Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1393349
(**2024**) **5**(2) 134 – 142
Research Article

On Some Positive Linear Operators Preserving the $\mathbb{B}^{-1}$-Convexity of Functions

Mustafa Uzun [1,*], **Tuncay Tunç** [2]
[1] Eskişehir Technical University, Faculty of Science, Department of Mathematics
Eskişehir, Türkiye
[2] Mersin University, Faculty of Science, Department of Mathematics
Mersin, Türkiye, ttunc@mersin.edu.tr

Received: 21 November 2023 **Accepted:** 19 February 2024

Abstract: In the study, we first give an inequality that non-negative differentiable functions must satisfy to be $\mathbb{B}^{-1}$-convex. Then, using the inequality, we show that the $\mathbb{B}^{-1}$-convexity property of functions is preserved by Bernstein-Stancu operators, Szász-Mirakjan operators and Baskakov operators. In addition, we compare the concepts $\mathbb{B}^{-1}$-convexity and $\mathbb{B}$-concavity of functions.

Keywords: $\mathbb{B}^{-1}$-convexity, Bernstein-Stancu operators, Szász-Mirakjan operators, Baskakov operators, shape preserving approximation.

1. Introduction and Preliminaries

Approximation to continuous functions with positive and linear operators is a useful method in the field of approximation theory of mathematics. According to necessity, it may also be required to preserve the convexity characteristics of functions in this type of approach. Therefore, it is significant to determine which kinds of convexities of functions are preserved by positive linear operators. There are many studies in this topic which is called shape preserving approximation: In [6] and references therein, one can find many results in detail for various operators and convexities.

Similarly, the main purpose of working is to determine whether the following positive and linear operators preserve the $\mathbb{B}^{-1}$-convexity property of the functions:

The Bernstein-Stancu operators which is generalization of the Bernstein operators, $B_n^{\alpha,\beta}$ on $C[0,1]$ are defined as follows [9]:

$$B_n^{\alpha,\beta}(h;t) = \sum_{j=0}^{n} \rho_{n,j}(t) h\left(\frac{j+\alpha}{n+\beta}\right), \qquad h \in C[0,1],$$

*Correspondence: mustafau@eskisehir.edu.tr
2020 *AMS Mathematics Subject Classification*: 26A51, 41A36

where $\rho_{n,j}(t) = \binom{n}{j} t^j (1-t)^{n-j}$, $n \in \mathbb{N}$ and $0 \le \alpha \le \beta$.

In [10], for a function h on $C[0,\infty)$, the Szász-Mirakjan Operators were defined by

$$S_n(h;t) = \sum_{j=0}^{\infty} s_{n,j}(t) h\left(\frac{j}{n}\right)$$

, where $s_{n,j}(t) = \frac{e^{-nt}(nt)^j}{j!}$ and $n \in \mathbb{N}$.

In [2], the Baskakov operators V_n were introduced as

$$V_n(h;t) = \sum_{j=0}^{\infty} v_{n,j}(t) h\left(\frac{j}{n}\right)$$

on $C[0,\infty)$, where $v_{n,j}(t) = \binom{n+j-1}{j} \frac{t^j}{(1+t)^{n+j}}$ and $n \in \mathbb{N}$.

We used the following notations throughout the study, for $t = (t_1, \ldots, t_k)$, $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$,

$$t \vee s := (\max\{t_1, s_1\}, \max\{t_2, s_2\} \ldots, \max\{t_k, s_k\}),$$

$$t \wedge s := (\min\{t_1, s_1\}, \min\{t_2, s_2\} \ldots, \min\{t_k, s_k\}).$$

$$\mathbb{R}^k_+ := \{t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \mid t_i \ge 0 \text{ for each } i \in \{1, 2, \ldots, k\}\},$$

$$\mathbb{R}^k_{++} := \{t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \mid t_i > 0 \text{ for each } i \in \{1, 2, \ldots, k\}\}.$$

Let us now recall the main definitions and theorems necessary for the following sections.

Definition 1.1 *[3, 7] A set $V \subset \mathbb{R}^k_+$ is called $\mathbb{B}$-convex if $\lambda t \vee s \in V$ for all $t, s \in V$, $\lambda \in [0,1]$.*

Remark 1.2 *[13] $\mathbb{B}$-convex subsets of $\mathbb{R}_+$ are intervals which are open, closed or half-open.*

Definition 1.3 *[7] A function $h : V \subset \mathbb{R}^k_+ \to \mathbb{R}_+$ is called $\mathbb{B}$-convex function if the set V is $\mathbb{B}$-convex and the inequality*

$$h(\lambda t \vee s) \le \lambda h(t) \vee h(s)$$

holds for all $t, s \in V$ and $\lambda \in [0,1]$. If V is $\mathbb{B}$-convex and the inequality

$$h(\lambda t \vee s) \ge \lambda h(t) \vee h(s)$$

holds for all $t, s \in V$, $\lambda \in [0,1]$, then h is called $\mathbb{B}$-concave function.

