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Research Article

A Note On ϕ- Strongly Quasi Primary Ideals

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Abstract

Our aim in this study is to define the ϕ-sq primary ideal, which is a prime ideal generalization, and investigate some of its fundamental characteristics. Given a commutative ring R that has an identity, $L(R)$ denotes the set of all ideals of R. Assume that $\phi : L(R) \to L(R) \cup \{\emptyset\}$ a function. A proper ideal I of R is called ϕ-sq primary ideal if $ab \in I - \phi(I)$ implies $a^2 \in I$ or $b \in \sqrt{I}$ for each $a, b \in R$. Afterwards, the basic features of this new structure were determined and its relationship with other ideals such as ϕ-2 prime ideal, strongly quasi primary ideal and ϕ-semiprimary ideal was examined.

Keywords: prime ideal, ϕ-sq primary ideal, ϕ-2 prime ideal, strongly quasi primary ideal, ϕ-semiprimary ideal

ϕ-kuvvetli Yarı Asalımsı İdealler Üzerine Bir Not

Öz

Bu çalışmanın amacı asal ideallerin bir genelleştirmesi olan ϕ-kuvvetli yarı asalımsı idealleri tanımlamak ve bu ideallerin bazı temel özelliklerini incelemektir. R birimli ve değişmeli bir halka ve $L(R)$, R nin tüm ideallerinin kümesi olsun. $\phi : L(R) \to L(R) \cup \{\emptyset\}$ olduğunu kabul edelim. Her $a, b \in R$ olmak üzere $ab \in I - \phi(I)$ iken $a^2 \in I$ veya $b \in \sqrt{I}$ ise R nin I has idealine ϕ-kuvvetli yarı asalımsı ideal denir. Tanımın ardından bu yeni yapının temel özellikleri belirlenerek ϕ-2 asal, kuvvetli yarı asalımsı ideal ve ϕ-yarıasal ideal gibi ideallerle olan ilişkisi incelenmiştir.

Anahtar Kelimeler: asal ideal, ϕ-kuvvetli yarı asalımsı ideal, ϕ-2 asal, kuvvetli yarı asalımsı ideal ve ϕ-yarıasal ideal

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1. Introduction

Prime ideals and their extensions play a significant role in the field of commutative algebra. A large number of authors have conducted research on prime ideals for this reason. The purpose of this investigation is to precisely outline the concept of the ϕ-sq primary ideal, which is a broader category of prime ideals, and thoroughly analyze its fundamental characteristics. The recently developed framework has been thoroughly analyzed and its associations with several significant ideals have been examined and determined. It is assumed that R is a commutative ring with identity throughout the paper. Let I be a proper ideal of R. $\sqrt{I} = \{a \in R : a^n \in I \quad \text{for some} \quad n \in \mathbb{N}\}$ denotes the radical of R. And also $(I : a) = \{r \in R : ra \in I\}$. Recall that weakly prime ideals were examined by Anderson and Smith in 2003 [2]. A proper ideal P of R is classified as a weakly prime ideal if, given that $0 \neq ab \in P$, it follows that either a belongs to P or b belongs to P. Anderson and Bataineh researched a type of ideal called ϕ-prime ideals, which combine prime ideals and weakly prime ideals[1]. A proper ideal Q of R is a ϕ-prime ideal if for $ab \in Q - \phi(Q)$ implies $a \in Q$ or $b \in Q$ for some $a, b \in R$. In a recent study [6] Koç and others defined the concept of a strongly quasi-primary ideal and provided a characterization of divided domains. They defined as follows: A proper ideal I of R is a strongly quasi-primary ideal if whenever $ab \in R$ provides $a^2 \in I$ or $b^n \in I$ for some $a, b \in I$ and $n \in \mathbb{N}$. In [3], researchers introduced the concept of wsq-primary ideals. A proper ideal I of R is said to be a wsq primary ideal if $0 \neq ab \in I$ implies that $a^2 \in I$ or $b \in \sqrt{I}$. Our goal in this article is to extend the concept of strongly quasi-primary and weakly prime ideals by introducing the notion of a ϕ-strongly quasi-primary ideal (abbreviated as ϕ-sq primary ideal). Additionally, Von Neumann regular rings, an important ring class, can be characterized using these ideals.

