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RESEARCH PAPER

Examination of Sturm-Liouville problem with proportional derivative in control theory

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Abstract

The current study is intended to provide a comprehensive application of Sturm-Liouville (S-L) problem by benefiting from the proportional derivative which is a crucial mathematical tool in control theory. This advantageous derivative, which has been presented to the literature with an interesting approach and a strong theoretical background, is defined by two tuning parameters in control theory and a proportional-derivative controller. Accordingly, this research is presented mainly to introduce the beneficial properties of the proportional derivative for analyzing the S-L initial value problem. In addition, we reach a novel representation of solutions for the S-L problem having an importing place in physics, supported by various graphs including different values of arbitrary order and eigenvalues under a specific potential function.

Keywords: Proportional-derivative controller; proportional integral; Sturm-Liouville problem; control theory; local derivative

AMS 2020 Classification: 00A69; 34B24; 00A05

1 Motivation

The advantage of using non-integer order integral-derivative operators lies in the fact that they express numerous real-world problems better than classical analysis tools. Fractional calculus provides a natural and intrinsic characterization of complex dynamical systems [1]. Also, the concepts in fractional calculus shed some new light on the solutions methods of differential equations, especially when the traditional tools are limited and insufficient. As a way of describing events in nature, this field whose history is as old as the classical differential has become quite interesting. Several fractional integral and derivative operators with various features have recently been introduced to the literature. While some researchers place a strong emphasis on the value of local derivatives, others highlight the benefits of non-local and singular kernel operators, while

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others make the case that non-local and non-singular kernel operators are also beneficial. Although this situation might be confusing, the availability of several derivative-integral definitions has evolved into a fundamental motivational tool for researchers in order to produce superior findings for the problems at hand. The amount of complex systems that have been studied from the perspective of fractional dynamics has significantly increased over the past few decades. Fractional calculus can be used to assess a variety of phenomena, including transmission line theory, heat transfer, diffusion, electrochemistry, fractal processes, deoxyribonucleic acid decoding for prototype systems, financial considerations, earthquake events, global warming, and even musical rhythms. In addition, the existence of numerous complex systems, both natural and human-made, shows the abundance of phenomena that can be described and studied with the help of concepts in fractional calculus. The major goal is to establish the analysis framework of the problems under consideration by enlarging it in the perspective of fractional calculus. Although fractional calculus helps to expand the traditional definitions of derivative and integral, which then obviously lead to fractional-type models, neither the restrictions of their application nor the processes and tools for comprehending them are well-defined at the current stage of scientific evaluation. With Caputo's formulation of the fractional derivative, the scope of applications for non-integer order differential operators has been widened, and exciting results have been obtained by using them more frequently. The usage of fractional derivatives, which is expanding rapidly today, is especially useful for characterizing processes and describing physical phenomena. It has also taken on crucial tasks like eliminating the deficiencies in differential equations created with classical derivatives.

The usage of local derivative and integral definitions defined in the limit form has also grown in popularity, in addition to fractional derivatives, which are non-local because they are defined in the integral form. The "proportional derivative" definition, which was developed with the proportional derivative controller used in control theory, is one of them and may be the most advantageous one. This derivative is defined with the help of two tuning parameters in control theory and a proportional-derivative (PD) controller given by

$$\mathbf{u}(t) = k_p E(t) + k_d E(t), \tag{1}$$

for the controller output **u** at time t [2]. PD is a successful control method that is straightforward to comprehend. Here, k_p stands for the proportional gain, k_d for the derivative gain, and E for the error between the state and process variables. It is well-recognized that the proportional derivative controller effectively addresses problems with real-world control. Also, the proportional term offers a general control action that, via the gain coefficient, is proportionate to the error signal. The derivative term improves the transient response through high-frequency compensation. Intuitively, for these concepts, it makes sense to say that P depends on the current error and D is an estimate of future errors. Controlling the considered system by the weighted sum of these two actions results in the system reaching the desired state. Suppose that for $\eta \in [0,1]$, $K_0, K_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ functions are continuous and satisfy the following conditions:

$$\lim_{\eta \to 0^+} K_1(\eta,t) = 1, \qquad \lim_{\eta \to 0^+} K_0(\eta,t) = 0, \tag{2}$$

$$\lim_{\eta \to 1^-} K_1(\eta,t) = 0, \qquad \lim_{\chi \to 1^-} K_0(\eta,t) = 1. \tag{3}$$

Then, for all $t \in \mathbb{R}$, $K_1(\eta,t) \neq 0$, $\eta \in [0,1)$ and $K_0(\eta,t) \neq 0$, $\eta \in (0,1]$, the proportional derivative is defined as

$$_{P}\mathbf{D}^{\eta}\omega(t) = K_1(\eta,t)\omega(t) + K_0(\eta,t)\omega'(t). \tag{4}$$

On the other hand, the proportional exponential function is given by

$$e_p(t,r) = e^{\int_r^t \frac{p(\tau)-K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}, \qquad e_0(t,r) = e^{-\int_r^t \frac{K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}, \tag{5}$$

where $\chi \in (0,1]$, $r,t \in \mathbb{R}$, $r \leq t$, $p : [r,t] \to \mathbb{R}$, and $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions. Also, p/k_0 and k_1/k_0 are Riemann integrable on $[s,t]$. Furthermore, for $\eta \in (0,1]$, proportional integral on $[a,b]$ is

$$_{P}\mathbf{I}^{\eta}\omega(t) = \int_a^t \omega(r)e_0(t,r)d_\eta r = \int_a^t \frac{\omega(r)e_0(t,r)}{K_0(\eta,r)}dr, \qquad d_\eta r = \frac{1}{K_0(\eta,r)}dr. \tag{6}$$

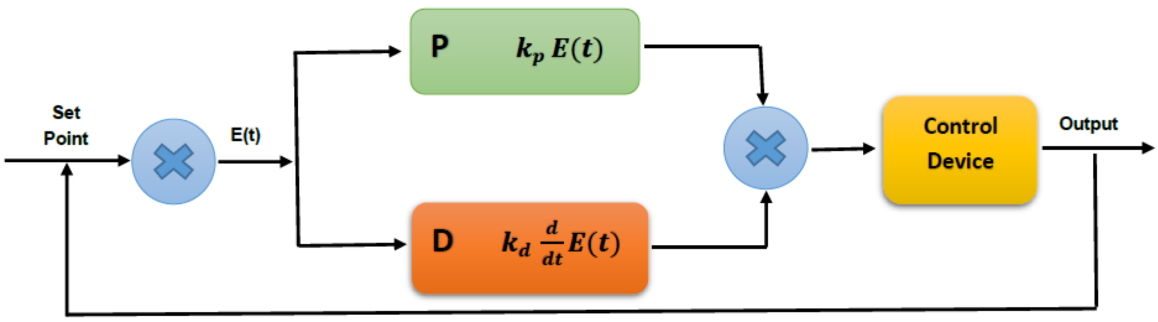


Figure 1. Block diagram of control system with proportional-derivative controller.

