integral curves of ξ are geodesics (i.e. $d\eta(\xi,X)=0$ for all vector fields X), then $\tilde{\xi} = (\xi,0)$ is Killing if and only if ξ is Killing. In warped product case, the integral curves of ξ need not be geodesics. If ξ is a Killing vector field, then (3.2) implies that $\tilde{\xi} = (\xi,0)$ is a Killing vector field.
In addition the vector field $\tilde{Y} = (0,\frac{d}{dt})$ satisfies the property $h(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) = h(\tilde{\nabla}_{\tilde{Z}}\tilde{Y},\tilde{X})$.
Now we evaluate the Riemannian curvature tensor $\tilde{R}$ of the product manifold in terms of the curvature tensor R of the almost contact metric manifold. For curvature properties of warped products, see for example [9]. Note that in our case, we consider the warped product of $\mathbb{R}$ with an almost contact metric manifold.

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} \quad = \quad \left(R(X,Y)Z + \left((\frac{d\sigma}{dt})^2 c + c\frac{d^2\sigma}{dt^2}\right)(aY-bX) - e^{2\sigma}(\frac{d\sigma}{dt})^2(g(Y,Z)X - g(X,Z)Y), \left\{e^{2\sigma}((\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2})g(-aY+bX,Z)\right\}\frac{d}{dt}\right).$$

To calculate the Ricci curvature $\tilde{Q}$ of the product manifold, we use that for any local orthonormal frame $\{e_i,\xi\}$ of the almost contact metric manifold M, the set

$$\{\tilde{e}_1,\ldots,\tilde{e}_{2n+2}\} = \{e^{-\sigma}(e_1,0),\ldots,e^{-\sigma}(e_{2n},0),e^{-\sigma}(\xi,0),(0,\frac{d}{dt})\}$$

is a local h-orthonormal frame of $M\times\mathbb{R}$. Then since

$$\tilde{Q}(\tilde{X},\tilde{Y}) \quad = \quad \sum_{i=1}^{2n+2} h(\tilde{R}(\tilde{e}_i,\tilde{X})\tilde{Y},\tilde{e}_i),$$

the Ricci curvature $\tilde{Q}$ of the product manifold is

$$\tilde{Q}(\tilde{X},\tilde{Y}) \quad = \quad Q(X,Y) - (2n+1)ab\left\{(\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2}\right\} - e^{2\sigma}g(X,Y)\left\{(2n+1)(\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2}\right\}, \tag{3.3}$$

where Q is the Ricci curvature of the almost contact metric manifold.
It is known that a warped product $N\times_f M$ of a 1-dimensional manifold N and an (n-1)-dimensional Einstein manifold M, where $n\leq 4$ is quasi-Einstein (that is, $rank(Ric-\lambda g)\leq 1$) [9]. In our case, if $N=\mathbb{R}$ and M is an almost contact metric manifold, it is possible to obtain Einstein almost-Hermitian manifolds from Einstein almost contact metric manifolds by choosing the function σ appropriately.

Theorem 3.1. *Let (M,φ,ξ,η,g) be an Einstein almost contact metric manifold with Einstein constant λ. If the function σ satisfies the differential equation*

$$-\frac{\lambda}{2n} = e^{2\sigma}\frac{d^2\sigma}{dt^2}, \tag{3.4}$$

then the almost-Hermitian manifold $M\times\mathbb{R}$ is also Einstein.

Proof. Let (M,φ,ξ,η,g) be Einstein with Einstein constant λ, that is
$Q(X,Y)=\lambda g(X,Y)$. To obtain an Einstein product manifold, from (3.3), the equation

$$\begin{aligned}\tilde{Q}(\tilde{X},\tilde{Y}) &= g(X,Y)\left\{\lambda - e^{2\sigma}(2n+1)(\frac{d\sigma}{dt})^2 - e^{2\sigma}\frac{d^2\sigma}{dt^2}\right\} - (2n+1)ab\left\{(\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2}\right\} \\ &= \nu h(\tilde{X},\tilde{Y}) \\ &= \nu\left(e^{2\sigma}g(X,Y)+ab\right)\end{aligned}$$

should be satisfied for a constant ν and for all vector fields $\tilde{X},\tilde{Y}$. In particular, for any $\tilde{X},\tilde{Y}$ such that $a=b=0$, we get

$$\nu e^{2\sigma} = \lambda - e^{2\sigma}(2n+1)(\frac{d\sigma}{dt})^2 - e^{2\sigma}\frac{d^2\sigma}{dt^2}. \tag{3.5}$$