2. Preliminaries

Definition 1. *A proper ideal I of R is called ϕ-sq primary ideal if $ab \in I - \phi(I)$ implies $a^2 \in I$ or $b \in \sqrt{I}$ for each $a, b \in R$.*

Remark 1. *(1) If $\phi(I) = \emptyset$, then I is sq primary ideal if and only if I is ϕ-sq primary ideal.*
(2) If $\phi(I) = \{0\}$, then I is wsq primary ideal if and only if I is ϕ-sq primary ideal.

Recall that an ideal I of R is called ϕ-primary ideal if $ab \in I - \phi(I)$ implies $a \in I$ or $b \in \sqrt{I}$ [1]. Recall from [4] that a proper ideal I of R is said to be a ϕ-2-absorbing primary ideal if whenever $abc \in I - \phi(I)$ then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. A proper ideal I of R is said to be ϕ-2 prime ideal if $ab \in I - \phi(I)$ implies $a^2 \in I$ or $b^2 \in I$.

Proposition 1. *Let I be a proper ideal of R. The following statements are satisfied.*
(i) Every ϕ-primary ideal is also ϕ-sq primary ideal.
(ii) Every ϕ-2 prime ideal is also ϕ-sq primary ideal.
(iii) Every sq primary ideal is also ϕ-sq primary ideal.
(iv) Every ϕ-sq primary ideal is ϕ-semiprimary ideal. In particular every ϕ-sq primary ideal is also ϕ-2-absorbing primary ideal.

Proof. (i), (ii): It is clear.

(iii): Let I be a sq primary ideal. Now we will show that I is ϕ-sq primary ideal. Let $ab \in I - \phi(I)$ for some $a, b \in R$. Then $ab \in I$ implies $a^2 \in I$ or $b \in \sqrt{I}$, which completes the proof.

(iv): Let I be ϕ-sq primary ideal and $ab \in I - \phi(I)$. Then we have $a^2 \in I$ or $b \in \sqrt{I}$. If $a^2 \in I$, then $a \in \sqrt{I}$. Thus, I is a ϕ-semiprimary ideal. Now we will show that I is ϕ-2-absorbing primary ideal. Let $a(bc) \in I - \phi(I)$ for some $a, b, c \in R$. Since I is ϕ-sq primary ideal, we have $a^2 \in I$ or $bc \in \sqrt{I}$. If $a^2 \in I$, then $a \in \sqrt{I}$ which implies $ac \in \sqrt{I}$. Hence I is ϕ-2-absorbing primary ideal. □

In Proposition 1, the converses of (i), (ii) and (iii) are not true in general (Take $\phi(I) = 0$ and see Example 1, Example 2, Example 3 of [6]).

Proposition 2. *Let I be an ideal of R such that $\sqrt{I}^2 \subseteq I$. Then I is ϕ-2-prime ideal if and only if I is ϕ-sq primary ideal if and only if I is ϕ-semiprimary ideal.*

Proof. The implication "I is ϕ-2-prime ideal $\Rightarrow$ I is ϕ-sq primary ideal$\Rightarrow$ I is ϕ-semiprimary ideal" follows from Proposition 1. Let I be a ϕ-semiprimary ideal and $ab \in I - \phi(I)$ for some $a, b \in R$. Then $a \in \sqrt{I}$ or $b \in \sqrt{I}$ which implies that $a^2 \in \sqrt{I}^2 \subseteq I$ or $b^2 \in \sqrt{I}^2 \subseteq I$. So I is ϕ-2-prime ideal. □

Proposition 3. *Let I be a ϕ-sq primary ideal which is not sq primary ideal. Then $I^2 \subseteq \phi(I)$. In this case $\sqrt{I} \subseteq \sqrt{\phi(I)}$.*