The derivative control method is known to change the controller output proportionally to the rate of the error signal change. Derivative control, on the other hand, observes how much the error has altered and tries to identify the current error. In order to minimize potential errors, it also generates control motion through using the rate of change. The integral technique is occasionally added in addition to the proportional method since the derivative method only affects the controller output when the error changes. In this context, it can be stated that the derivative control approaches can never be employed alone. The derivative value is determined by the rate of change of the error signal, that is, by the slope of the error signal. An ideal derivative technique is expected to respond with an infinite variation to the controller output and the derivative effect for quickly changing signals is constrained. In the derivative receiver circuit, the frequency of the signal applied at the input must be smaller than the cutoff frequency of the circuit, while the period of the signal applied at the input is desired to be close to the derivative time for the differentiation process to take place.

The difference signal between the set value and the measured value is subjected to a derivative operation in proportional-derivative control. After the error signal first passes through the proportional controller, the derivative signal, balancing voltage, and proportional signal are collected in the collector circuit. Figure 1 depicts the control system diagram with a PD controller. As shown in the diagram, the PD controller continuously determines the error value $E(t)$ [3].

2 Introduction

The study of Sturm between 1829 and 1836 serves as the basis of the Sturm-Liouville (S-L) theory. Later, the brief but crucial study of Sturm-Liouville was published in 1837. In this study, they addressed the boundary value problem (BVP) for the differential equation given as

$$-y'' + q(x)y = \lambda y, \qquad 0 \leq x \leq 1, \tag{7}$$

where λ is a complex parameter and q is a real-valued function that can be quadratically integrated over the integral $[0, 1]$. Sturm and Liouville examined whether there are nontrivial solutions of Eq. (7) satisfying the following boundary conditions [4]:

$$\begin{aligned} y(0)\cos\gamma_1 + y'(0)\sin\gamma_1 &= 0, \\ y(1)\cos\gamma_2 + y'(1)\sin\gamma_2 &= 0. \end{aligned} \tag{8}$$

Here, γ_1 and γ_2 are real numbers between 0 and π. If (7)-(8) BVP is solved, the complex number λ is called the eigenvalue of q, γ_1, and γ_2. Also, the nontrivial solutions for λ are called eigenfunctions of q, γ_1, and γ_2, The set of all eigenvalues is the spectrum of the BVP given by (7)-(8). Significant advances in spectral theory have been achieved for the Sturm-Liouville operator as follows

$$l = -\frac{d^2}{dx^2} + q(x), \tag{9}$$

sometimes also called the one-dimensional time-dependent Schrödinger operator.

The first investigations on spectral theory for such operators were performed by Bernoulli, D'alambert, Euler, Sturm, and Liouville for rod vibration problems. In the 20th century, spectral theory developed rapidly for different classes of differential and integral operators. Famous mathematicians including Birkhoff, Hilbert, Neumann, Steklov, Stone, and Weyl as well as many others have made major contributions to this topic through outstanding ideas. On the other hand, the main conclusions regarding the inverse problems of spectral theory were obtained in the second half of the 20th century. Particularly in the latter half of the 20th century, the techniques employed to study the Sturm-Liouvile operator have continuously improved. For instance, in 1967, a group of American physicists and mathematicians Gardner, Greene, Kruskal, and Miura developed an important method by solving the Korteweg-de Vries (KDV) equation for a proposed initial condition through using the inverse scattering method. In 1968, Lax evaluated the inverse scattering method in a more general frame by solving the KDV equation with the help of linear equations, and this frame later opened the way for generalizing the technique as a method for solving other partial differential equations. The initial value problems of nonlinear partial differential equations can be solved utilizing the inverse scattering method. The approach is based on converting the initial value problem into a linear integral equation. Both mathematicians and physicists continue to focus more on the inverse scattering problems of quantum theory for singular Sturm-Liouville operators, which have numerous applications in this area and geophysics [4]. For more information of fractional calculus in application, S-L problem and to see the S-L problem in fractional calculus we refer the reader to [5–14].

This manuscript is organized as follows: In Section 1, we give a motivation part on the proportional derivative by mentioning its importance in control theory before writing the introduction part in Section 2. Then, the model description and solution method in order to solve the S-L problem are given in Section 3. Additionally, we obtain the representation of the solution for the S-L problem through the proportional derivative operator in Section 4. On the other hand, in Section 5, various graphs are shown for different values of arbitrary order η and eigenvalues. Finally, we introduce some crucial concluding remarks of this study in Section 6.

3 Model description and solution method

The Sturm-Liouville operator T can be expressed through the proportional derivative as below:

$$T \equiv -{}_P\mathbf{D}^{\eta}({}_P\mathbf{D}^{\eta}) + q(x), \tag{10}$$

where $\eta \in (0,1]$ and $q(x)$ is a real-valued continuous function on interval $[a,b]$. Here, the main objective is to consider the S-L problem having separated boundary conditions given by

$$Ty(x) = -{}_P\mathbf{D}^{\eta}[{}_P\mathbf{D}^{\eta}y(x)] + q(x)y(x) = \lambda y(x), \tag{11}$$

$$\begin{aligned} y(a)\cos\gamma_1 + {}_P\mathbf{D}^{\eta}y(a)\sin\gamma_1 = 0,\\ y(b)\cos\gamma_2 + {}_P\mathbf{D}^{\eta}y(b)\sin\gamma_2 = 0. \end{aligned} \tag{12}$$

If we take $\cot\gamma_1 = -h$ and $\cot\gamma_2 = H$ for $a = 0$ and $b = \pi$, that is $x \in [0,\pi]$, the boundary condition (12) takes the following form

$$\begin{aligned} {}_P\mathbf{D}^{\eta}y(0) - hy(0) = 0,\\ {}_P\mathbf{D}^{\eta}y(\pi) + Hy(\pi) = 0. \end{aligned} \tag{13}$$

Furthermore, the BVP (11)-(12) has a nontrivial solution denoted by $y(x,\lambda_n)$ for any λ_n. Also, λ_n and $y(x,\lambda_n)$ are called as eigenvalue and eigenfunction, respectively. In [15], the variation of parameters method is defined by means of the proportional derivative. While this generalization can be used to solve many real-life problems, it also enables the behavior of the problems to be examined in more detail by obtaining more general solutions.