On the other hand if $\tilde{X}=\tilde{Y}=(0,\frac{d}{dt})$, that is, if $X=Y=0$ and $a=b=1$, we obtain

$$\nu = -(2n+1)\left\{(\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2}\right\}. \tag{3.6}$$

Comparing (3.5) and (3.6), we have

$$\nu = e^{-2\sigma}\lambda - (2n+1)(\frac{d\sigma}{dt})^2 - \frac{d^2\sigma}{dt^2} = -(2n+1)\left\{(\frac{d\sigma}{dt})^2 + \frac{d^2\sigma}{dt^2}\right\}$$

and thus

$$\frac{\lambda}{2n} + e^{2\sigma}\frac{d^2\sigma}{dt^2} = 0.$$

If the function σ is chosen so that $-\frac{\lambda}{2n} = e^{2\sigma}\frac{d^2\sigma}{dt^2}$, then it can be seen that

$$\tilde{Q}(\tilde{X},\tilde{Y}) = \nu h(\tilde{X},\tilde{Y})$$

for all vector fields $\tilde{X},\tilde{Y}$ on the product manifold. □

Note that the differential equation (3.4) has the solution

$$\sigma(t) = \ln\left(\frac{\sqrt{2c(1-\cosh(2\sqrt{k_1}t+\sqrt{k_1}k_2))}}{2\sqrt{k_1}}\right),$$

where $k_1 > 0$, k_2 are constants and $c = -\frac{\lambda}{2n}$ and if $\lambda > 0$, then σ is defined for all real numbers t.
It is known that any 3-Sasakian manifold M is Einstein with positive Einstein constant [12] and scalar curvature $s = 42$ [1]. By Theorem 3.1, $M\times\mathbb{R}$ is also Einstein.
The scalar curvature $\tilde{s}$ of the product manifold is

$$\tilde{s} = e^{-2\sigma}s - (2n+1)(2n+2)(\frac{d\sigma}{dt})^2 - 2(2n+1)\frac{d^2\sigma}{dt^2} \tag{3.7}$$

from (3.3). The equation (3.7) implies

Proposition 3.1. *Let (M,φ,ξ,η,g) be an almost contact metric manifold with scalar curvature $s = 0$. Then the scalar curvature of the product manifold is nonzero for a nonconstant function σ.*

Proof. Let $s = 0$. By equation (3.7), $\tilde{s} = 0$ if and only if the function σ satisfies

$$(\frac{d\sigma}{dt})^2 = -\frac{1}{n+1}\frac{d^2\sigma}{dt^2}.$$

If σ is nonconstant, the solution of this differential equation is

$$\sigma(t) = \frac{1}{n+1}\ln\left(\frac{1}{n+1}k_1 + t\right) + k_2$$

for constants k_1, k_2. Since σ is not defined for all real numbers, there can not be obtained a function defined for all real numbers such that the scalar curvature of the product manifold is zero. □

Now we study the relations between some classes of almost-Hermitian manifolds and almost contact metric manifolds. For this we evaluate the covariant derivative of the almost complex structure J by (3.2) and (3.1).

$$\left(\tilde{\nabla}_{(X,a\frac{d}{dt})}J\right)(Y,b\frac{d}{dt}) = \left((\nabla_X\varphi)(Y) - be^{-\sigma}\nabla_X\xi - b\frac{d\sigma}{dt}\varphi(X) + e^{\sigma}\frac{d\sigma}{dt}(\eta(Y)X - g(X,Y)\xi), \left\{e^{\sigma}(\nabla_X\eta)(Y) - e^{2\sigma}\frac{d\sigma}{dt}g(X,\varphi(Y))\right\}\frac{d}{dt}\right).$$