Proof. Suppose that $I^2 \nsubseteq \phi(I)$. Now, we will show that I is sq primary ideal. Let $xy \in I$ for some $x, y \in R$. Assume that $x^2 \notin I$. If $xy \notin \phi(I)$, then $xy \in I - \phi(I)$ which implies that $y \in \sqrt{I}$ that completes the proof. Thus, we assume that $xy \in \phi(I)$. If $xI \nsubseteq \phi(I)$, then there exists $c \in I$ such that $xc \notin \phi(I)$. This implies that $x(y + c) \in I - \phi(I)$. Since I is ϕ-sq primary ideal, we have $(y + c) \in \sqrt{I}$. Thus we may assume that $xI \subseteq \phi(I)$. Now we will show that $yI \subseteq \phi(I)$. If $yI \nsubseteq \phi(I)$, then we can choose $d \in I$ such that $yd \notin \phi(I)$. Then $y(x + d) \in I - \phi(I)$ which implies that $(x + d)^2 = x^2 + 2xd + d^2 \in I$ or $y \in \sqrt{I}$. If $y \in \sqrt{I}$, then we're through. If $(x + d)^2 = x^2 + 2xd + d^2 \in I$, then $x^2 \in I$ which is a contradiction. Thus, we have $yI \subseteq \phi(I)$. Since $I^2 \nsubseteq \phi(I)$, we have $p, q \in I$ such that $pq \notin \phi(I)$. Then $(x + p)(y + q) = xy + xq + yp + pq \in I - \phi(I)$. This gives $(x + p)^2 = x^2 + 2xp + p^2 \in I$ or $y + q \in \sqrt{I}$. Thus $x^2 \in I$ or $y \in \sqrt{I}$ which completes the proof. □

Proposition 4. *Let R be a ring and I be a proper ideal of R. Then I is a ϕ-sq primary ideal if and only if $I/\phi(I)$ is a wsq primary ideal of $R/\phi(I)$.*

Proof. Let I be a ϕ-sq primary ideal of R and choose $a, b \in R$ such that $0_{R/\phi(I)} \neq (a+\phi(I))(b+\phi(I)) \in I/\phi(I)$. Then we conclude that $ab \in I - \phi(I)$. Since I is a ϕ-sq primary ideal, we obtain $a^2 \in I$ or $b \in \sqrt{I}$. This implies that $(a + \phi(I))^2 = a^2 + \phi(I) \in I/\phi(I)$ or $b + \phi(I) \in \sqrt{I}/\phi(I) = \sqrt{I/\phi(I)}$. Thus, $I/\phi(I)$ is a wsq primary ideal of $R/\phi(I)$. Contrarily, assume that $I/\phi(I)$ is a wsq primary ideal of $R/\phi(I)$. Now, we will show that I is a ϕ-sq primary ideal. Let $ab \in I - \phi(I)$ for some $a, b \in R$. Then we have $0_{R/\phi(I)} \neq (a + \phi(I))(b + \phi(I)) \in I/\phi(I)$. As $I/\phi(I)$ is a wsq primary ideal of $R/\phi(I)$, we conclude that $(a + \phi(I))^2 = a^2 + \phi(I) \in I/\phi(I)$ or $b + \phi(I) \in \sqrt{I}/\phi(I) = \sqrt{I/\phi(I)}$. Then we get $a^2 \in I$ or $b \in \sqrt{I}$ which completes the proof. □

Theorem 1. *The followings are equivalent for a proper ideal I of R :*

(i) I is a ϕ-sq primary ideal.

(ii) $(a) \subseteq (I : a)$ or $(I : a) \subseteq \sqrt{I}$ or $(I : a) \subseteq (\phi(I) : a)$ for every $a \in R$.

(iii) $aJ \subseteq I$ and $aJ \nsubseteq \phi(I)$ implies $a^2 \in I$ or $J \subseteq \sqrt{I}$ for every $a \in R$ and ideal J of R.