Let $0 \le \eta \le 1$ and $n \in \{1,2,3,...\}$, then ${}_P\mathbf{D}^{n\eta}y(x)$ is given by ${}_P\mathbf{D}^{n\eta}y = {}_P\mathbf{D}^{\eta}\,{}_P\mathbf{D}^{\eta}\ldots{}_P\mathbf{D}^{\eta}y$. For simplicity of notation, one can write $y^{(n\eta)}(x)$ instead of ${}_P\mathbf{D}^{n\eta}y(x)$. Hence, here, the expression of $y^{(2\eta)}(x)$ means that $\frac{d^{\eta}}{dt^{\eta}}\left(\frac{d^{\eta}y}{dx^{\eta}}\right)$.

The variation of parameters method, which is often used to find a particular solution of non-homogeneous linear differential equations with constant or variable coefficients, is defined by the proportional derivative as follows. It is well-known that the homogeneous part of a differential equation of form (11) has two linearly independent solutions, $y_1(x)$ and $y_2(x)$. In this situation, we have a particular solution of the proposed equation as $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$. Hence, with respect to the proportional variation of parameters method, we have the formulas addressed by

$$v_1'(x) = \frac{q(x)y(x)y_2(x)}{K_0^2(\eta,x)W_p(y_1,y_2)(x)}, \qquad v_2'(x) = \frac{-q(x)y(x)y_1(x)}{K_0^2(\eta,x)W_p(y_1,y_2)(x)}, \tag{14}$$

where $W_p(y_1,y_2)(x)$ is the proportional Wronskian defined as

$$W_p(y_1,y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ {}_P\mathbf{D}^{\eta}y_1(x) & {}_P\mathbf{D}^{\eta}y_2(x) \end{vmatrix}. \tag{15}$$

Therefore, if we apply the integral to the functions $\nu_1'(x)$ and $\nu_2'(x)$, we get the functions $\nu_1(x)$ and $\nu_2(x)$. By substituting these functions to the $y_p(x)$, we reach the particular solution. As a result, the general solution is obtained by calculating the sum of the solution of the homogeneous part of the equation under consideration and particular solution $y_p(x)$. For more information on proportional derivatives and applications of different types of fractional derivatives, we refer the reader to [16–20].

4 Main results

In the current section, we introduce the representation of the solution for the S-L problem employing the proportional derivative. Here, we use two suitable initial conditions and so we get two representations of the solution by utilizing the proportional variation of parameters method. Let $\varphi(x,\lambda)$ be the solution of Eq. (11) with the initial condition given as

$$\varphi(0,\lambda)=1, \qquad {}_P\mathbf{D}^{\eta}\varphi(0,\lambda)=h, \tag{16}$$

and the other solution is $\Phi(x,\lambda)$ under the following initial condition

$$\Phi(0,\lambda)=0, \qquad {}_P\mathbf{D}^{\eta}\Phi(0,\lambda)=1. \tag{17}$$

In order to obtain the solutions $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$, we benefit from the proportional variation of parameters method. For this purpose, we employ the solution of the homogeneous counterpart of Eq. (11) obtained as

$$y_h(x)=c_1e_0(x,0)\cos\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right)+c_2e_0(x,0)\sin\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right). \tag{18}$$

On the other hand, for the non-homogeneous equation (11), we assume that

$$y_p(x)=\nu_1(x)e_0(x,0)\cos\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right)+\nu_2(x)e_0(x,0)\sin\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right). \tag{19}$$

Also, the p-Wronskian can be computed as below:

$$W_p=\begin{vmatrix} e^{-\int_0^x\frac{K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}\cos\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right) & e^{-\int_0^x\frac{K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}\sin\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right)\\ {}_P\mathbf{D}^{\eta}\left[e^{-\int_0^x\frac{K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}\cos\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right)\right] & {}_P\mathbf{D}^{\eta}\left[e^{-\int_0^x\frac{K_1(\eta,\tau)}{K_0(\eta,\tau)}d\tau}\sin\left(\int_0^x\frac{\sqrt{\lambda}}{K_0(\eta,s)}ds\right)\right]\end{vmatrix}, \tag{20}$$

and if we choice $K_1(\eta,s)=1-\eta$ and $K_0(\eta,s)=\eta$, we reach

$$W_p=\begin{vmatrix} e^{-\frac{(1-\eta)}{\eta}x}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right) & e^{-\frac{(1-\eta)}{\eta}x}\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)\\ {}_P\mathbf{D}^{\eta}\left[e^{-\frac{(1-\eta)}{\eta}x}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right] & {}_P\mathbf{D}^{\eta}\left[e^{-\frac{(1-\eta)}{\eta}x}\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right]\end{vmatrix}, \tag{21}$$

$${}_P\mathbf{D}^{\eta}\left[e^{-\frac{(1-\eta)}{\eta}x}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right]=e^{-\frac{(1-\eta)}{\eta}x}\left[\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)-\sqrt{\lambda}\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right], \tag{22}$$

and

$$ {}_{p}\mathbf{D}^{\eta}\left[e^{-\frac{(1-\eta)}{\eta}x}\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right]=e^{-\frac{(1-\eta)}{\eta}x}\left[\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)-\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)\right]. \tag{23} $$