Then the covariant derivative of the Kaehler form F is

$$\begin{aligned}\beta(\tilde{X},\tilde{Y},\tilde{Z}) &= (\tilde{\nabla}_{\tilde{X}}F)(\tilde{Y},\tilde{Z}) \\ &= h\left((\tilde{\nabla}_{\tilde{X}}J)(\tilde{Y}),\tilde{Z}\right) \\ &= e^{2\sigma}\alpha(X,Y,Z) + e^{\sigma}(c(\nabla_X\eta)(Y) - b(\nabla_X\eta)(Z)) + e^{2\sigma}\frac{d\sigma}{dt}(cg(\varphi(X),Y) - bg(\varphi(X),Z)) + e^{3\sigma}\frac{d\sigma}{dt}(\eta(Y)g(X,Z) - \eta(Z)g(X,Y)).\end{aligned} \tag{3.8}$$

Also the co-derivative of the Kaehler form is

$$\delta F(\tilde{X}) = -f(X) + e^{\sigma}\frac{d\sigma}{dt}2n\eta(X) - ae^{-\sigma}f^*(\xi). \tag{3.9}$$

It is known that the almost contact structure is Sasakian if and only if $(M\times\mathbb{R},J,h^0)$ is Kaehlerian according to construction in [13]. In addition, in [6], $\mathscr{D}$-homothetic bi-warped metric $\tilde{g} = dt^2 + f^2g + f^2(h^2-1)\eta\otimes\eta$ is studied on $\mathbb{R}\times M$, where M is an almost contact metric manifold and it is proved that if $ff' \neq 0$, then M is Sasakian if and only if $\mathbb{R}\times M$ is Kaehlerian. Our next two results show that for warped product if M is Sasakian, then the product manifold need not be Kaehlerian. Conversely if $M\times\mathbb{R}$ is Kaehlerian, then M can be γ-Sasakian depending on σ.

Theorem 3.2. *If the product manifold $M\times\mathbb{R}$ is Kaehlerian (trivial class) and $e^{\sigma}\frac{d\sigma}{dt} = \frac{f(\xi)}{2n}$, then the almost contact metric manifold M is in $\mathscr{F}_2$.*

Proof. If $M\times\mathbb{R}$ is Kaehlerian, then $\beta = 0$. Replacing $\tilde{X} = (X,0)$, $\tilde{Y} = (Y,0)$, $\tilde{Z} = (Z,0)$ in (3.8) implies

$$\alpha(X,Y,Z) = e^{\sigma}\frac{d\sigma}{dt}\{\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\}$$

and since $e^{\sigma}\frac{d\sigma}{dt} = \frac{f(\xi)}{2n}$, M satisfies the defining relation (2.7) of $\mathscr{F}_2$. □

The converse also holds depending on σ.

Theorem 3.3. *Let (M,φ,ξ,η,g) be in $\mathscr{F}_2$. If $e^{\sigma}\frac{d\sigma}{dt}=\frac{f(\xi)}{2n}$, then $M\times\mathbb{R}$ is Kaehlerian (trivial class). If $e^{\sigma}\frac{d\sigma}{dt}\neq\frac{f(\xi)}{2n}$, then $M\times\mathbb{R}$ is in $\mathscr{W}_4$.*

Proof. Since α satisfies the defining relation (2.7), from (3.8), we get

$$\beta(\tilde{X},\tilde{Y},\tilde{Z})=e^{2\sigma}(\frac{f(\xi)}{2n}-e^{\sigma}\frac{d\sigma}{dt})(\eta(Z)g(X,Y)-\eta(Y)g(X,Z)). \tag{3.10}$$

Then it is clear that if $e^{\sigma}\frac{d\sigma}{dt}=\frac{f(\xi)}{2n}$, then $M\times\mathbb{R}$ is Kaehlerian. Otherwise, if $e^{\sigma}\frac{d\sigma}{dt}\neq\frac{f(\xi)}{2n}$, it can be checked that (3.10) satisfies the defining relation (2.8) of $\mathscr{W}_4$. □

Note that in [10], the warped product metric $\tilde{g}=f^2g+dt^2$ is considered on $M\times\mathbb{R}$, where M is a γ-Sasakian manifold and it is proved that the product manifold is locally conformal Kaehler (a subclass of $\mathscr{W}_4$). The complex structure J on the product manifold in [10] satisfies $J(0,\frac{d}{dt})=(\xi,0)$ and $J(\xi,0)=-(0,\frac{d}{dt})$. In our case, $J(0,\frac{d}{dt})=-(e^{-\sigma}\xi,0)$ and $J(\xi,0)=(0,e^{\sigma}\frac{d}{dt})$.