Proof. $(i) \Rightarrow (ii)$: Suppose that I is a ϕ-sq primary ideal. Choose an element $a \in R$. If $a^2 \in I$, then one can easily see that $(a) \subseteq (I : a)$. So, assuming that $a^2 \notin I$. Let $x \in (I : a)$. Then we have $ax \in I$. If $ax \in \phi(I)$, then we have $x \in (\phi(I) : a)$. If $ax \notin \phi(I)$, then we conclude that $x \in \sqrt{I}$ since I is a ϕ-sq primary ideal and $ax \in I - \phi(I)$. Thus we conclude that $(I : a) \subseteq \sqrt{I} \cup (\phi(I) : a)$. The rest follows from the fact that if an ideal is contained in the union of two ideals, then it must be contained in one of them.

$(ii) \Rightarrow (iii)$: Let $aJ \subseteq I$ and $aJ \nsubseteq \phi(I)$ for some $a \in R$ and ideal J of R. It is possible to make a general assumption without any loss of relevance that $a^2 \notin I$. Since $J \subseteq (I : a)$ and $J \nsubseteq (\phi(I) : a)$, by (ii), we have $J \subseteq (I : a) \subseteq \sqrt{I}$, as needed.

$(iii) \Rightarrow (i)$: It is straightforward. □

Definition 2. *Let R, S be two commutative rings with unity, $\phi : L(R) \to L(R) \cup \{\emptyset\}$ and $\psi : L(S) \to L(S) \cup \{\emptyset\}$ be two functions. A ring homomorphism $f : R \to S$ is said to be a (ψ, ϕ)-homomorphism if $f^{-1}(\psi(J)) = \phi(f^{-1}(J))$ for every $J \in L(S)$.*

Theorem 2. *Let R, S be two commutative rings with unity, $\phi : L(R) \to L(R) \cup \{\emptyset\}$ and $\psi : L(S) \to L(S) \cup \{\emptyset\}$ be two functions. Assume that $f : R \to S$ is a (ψ, ϕ)-homomorphism of rings. The following expressions are provided.*

(i) If J is a ψ-sq primary ideal of S, then $f^{-1}(J)$ is a ϕ-sq primary ideal of R.

(ii) If I is a ϕ-sq primary ideal of R containing $Ker(f)$ and f is surjective, then $f(I)$ is a ψ-sq primary ideal of S.

Proof. (i) : Let J be a ψ-sq primary ideal of S and $ab \in f^{-1}(J) - \phi(f^{-1}(J))$ for some $a, b \in R$. Since $\phi(f^{-1}(J)) = f^{-1}(\psi(J))$, we conclude that $f(ab) = f(a)f(b) \in J - \psi(J)$. As J is a ψ-sq primary ideal of S, we get $f(a)^2 = f(a^2) \in J$ or $f(b)^n = f(b^n) \in J$ for some $n \in \mathbb{N}$. This gives that $a^2 \in f^{-1}(J)$ or $b^n \in f^{-1}(J)$. Thus, $f^{-1}(J)$ is a ϕ-sq primary ideal of R.

(ii) : Suppose that I is a ϕ-sq primary ideal of R containing $Ker(f)$ and f is surjective. Let $yz \in f(I) - \psi(f(I))$ for some $y, z \in S$. Since f is surjective, $f(a) = y$ and $f(b) = z$ for some $a, b \in R$. This implies that $f(ab) = yz \in f(I)$, and this yields $ab \in I$. Now, first note that $I = f^{-1}(f(I))$ since I contains $Ker(f)$. Put $J = f(I)$, then by (ψ, ϕ)-homomorphism, we have $f^{-1}(\psi(J)) = f^{-1}(\psi(f(I))) = \phi(f^{-1}(f(I)))$. Then we get $f^{-1}(\psi(f(I))) = \phi(I)$. Since f is surjective, we conclude that $\psi(f(I)) = f(\phi(I))$. If $ab \in \phi(I)$, then we have $f(ab) = yz \in f(\phi(I)) = \psi(f(I))$ which is a contradiction. Thus we have $ab \in I - \phi(I)$. Since I is a ϕ-sq primary ideal of R, we get $a^2 \in I$ or $b \in \sqrt{I}$ which implies that $y^2 = f(a^2) \in f(I)$ or $z = f(b) \in f(\sqrt{I}) = \sqrt{f(I)}$. Consequently, $f(I)$ is a ψ-sq primary ideal of S. □