Hence we get p-Wronskian as

$$ W_p=\sqrt{\lambda}e^{\frac{2(\eta-1)}{\eta}x}. \tag{24} $$

By taking the integral of following expressions

$$ \nu_1'(x)=\frac{y_2(x)q(x)y(x)}{\eta^2 W_p}, \qquad \nu_2'(x)=\frac{-y_1(x)q(x)y(x)}{\eta^2 W_p}, \tag{25} $$

it can be reached the functions $\nu_1(x)$ and $\nu_2(x)$ as follows

$$ \nu_1(x)=\int_0^x\frac{e^{-\frac{(1-\eta)}{\eta}t}\sin\left(\frac{\sqrt{\lambda}}{\eta}t\right)}{\eta^2\sqrt{\lambda}e^{\frac{2(\eta-1)}{\eta}}}q(t)y(t)dt, \qquad \nu_2(x)=-\int_0^x\frac{e^{-\frac{(1-\eta)}{\eta}t}\cos\left(\frac{\sqrt{\lambda}}{\eta}t\right)}{\eta^2\sqrt{\lambda}e^{\frac{2(\eta-1)}{\eta}t}}q(t)y(t)dt. \tag{26} $$

If we arrange the above formulas, we get

$$ \nu_1(x)=\frac{1}{\eta^2\sqrt{\lambda}}\int_0^x e^{\frac{(1-\eta)}{\eta}t}\sin\left(\frac{\sqrt{\lambda}}{\eta}t\right)q(t)y(t)dt, \tag{27} $$

and

$$ \nu_2(x)=\frac{-1}{\eta^2\sqrt{\lambda}}\int_0^x e^{\frac{(1-\eta)}{\eta}t}\cos\left(\frac{\sqrt{\lambda}}{\eta}t\right)q(t)y(t)dt. \tag{28} $$

Substituting the functions $\nu_1(x)$ and $\nu_2(x)$ into Eq. (19), one can readily have

$$ \begin{aligned} y_p(x)=&\,e^{-\frac{(1-\eta)}{\eta}x}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)\frac{1}{\eta^2\sqrt{\lambda}}\int_0^x e^{\frac{(1-\eta)}{\eta}t}\sin\left(\frac{\sqrt{\lambda}}{\eta}t\right)q(t)y(t)dt\\ &-e^{-\frac{(1-\eta)}{\eta}x}\sin\left(\frac{\sqrt{\lambda}}{\eta}\right)\frac{1}{\eta^2\sqrt{\lambda}}\int_0^x e^{\frac{(1-\eta)}{\eta}t}\cos\left(\frac{\sqrt{\lambda}}{\eta}t\right)q(t)y(t)dt. \end{aligned} \tag{29} $$

Thereby, the general solution is obtained as

$$ \begin{aligned} y(x)=&\,c_1e^{-\frac{(1-\eta)}{\eta}x}\cos\left(\frac{\sqrt{\lambda}}{\eta}x\right)+c_2e^{-\frac{(1-\eta)}{\eta}x}\sin\left(\frac{\sqrt{\lambda}}{\eta}x\right)\\ &+\frac{1}{\eta^2\sqrt{\lambda}}\int_0^x q(t)y(t)e^{\frac{(1-\eta)}{\eta}t}\sin\left[\frac{\sqrt{\lambda}}{\eta}(x-t)\right]dt. \end{aligned} \tag{30} $$

Let λ be s^2, then by applying the initial condition (16), we have the solution as follows

$$\varphi(x,s) = e^{-\frac{(1-\eta)}{\eta}x} \cos\left(\frac{s}{\eta}x\right) + \frac{h}{s}e^{-\frac{(1-\eta)}{\eta}x} \sin\left(\frac{s}{\eta}x\right) + \frac{1}{s\eta^2}\int_0^x q(t)\varphi(t)e^{\frac{(1-\eta)}{\eta}t} \sin\left[\frac{s}{\eta}(x-t)\right] dt, \tag{31}$$

and utilizing the initial condition (17), we can get the solution

$$\Phi(x,s) = \frac{1}{s}e^{-\frac{(1-\eta)}{\eta}x} \sin\left(\frac{s}{\eta}x\right) + \frac{1}{s\eta^2}\int_0^x q(t)\Phi(t)e^{\frac{(1-\eta)}{\eta}t} \sin\left[\frac{s}{\eta}(x-t)\right] dt. \tag{32}$$

5 Visual results and discussions

This section includes graphs of the solution functions of S-L problem that are achieved by employing the benefits of proportional derivative. The behavior of the representation of solution function $\varphi(x,s)$ for the classical situation is first demonstrated when $s = 1, 2, 3$, and then it is shown how the solution curve motions vary for $\eta = 0.9, 0.7, 0.5$ in Figure 2 and Figure 3. On the S-L problem, which has physically crucial meanings, it has been clearly observed how different order values of the proportional derivative affect the problem and how they change the behavior of the solution functions.

On the other hand, it should be expressed that the reason for using the same eigenvalues is to see the effect of different order values. In Figure 4-Figure 5, we demonstrate how the solutions change as the η parameter takes different values for s=1, s=3 and s=5, respectively. Additionally, Figure 6 shows the behavior of the function $\varphi(x,s)$ for $\eta = 1, 0.8, 0.6, 0.4$ when $s = \sqrt{0.1}$. Afterwards, similarly, we plot the graphs for the solution function $\varphi(x,s)$ by using the same parameter values for the solution function $\Phi(x,s)$ in Figure 7-Figure 10. Here, the representation of solution function $\varphi(x,s)$ under the condition (16) is

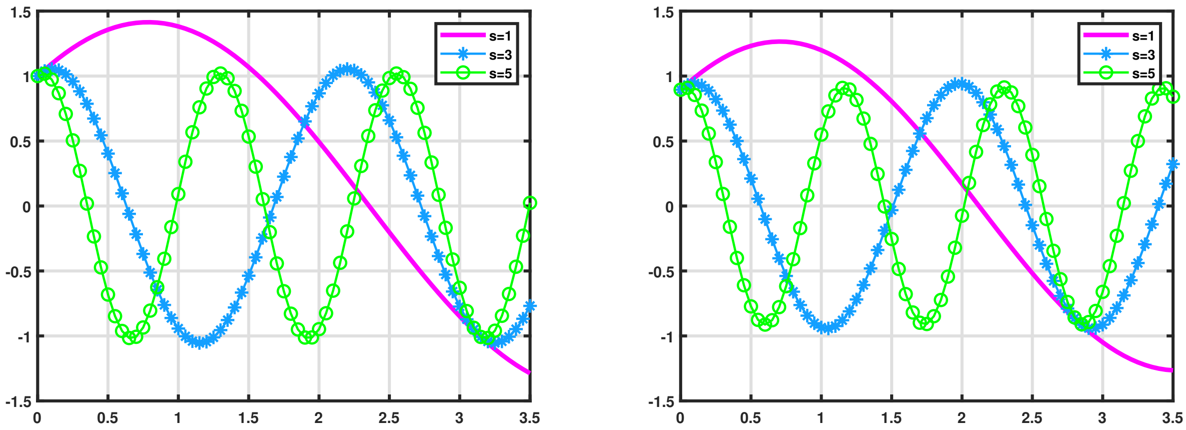
$$\varphi(x,s) = e^{-\frac{(1-\eta)}{\eta}x} \cos\left(\frac{s}{\eta}x\right) + \frac{h}{s}e^{-\frac{(1-\eta)}{\eta}x} \sin\left(\frac{s}{\eta}x\right) + \frac{1}{s\eta^2}\int_0^x q(t)\varphi(t)e^{\frac{(1-\eta)}{\eta}t} \sin\left[\frac{s}{\eta}(x-t)\right] dt, \tag{33}$$