It is known that S^{2n+1} has a Sasakian (1-Sasakian) structure [18]. Since $e^{\sigma}\frac{d\sigma}{dt}=1=\frac{f(\xi)}{2n}$, then $\sigma(t)=\ln(k_1+t)$ for k_1 constant. Since $\sigma(t)$ is not defined for all real numbers, $S^{2n+1}\times\mathbb{R}$ is not Kaehlerian for any function σ. If for example $\sigma(t)=t$, then $S^{2n+1}\times\mathbb{R}$ is in $\mathscr{W}_4$ by Theorem 3.3.

Let (M,φ,ξ,η,g) be in $\mathscr{F}_3$, that is, γ-Kenmotsu. It is known that if $\gamma\eta$ is closed, then the warped product manifold is locally conformal Kaehler [3]. We also have the following.

Theorem 3.4. *Let (M,φ,ξ,η,g) be in $\mathscr{F}_3$, that is, γ-Kenmotsu. Then $M\times\mathbb{R}$ is in $\mathscr{W}_4$ for all functions σ.*

Corollary 3.5. *Let (M,φ,ξ,η,g) be in $\mathscr{F}_2\oplus\mathscr{F}_3$, that is, M is trans-Sasakian. Then $M\times\mathbb{R}$ is in $\mathscr{W}_4$.*

The converse also holds.

Theorem 3.6. *If the product manifold $M\times\mathbb{R}$ is in $\mathscr{W}_4$, then M is in $\mathscr{F}_2\oplus\mathscr{F}_3$.*

Proof. Let $M\times\mathbb{R}$ be in $\mathscr{W}_4$. Then the defining relation (2.8) of $\mathscr{W}_4$ holds. For $\tilde{X}=(0,\frac{d}{dt})$, $\tilde{Y}=(0,\frac{d}{dt})$, $\tilde{Z}=(Z,0)$, we evaluate the left and right handsides of (2.8) by using (3.2) and we obtain

$$f(Z)=\eta(Z)f(\xi) \tag{3.11}$$

and thus

$$f(\varphi(Z))=0. \tag{3.12}$$

Doing similar calculations for $\tilde{X}=(X,0)$, $\tilde{Y}=(Y,0)$, $\tilde{Z}=(Z,0)$ and using (3.11) and (3.12) yields

$$\alpha(X,Y,Z) = \frac{f(\xi)}{2n}\{\eta(Z)g(X,Y)-\eta(Y)g(X,Z)\}-\frac{f^*(\xi)}{2n}\{\eta(Z)g(X,\varphi Y)-\eta(Y)g(X,\varphi Z)\},$$

which is the defining relation of the class $\mathscr{F}_2\oplus\mathscr{F}_3$. □

The results for trans-Sasakian manifolds are in accordance with [13]. However warped product gives an almost-Hermitian manifold for any function σ defined on $\mathbb{R}$.

It is known that an almost contact metric structure is nearly-K-cosymplectic if and only if the almost Hermitian structure (2.11) is $\mathscr{W}_1$ (nearly Kaehlerian) [13]. For the almost Hermitian structure (3.1), we have

Theorem 3.7. *Let (M,φ,ξ,η,g) be in $\mathscr{F}_{11}$, that is, M is nearly-K-cosymplectic. Then $M\times\mathbb{R}$ is in $\mathscr{W}_1\oplus\mathscr{W}_4$ and not in a subclass.*

Proof. From (3.8), it can be seen that β satisfies the defining relation (2.9) of $\mathscr{W}_1\oplus\mathscr{W}_4$. In addition β does not satisfy defining relation of any subclass of $\mathscr{W}_1\oplus\mathscr{W}_4$. □

Consider $S^6\times\mathbb{R}$ with its nearly-K-cosymplectic structure [13]. Then Theorem 3.7 implies that $(S^6\times\mathbb{R})\times\mathbb{R}$ is in $\mathscr{W}_1\oplus\mathscr{W}_4$ for any function σ. Thus infinitely many almost-Hermitian manifolds of class $\mathscr{W}_1\oplus\mathscr{W}_4$ can be obtained for any choice of σ.