Theorem 3. *Let S be a multiplicative closed subset of R and $\phi_q : L(S^{-1}R) \to L(S^{-1}R) \cup \{\emptyset\}$, defined by $\phi_q(S^{-1}I) = S^{-1}(\phi(I))$ for each ideal I of R, be a function. Then the statements below are true.:*

(i) *If I is a ϕ-sq primary ideal of R with $S \cap I = \emptyset$, then $S^{-1}I$ is a ϕ_q−sq primary ideal of $S^{-1}R$.*

(ii) Let I be an ideal of R such that $Z_{\phi(I)}(R) \cap S = \emptyset$ and $Z_I(R) \cap S = \emptyset$. If $S^{-1}I$ is a ϕ_q-sq primary ideal of $S^{-1}R$, then I is a ϕ-sq primary ideal of R.

Proof. (i) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}I - \phi_q(S^{-1}I)$ for any $a, b \in R$ and $s, t \in S$. Since $\phi_q(S^{-1}I) = S^{-1}(\phi(I))$, we have $uab = (ua)b \in I - \phi(I)$ for some $u \in S$. As I is a ϕ-sq primary ideal of R, we get $(ua)^2 \in I$ or $b \in \sqrt{I}$. This implies that $\frac{u^2a^2}{u^2s^2} = \frac{a^2}{s^2} \in S^{-1}I$ or $\frac{ub}{ut} = \frac{b}{t} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$. Therefore $S^{-1}I$ is a ϕ_q-sq primary ideal of $S^{-1}R$.

(ii) Let $ab \in I - \phi(I)$ for some $a, b \in R$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$. As $Z_{\phi(I)}(R) \cap S = \emptyset$, it is obvious that $\frac{a}{1}\frac{b}{1} \notin S^{-1}(\phi(I)) = \phi_q(S^{-1}I)$. Since $S^{-1}I$ is a ϕ_q-sq primary ideal of $S^{-1}R$, we have $\frac{a^2}{1} \in S^{-1}I$ or $\frac{b}{1} \in S^{-1}\sqrt{I}$. If $\frac{a^2}{1} \in S^{-1}I$, then $ua^2 \in I$ for some $u \in S$. Since $Z_I(R) \cap S = \emptyset$, we get $a^2 \in I$ and so I is a ϕ-sq primary ideal of R. If $\frac{b}{1} \in S^{-1}\sqrt{I}$, then $ub \in \sqrt{I}$ and hence $u^nb^n \in I$ for some $n \in \mathbb{N}$. Since $Z_I(R) \cap S = \emptyset$, we have $b^n \in I$ and so I is a ϕ-sq primary ideal of R. □

Theorem 4. *Let R_1, R_2 be two commutative rings and $\psi_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i) \cup \{\emptyset\}$ be function for i $= 1, 2$. Let $\phi_\times = \psi_1 \times \psi_2$. Then $\phi_\times$-sq primary ideals of $R_1 \times R_2$ have exactly one of the following three forms:*

(i) $I_1 \times I_2$ where I_i is a proper ideal of R_i with $\psi_i(I_i) = I_i$.

(ii) $I_1 \times R_2$ where I_1 is a ψ_1-sq primary ideal of R_1 which must be strongly quasi primary ideal if $\psi_2(R_2) \neq R_2$.

(iii) $R_1 \times I_2$ where I_2 is a ψ_2-sq primary ideal of R_2 which must be strongly quasi primary ideal if $\psi_1(R_1) \neq R_1$.