and the representation of solution function $\Phi(x,s)$ under the condition (17) is

$$\Phi(x,s) = \frac{1}{s}e^{-\frac{(1-\eta)}{\eta}x} \sin\left(\frac{s}{\eta}x\right) + \frac{1}{s\eta^2}\int_0^x q(t)\Phi(t)e^{\frac{(1-\eta)}{\eta}t} \sin\left[\frac{s}{\eta}(x-t)\right] dt. \tag{34}$$

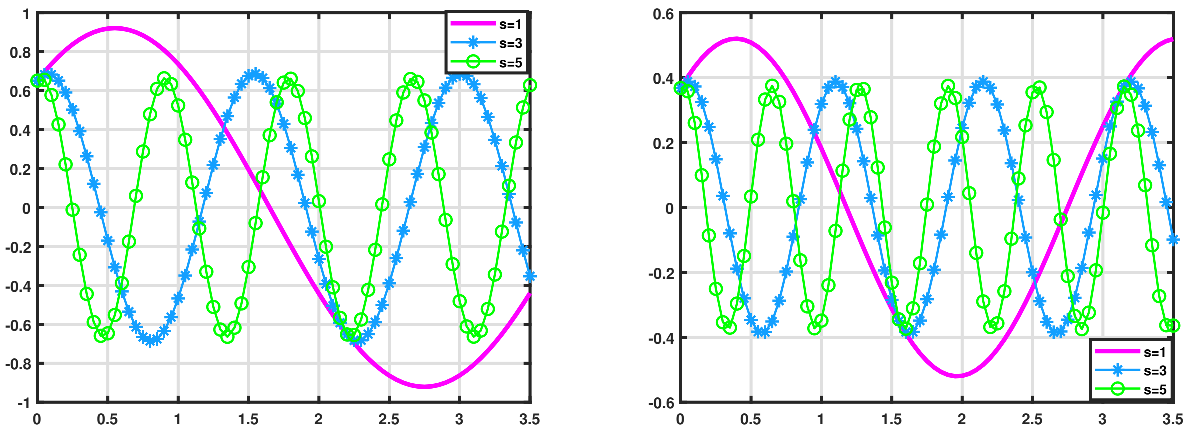
All graphs are obtained by the various values of arbitrary order and eigenvalues when the potential function $q(t) = 0$. Accordingly, the main objective of the graphs is to see the effect of the eigenvalues, which are important for the problem under investigation, on the solution functions and to observe the effect of the proportional derivative on the S-L problem. To observe these two situations separately, which are important for the current study, in some graphs, eigenvalues are not changed, while arbitrary order of proportional derivative values are changed.

In a similar way, to see the effect of the eigenvalues, the derivative order is not changed and the eigenvalues are changed.



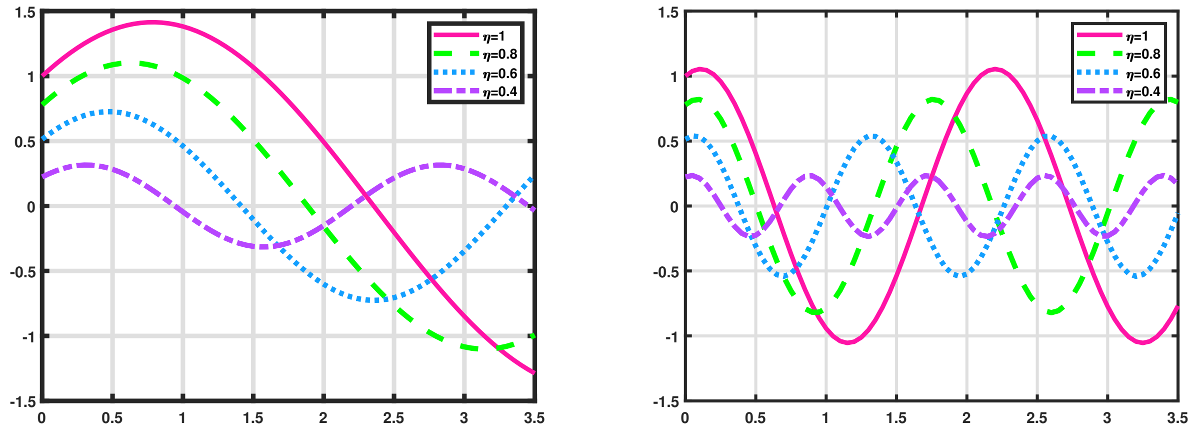
(a) **(b)**

Figure 2. The solutions curves of the function $\varphi(x,s)$ when $\eta = 1$ (classical case) (a) and $\eta = 0.9$ (arbitrary order case) (b) for the values of $s = 1, 3, 5$ (this corresponds to the $\lambda = 1, 9, 25$ eigenvalues) under the condition (16).



(a) **(b)**

Figure 3. The solutions curves of the function $\varphi(x,s)$ when $\eta = 0.7$ (a) and $\eta = 0.5$ (b) for the values of $s = 1, 3, 5$ (this corresponds to the $\lambda = 1, 9, 25$ eigenvalues) under the condition (16).



(a) (b)

Figure 4. The solutions curves of the function $\varphi(x, s)$ when $s = 1$ (a) and $s = 3$ (b) for different values of arbitrary order $\eta = 1, 0.8, 0.6, 0.4$.