Theorem 3.8. *Let (M,φ,ξ,η,g) be in $\mathscr{F}_1$. Then $M\times\mathbb{R}$ is in the widest class $\mathscr{W}$ and not in a subclass.*

Proof. Let (M,φ,ξ,η,g) be in $\mathscr{F}_1$. Then the defining relation (2.6) is satisfied. For any orthonormal frame $\{e_1,\dots,e_{2n},\xi\}$, $f(X)=\sum\alpha(e_i,e_i,X)=0$ and $f^*(\xi)=\sum\alpha(e_i,\varphi(e_i),\xi)=0$ by (2.6). Then $\delta F(\tilde{X})=e^{\sigma}\frac{d\sigma}{dt}2n\eta(X)$ from (3.9). In particular, $\delta F(\xi,0)=e^{\sigma}\frac{d\sigma}{dt}2n\neq0$ for a nonconstant function σ. Thus $M\times\mathbb{R}\notin\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3$ and this implies that $M\times\mathbb{R}$ is not in any subclass of $\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3$.
Assume that $M\times\mathbb{R}\in\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$. Then from (2.10),

$$\beta(\tilde{X},\tilde{X},\tilde{Y})=\beta(J(\tilde{X}),J(\tilde{X}),\tilde{Y}) \tag{3.13}$$

for all vector fields $\tilde{X},\tilde{Y}$. We evaluate $\beta(\tilde{X},\tilde{X},\tilde{Y})$ from (3.8). In particular, for $\tilde{X}=(\xi,0)$, $\tilde{Y}=(Y,0)$, we get $\beta(\tilde{X},\tilde{X},\tilde{Y})=e^{2\sigma}\alpha(\xi,\xi,Y)$ and $\beta(J(\tilde{X}),J(\tilde{X}),\tilde{Y})=0$, which yields $\alpha(\xi,\xi,Y)=0$ from (3.13). Since $\alpha(\xi,\xi,Y)\neq0$ in $\mathscr{F}_1$, this is a contradiction. Thus $M\times\mathbb{R}\notin\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$ and $M\times\mathbb{R}$ is not in any subclass of $\mathscr{W}_1\oplus\mathscr{W}_3\oplus\mathscr{W}_4$.
Similarly it can be seen that $M\times\mathbb{R}\notin\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_4$ and $M\times\mathbb{R}\notin\mathscr{W}_2\oplus\mathscr{W}_3\oplus\mathscr{W}_4$. As a result $M\times\mathbb{R}$ is in the widest class $\mathscr{W}$. □

Example 3.9. *Let $M = \mathbb{R}^3$ with coordinates (x,y,z). Consider linearly independent vectors*

$$e_1 = e^z \frac{\delta}{\delta x}, \quad e_2 = e^{-z} \frac{\delta}{\delta y}, \quad e_3 = \frac{\delta}{\delta z},$$

which are g-orthonormal with respect to the metric

$$g = e^{-2z} dx \otimes dx + e^{2z} dy \otimes dy + dz \otimes dz.$$

Nonzero brackets and nonzero covariant derivatives are

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = e_2,$$

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2.$$

Let $\xi = e_1$, η the metric dual of ξ and φ be the endomorphism defined by

$$\varphi(e_1) = 0, \quad \varphi(e_2) = e_3, \quad \varphi(e_3) = -e_2.$$

It can be checked that (φ, ξ, η, g) is a nontrivial $((\nabla_{e_1} \Phi)(e_1, e_2) \neq 0)$ almost contact metric structure on M which satisfies the defining relation (2.6) of $\mathscr{F}_1$. Then by Theorem 3.8, $M \times \mathbb{R}$ is a 4-dimensional almost-Hermitian manifold which is in the widest class $\mathscr{W}$ for any non-constant function σ.

In a similar manner it can be seen that if M is in $\mathscr{F}_8$, then $M \times \mathbb{R}$ is in $\mathscr{W}$ and not in a subclass. Thus infinitely many almost-Hermitian manifolds in the widest class can be obtained from almost contact metric ones by using a function σ defined on $\mathbb{R}$.

4. Conclusion

In this paper, almost Hermitian manifolds from the product of almost contact metric manifolds with $\mathbb{R}$ by warped product are constructed. Depending on the function of warped product, the curvature properties of the almost Hermitian manifolds obtained in this way. In particular, Einstein almost Hermitian manifolds are considered. In addition, the relationships between some classes of almost contact metric manifolds and almost Hermitian manifolds are investigated.

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