Proof. Suppose that $I = I_1 \times I_2$ is a $\phi_\times$-sq primary ideal of $R_1 \times R_2$. Without loss of generality, we may assume that $I_1 \times I_2 \neq \psi_1(I_1) \times \psi_2(I_2)$. We first demonstrate that I_1 is a ψ_1-sq primary ideal of R_1. Let $xy \in I_1 - \psi_1(I_1)$ for some $x, y \in R_1$. Then we have $(x, 0)(y, 0) = (xy, 0) \in I - \phi_\times(I)$. By assumption, we conclude that $(x, 0)^2 \in I$ or $(y, 0) \in \sqrt{I}$. Then we have $x^2 \in I_1$ or $y \in \sqrt{I_1}$. Thus, I_1 is a ψ_1-sq primary ideal of R_1. Similar argument shows that I_2 is a ψ_2-sq primary ideal of R_2. Since $I_1 \times I_2 \neq \psi_1(I_1) \times \psi_2(I_2)$, we may assume that $I_1 \neq \psi_1(I_1)$. Then there exists $a \in I_1 - \psi_1(I_1)$. Choose $b \in I_2$. Then we have $(a, 1)(1, b) = (a, b) \in I - \phi_\times(I)$ which implies that $(a, 1)^2 \in I$ or $(1, b) \in \sqrt{I}$. This gives $I_1 = R_1$ or $I_2 = R_2$. Assume that $I_2 = R_2$. Now, we will show that I_1 is quasi primary if $\psi_2(R_2) \neq R_2$. Now, choose $c \in R_2 - \psi_2(R_2)$ and $xy \in I_1$. Then $(x, c)(y, 1) = (xy, c) \in I - \phi_\times(I)$ which implies that $(x, c)^2 \in I$ or $(y, 1) \in \sqrt{I}$. Thus we have $x^2 \in I_1$ or $y \in \sqrt{I_1}$ which completes the proof. Conversely, nothing needs to be proven if (i) is satisfied. We may presume that without losing generality (ii) is satisfied. If $\psi_2(R_2) \neq R_2$, then I_1 is strongly quasi primary ideal of R_1. Then by [6, Lemma 2.1], $I = I_1 \times R_2$ is a strongly quasi primary ideal of $R_1 \times R_2$. The rest follows from Proposition 1. Now, assume that $\psi_2(R_2) = R_2$ and I_1 is a ψ_1-sq primary ideal of R_1. Let $(a, b)(x, y) \in I - \phi_\times(I)$. Then we have $ax \in I_1 - \psi_1(I_1)$. Since I_1 is a ψ_1-sq primary ideal of R_1, we have $a^2 \in I_1$ or $x \in \sqrt{I_1}$. Then we obtain $(a, b)^2 \in I$ or $(x, y) \in \sqrt{I}$. Hence, I is a $\phi_\times$-sq primary ideal. In the other case, it is evident that I is a $\phi_\times$-sq primary ideal □

Let $\phi_n : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function defined by $\phi_n(I) = I^n$ for a fixed natural number $n \in \mathbb{N}$. Then we say that I is an n-almost sq primary ideal if I is a ϕ_n-sq primary ideal of R. Recall from [7] that a ring R is said to be a von Neumann regular ring if $I = I^2$ for every ideal I of R. In this case, $I = I^n$ for all $n \in \mathbb{N}$. By [5, Theorem 1], a ring R is a von Neumann regular ring if and only if $I = \sqrt{I}$ for every ideal I of R if and only if $IJ = I \cap J$ for every ideals I, J of R. Now, we will give a new characterization of von Neumann regular rings in terms of n-almost sq primary ideals of R.

Theorem 5. *Let $R_1, R_2, \ldots, R_m$ be commutative rings and $R = R_1 \times R_2 \times \cdots \times R_m$, where $3 \leq m < \infty$. Suppose that $n \geq 2$. Then the following expressions are equivalent.*

(i) Every proper ideal of R is an n-almost sq primary ideal.

(ii) $R_1, R_2, \ldots, R_m$ are von Neumann regular rings.