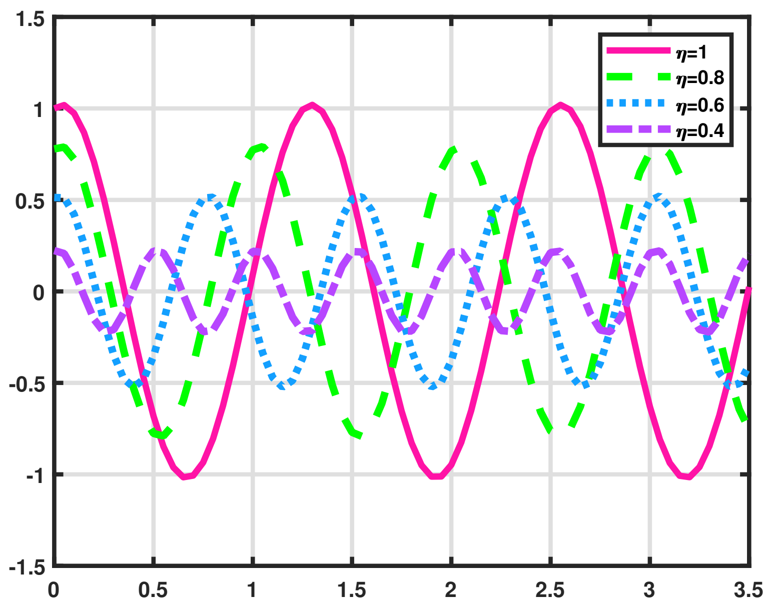


Figure 5. The solutions curves of the function $\varphi(x, s)$ when $s = 5$ for different values of arbitrary order $\eta = 1, 0.8, 0.6, 0.4$.

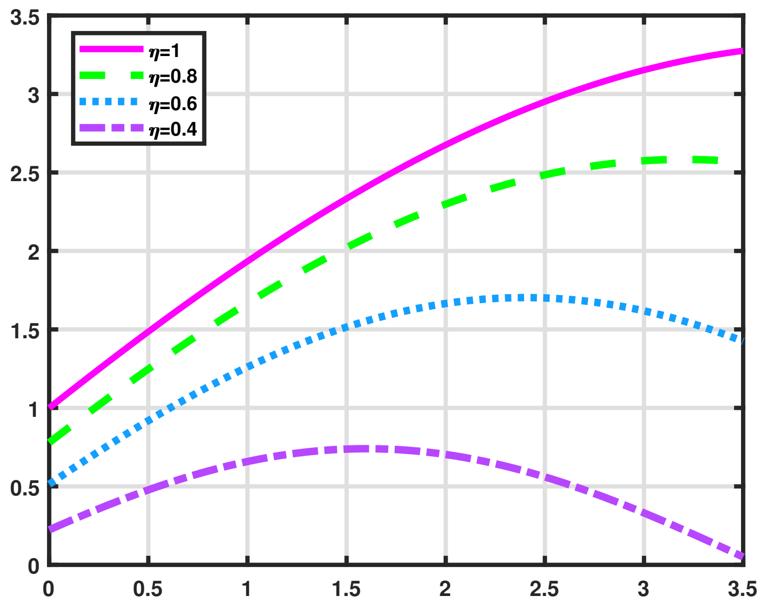
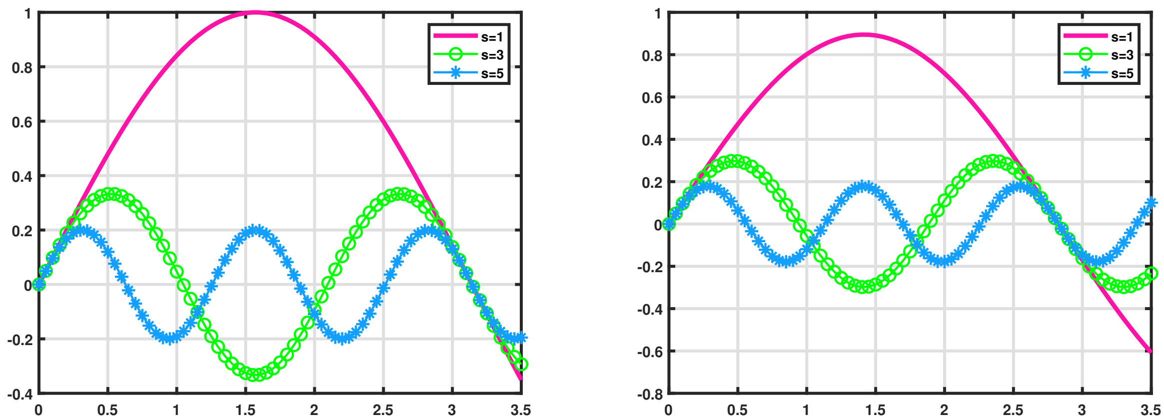
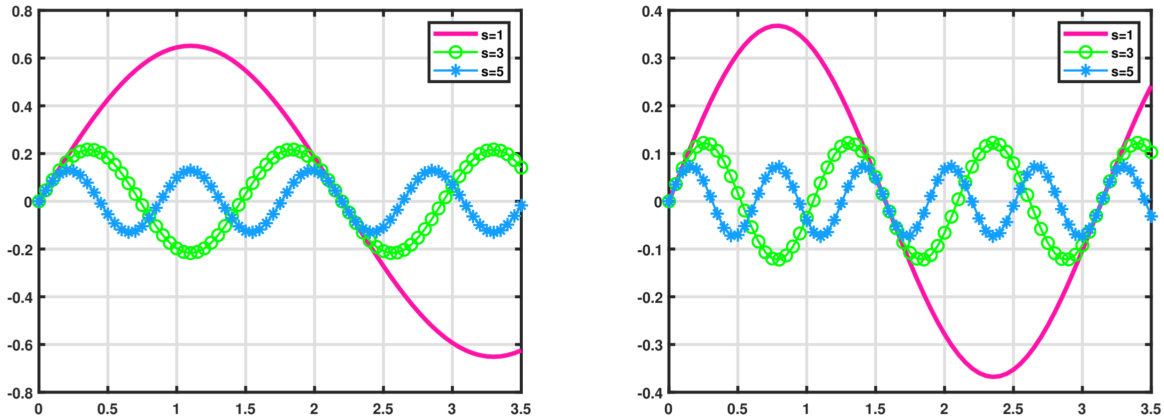


Figure 6. The solution curves of the function $\varphi(x,s)$ when $s = \sqrt{0.1}$ for different values of arbitrary order $\eta = 1, 0.8, 0.6, 0.4$.