Proof. $(i) \Rightarrow (ii)$: Suppose that every proper ideal of R is an n-almost sq primary ideal. Now, we will show that $R_1, R_2, \ldots, R_m$ are von Neumann regular rings.Without loss of generality R_1 is not von Neumann regular. Then there exists a proper ideal I_1 of R_1 such that $I_1^n \neq I_1$. Then there exists $x \in I_1 - I_1^n$. Let $I = I_1 \times 0 \times 0 \times R_4 \times R_5 \times \cdots \times R_m$. Now, put $a = (x, 0, 1, 1, \ldots, 1)$ and $b = (1, 1, 0, 1, \ldots, 1)$. Then note that $ab = (x, 0, 0, 1, \ldots, 1) \in I - I^n$. However, $a^2 = (x^2, 0, 1, \ldots, 1) \notin I$ and $b = (1, 1, 0, 1, \ldots, 1) \notin \sqrt{I}$. Thus, we have a non n-almost sq primary ideal I of R which is a contradiction. Hence, $R_1, R_2, \ldots, R_m$ are von Neumann regular rings.

$(ii) \Rightarrow (i)$: Let $R_1, R_2, \ldots, R_m$ be von Neumann regular rings. Then by [5, Proposition 4] $R = R_1 \times R_2 \times \cdots \times R_m$ is a von Neumann regular ring. In this case, $I = I^n = \phi_n(I)$ for every ideal I of R. Thus, every every proper ideal of R is trivially an n-almost sq primary ideal. □

Theorem 6. *Let $f : Y \to Z$ be a ring homomorphism. Assume that δ is an ideal expansion of $I(Y)$, φ is a reduction function of $I(Y)$ and also γ is an ideal expansion of $I(Z)$, ψ is a reduction function of $I(Z)$. Then f is said to be $(\delta, \varphi) - (\gamma, \psi)$ homomorphism if $\varphi(f^{-1}(J)) = f^{-1}(\psi(J))$ for every $J \in I(Z)$. Let $f : Y \to Z$ be a (γ, ψ) homomorphism.*

(i) If J is a ϕ-sq primary ideal of Z, then $f^{-1}(J)$ is a ϕ-sq primary ideal of Y.

(ii) If I is ϕ-sq primary ideal of Y containing $Ker(f)$ and f is a surjective, then $f(I)$ is a ϕ-sq primary ideal of Z.

Proof. (i) : Let J be a ϕ-sq primary ideal of Z. Take $a, b \in Y$ such that $ab \in f^{-1}(J) - \phi(f^{-1}(J))$. Then we have $f(a)f(b) \in J - \psi(J)$. Since is a ϕ-sq primary ideal, $(f(a))^2 \in J$ or $(f(b))^n \in J$ which implies that $a^2 \in f^{-1}(J)$ or $b^n \in f^{-1}(J)$. So $f^{-1}(J)$ is a ϕ-sq primary ideal of Y.

(ii) : Suppose that I is a ϕ-sq primary ideal of R containing $Ker(f)$ and f is surjective. Let $mn \in f(I) - \psi(f(I))$ for some $m, n \in S$. Since f is surjective, $f(a) = m$ and $f(b) = n$ for some $a, b \in R$. This implies that $f(ab) = mn \in f(I)$, and this yields $ab \in I$. Now, first note that $I = f^{-1}(f(I))$ since I contains $Ker(f)$. Set $J = f(I)$, then by (ψ, ϕ)-homomorphism, we have $f^{-1}(\psi(J)) = f^{-1}(\psi(f(I))) = \phi(f^{-1}(f(I)))$. Then we get $f^{-1}(\psi(f(I))) = \phi(I)$. Since f is surjective, we conclude that $\psi(f(I)) = f(\phi(I))$. If $ab \in \phi(I)$, then we have $f(ab) = mn \in f(\phi(I)) = \psi(f(I))$ which is a contradiction. Thus we have $ab \in I - \phi(I)$. Since I is a ϕ-sq primary ideal of R, we get $a^2 \in I$ or $b \in \sqrt{I}$ which implies that $m^2 = f(a^2) \in f(I)$ or $n = f(b) \in f(\sqrt{I}) = \sqrt{f(I)}$. Consequently, $f(I)$ is a ψ-sq primary ideal of S. □

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