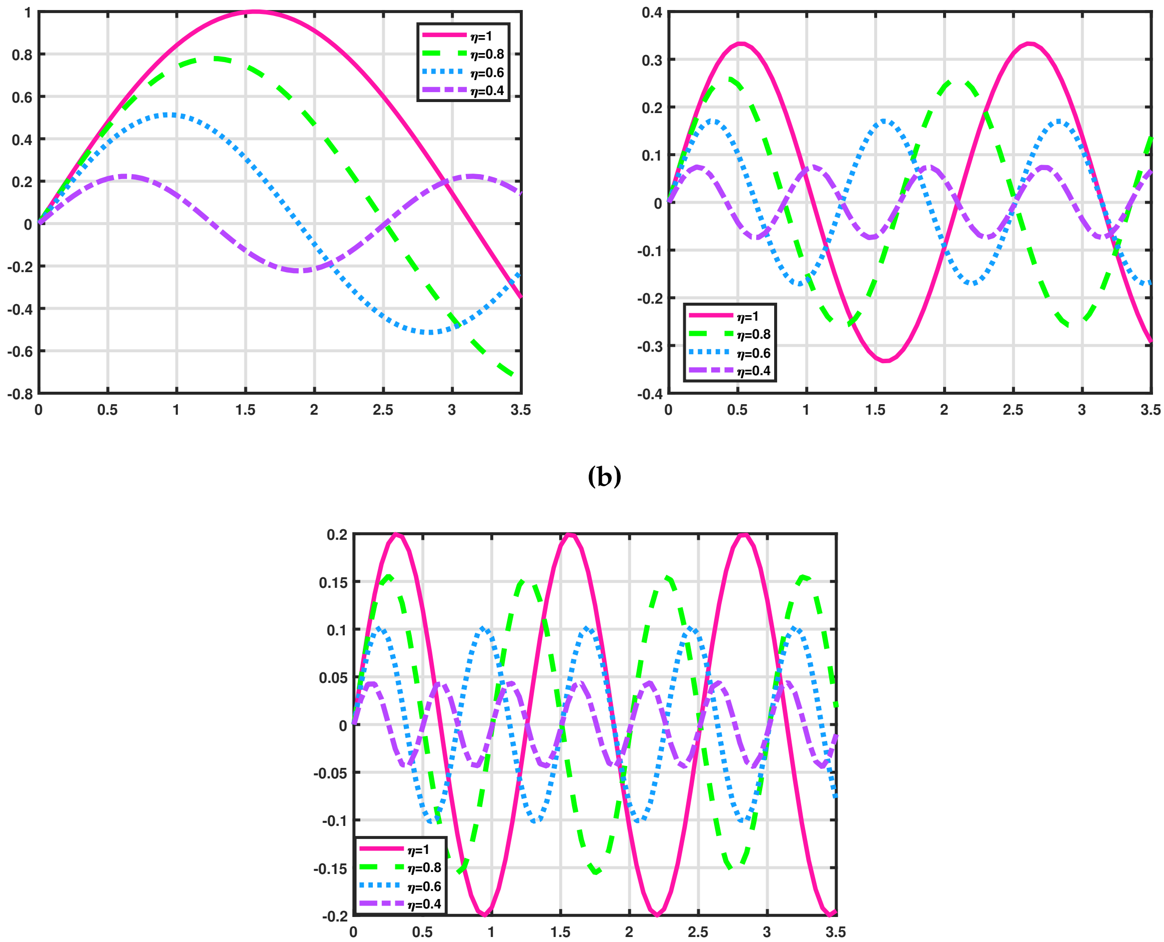
(a) **(b)**

Figure 7. The solutions curves of the function $\Phi(x,s)$ when $\eta = 1$ (classical case) (a) and $\eta = 0.9$ (arbitrary order case) (b) for the values of $s = 1, 3, 5$ (this corresponds to the $\lambda = 1, 9, 25$ eigenvalues) under the condition (17).



(a) **(b)**

Figure 8. The solutions curves of the function $\Phi(x,s)$ when $\eta = 0.7$ (a) and $\eta = 0.5$ (b) for the values of $s = 1, 3, 5$ (this corresponds to the $\lambda = 1, 9, 25$ eigenvalues) under the condition (17).



(c)

Figure 9. The solutions curves of the function $\Phi(x,s)$ when $s = 1$ (a), $s = 3$ (b), and $s = 5$ (c) for different values of arbitrary order $\eta = 1, 0.8, 0.6, 0.4$.

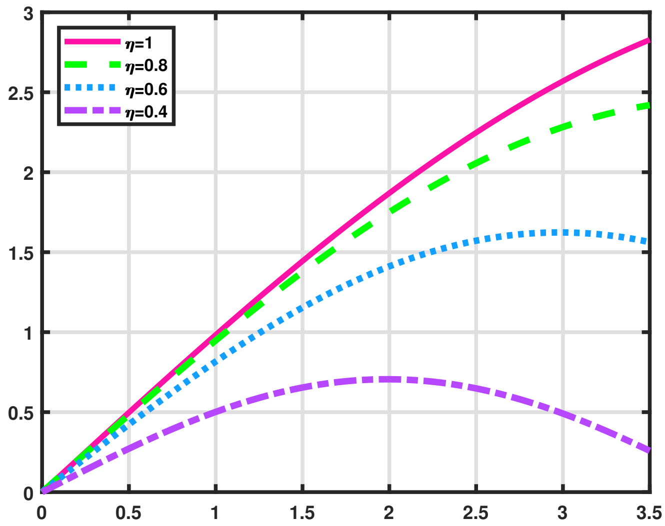


Figure 10. The solution curves of the function $\Phi(x,s)$ when $s=\sqrt{0.1}$ for different values of arbitrary order $\eta=1,0.8,0.6,0.4$.

6 Concluding remarks

The proportional derivative, which is considered in the class of local derivatives including arbitrary order, is considered more advantageous than other local derivatives in terms of its features. Since it is based on control theory, it has an important place, especially in engineering. In [2], the authors state that since the unit operator cannot be obtained for the other local derivatives when $\mathcal{D}^0\omega \neq \omega$, that is, $\chi \to 0$, and on the other hand, there is a $t \geq 0$ condition to satisfy the $\mathbf{D}^\chi\omega(t) = t^{1-\chi}\omega'(t)$ formula, they have introduced a novel definition of local derivative called proportional derivative in order to overcome these restrictions. This new and seemingly more well-founded local derivative definition is created in such a way that $\mathbf{D}^0$ corresponds to the unit operator and $\mathbf{D}^1$ corresponds to the integer-order classical derivative, while $0 \leq \chi \leq 1$ and $t \in \mathbb{R}$. In the definition of the proportional derivative, various special cases can be obtained for different choices of the functions $K_1(\eta,t)$ and $K_0(\eta,t)$. For example, proportional derivatives of special types can be obtained by choosing for any $\omega \in (0,\infty)$, $K_1 \equiv (1-\chi)\omega^\chi$ and $K_0 \equiv \chi\omega^{1-\chi}$, $K_1 = (1-\chi)|t|^\chi$ and $K_0 = \chi|t|^{1-\chi}$ on $\mathbb{R}\backslash\{0\}$, or $K_1 = \cos(\chi\pi/2)|t|^\chi$ and $K_0 = \sin(\chi\pi/2)|t|^{1-\chi}$. This can be seen as another advantage of the proportional derivative. Because, in application, one can have the opportunity to obtain better results by making the special choices needed according to the behavior of the problem under consideration. Therefore, attention should be paid to whether the special choices made are useful and meaningful in application. Due to all these advantages, the proportional derivative is preferred in solving the S-L equation in this study. It is thought that the results obtained as an alternative to the classical derivative will be useful for experts in the field.

Also, it should be emphasized that addressing and examining the S-L problem, which is of great physical importance, with the help of proportional derivatives used in control theory, can make a significant contribution to the literature. It is known that there are many different S-L problems in the literature. Therefore, this study is important in terms of encouraging the application of proportional derivative to different problems in this field.

Declarations

List of abbreviations

Not applicable.

Ethical approval

The author states that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable.

Conflicts of interest

The author confirms that there is no competing interest in this study.

Data availability statement

Data availability is not applicable to this article as no new data were created or analyzed in this study.

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Author's contributions

The author has made substantial contributions to the conception, design of the work, the acquisition, analysis, interpretation of data, and the creation of new software used in the work.

References

[1] Miller, K.S. and Ross, B. *An introduction to the fractional calculus and fractional differential equations.* New York: John Wiley and Sons, (1993).

[2] Anderson, D.R. and Ulness, D.J. Newly defined conformable derivatives. *Advances in Dynamical Systems and Applications*, 10(2), 109-137, (2015). [CrossRef]

[3] Li, Y., Ang, K.H. and Chong, G.C.Y. PID control system analysis and design. *IEEE Control Systems Magazine*, 26(1), 32-41, (2006). [CrossRef]

[4] Levitan, B.M. and Sargsujan, I.S. *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators* (Vol. 39). American Mathematical Society: Providence, (1975).

[5] Klimek, M. and Agrawal, O.P. Fractional Sturm–Liouville problem. *Computers & Mathematics with Applications*, 66(5), 795-812, (2013). [CrossRef]

[6] Zayernouri, M. and Karniadakis, G.E. Fractional Sturm–Liouville eigen-problems: Theory and numerical approximation. *Journal of Computational Physics*, 252, 495-517, (2013). [CrossRef]

[7] Al-Mdallal, Q.M. An efficient method for solving fractional Sturm–Liouville problems. *Chaos, Solitons & Fractals*, 40(1), 183-189, (2009). [CrossRef]

[8] Allahverdiev, B.P., Tuna, H. and Yalçinkaya, Y. Conformable fractional Sturm-Liouville equation. *Mathematical Methods in the Applied Sciences*, 42(10), 3508-3526, (2019). [CrossRef]

[9] Bas, E. and Acay, B. The direct spectral problem via local derivative including truncated Mittag-Leffler function. *Applied Mathematics and Computation*, 367, 124787, (2020). [CrossRef]

[10] Ercan, A. Comparative analysis for fractional nonlinear Sturm-Liouville equations with singular and non-singular kernels. *AIMS Mathematics*, 7(7), 13325-13343, (2022). [CrossRef]

[11] Ercan, A. Conformable Discontinuous Sturm-Liouville Problem with Applied Results. *International Journal of Applied Mathematics and Statistics*, 61(1), 71-81, (2022). [CrossRef]

[12] Ercan, A. and Panakhov, E. Stability of the reconstruction discontinuous Sturm-Liouville problem. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 68(1), 484-499, (2019). [CrossRef]

[13] Hammouch, Z., Yavuz, M. and Özdemir, N. Numerical solutions and synchronization of a variable-order fractional chaotic system. *Mathematical Modelling and Numerical Simulation with Applications*, 1(1), 11-23, (2021). [CrossRef]

[14] Yavuz, M., Ozdemir, N. and Baskonus, H.M. Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel. *The European Physical Journal Plus*, 133, 215, (2018). [CrossRef]

[15] Acay, B., Inc, M., Chu, Y.M. and Almohsen, B. Modeling of pressure–volume controlled artificial respiration with local derivatives. *Advances in Difference Equations*, 2021, 1-21, (2021). [CrossRef]

[16] Baleanu, D., Fernandez, A. and Akgül, A. On a fractional operator combining proportional and classical differintegrals. *Mathematics*, 8(3), 1-13, (2020). [CrossRef]

[17] Jarad, F., Alqudah, M.A., Abdeljawad, T. On more general forms of proportional fractional operators. *Open Mathematics*, 18(1), 167-176, (2020). [CrossRef]

[18] Jarad, F., Abdeljawad, T. and Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *The European Physical Journal Special Topics*, 226, 3457-3471, (2017). [CrossRef]

[19] Acay, B. and Inc, M. Electrical circuits RC, LC, and RLC under generalized type non-local singular fractional operator. *Fractal and Fractional*, 5(1), 9, (2021). [CrossRef]

[20] Acay, B., Bas, E. and Abdeljawad, T. Non-local fractional calculus from different viewpoint generated by truncated M-derivative. *Journal of Computational and Applied Mathematics*, 366, 112410, (2020). [CrossRef]

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