For $\mathbb{B}$-concave functions, the following expression was obtained in [11]:

Theorem 1.4 *[11] A function* $h:[0,1]\to\mathbb{R}_+$ *is* $\mathbb{B}$*-concave if and only if the following conditions hold:*

(i) *h is increasing on* $[0,1]$,

(ii) *The inequality* $h(\lambda t)\geq\lambda h(t)$ *holds for all* $\lambda,t\in[0,1]$. *Also, let h be a differentiable function. Then h is* $\mathbb{B}$*-concave if and only if h is increasing and the inequality* $th'(t)-h(t)\leq 0$ *holds for all* $t\in[0,1]$.

Theorem 1.5 *[12] For a* $\mathbb{B}$*-concave function* $h:[0,1]\to\mathbb{R}_+$, *the continuity at* $t=0$ *is sufficient for continuity on* $[0,1]$ *of* h.

The $\mathbb{B}$-concavity and $\mathbb{B}$-convexity preserving properties of functions by one and two dimensional Bernstein operators were studied in recent years. Some results in the studies as follows:

- Bernstein operators with one variable do not preserve the $\mathbb{B}$-convexity of functions, but preserve the $\mathbb{B}$-concavity of functions [11],
- Two dimensional Bernstein operators do not preserve both of the convexities [12],
- In [8], author presented a sufficient condition for Bernstein operators on bidemensional simplex to be $\mathbb{B}$-concave.

$\mathbb{B}^{-1}$-convex sets were introduced and studied in [1, 4]. One of the results in the studies was obtained for the any subset of $\mathbb{R}_{++}$. See the following theorem:

Theorem 1.6 *Let* $W\subset\mathbb{R}_{++}$. *W is* $\mathbb{B}^{-1}$*-convex set if and only if* $\lambda t\wedge s\in W$ *for all* $t,s\in W$ *and* $\lambda\in[1,\infty)$.

Remark 1.7 *[13]* $\mathbb{B}^{-1}$*-convex subsets of* $\mathbb{R}_{++}$ *are intervals which are open, closed or half-open.*

Then, in [7], $\mathbb{B}^{-1}$-convex functions were defined as the following theorem:

Theorem 1.8 *If a set* $W\subset\mathbb{R}_{++}$ *is* $\mathbb{B}^{-1}$*-convex and* $h:W\to\mathbb{R}_{++}$, *then h is* $\mathbb{B}^{-1}$*-convex function if and only if the inequality*

$$h(\lambda t\wedge s)\leq\lambda h(t)\wedge h(s) \tag{1}$$

holds for all $t,s\in W$ *and* $\lambda\in[1,\infty)$.

Corollary 1.9 *[5] If a set* $W\subset\mathbb{R}_{++}$ *is* $\mathbb{B}^{-1}$*-convex, then* $h:W\to\mathbb{R}_{++}$ *is* $\mathbb{B}^{-1}$*-convex if and only if the following conditions hold:*

(i) *h is increasing on W*,

(ii) *The inequality* $h(\lambda t)\leq\lambda h(t)$ *holds for all* $\lambda t\in W$, *where* $t\in W$ *and* $\lambda\in[1,\infty)$.

2. Preservation of $\mathbb{B}^{-1}$-Convex Functions

In this section, an alternative way is given to determine whether the differentiable functions defined on a subset of $\mathbb{R}_{++}$ are $\mathbb{B}^{-1}$-convex. Then using this, results are given regarding the positive linear operators mentioned above preserve the $\mathbb{B}^{-1}$-convexity property of functions.

Lemma 2.1 *Let $W \subset \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex set and $h : W \to \mathbb{R}_{++}$ be a differentiable function. Then, h is $\mathbb{B}^{-1}$-convex function if and only if h is increasing on W and the inequality $th'(t) - h(t) \le 0$ holds for all $t \in W$.*

Proof Assume h is $\mathbb{B}^{-1}$-convex function. Let $t \in W$ and (λ_n) be a sequence in $(1,\infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ and $\lambda_n t \in W$ for each $n \in \mathbb{N}$. Due to the $\mathbb{B}^{-1}$-convexity of h, we have $h(\lambda_n t) \le \lambda_n h(t)$ for all $n \in \mathbb{N}$. Then, we get

$$\frac{h(\lambda_n t) - h(t)}{(\lambda_n - 1)t} \le \frac{h(t)}{t} \Rightarrow h'(t) = \lim_{n\to\infty} \frac{h(\lambda_n t) - h(t)}{(\lambda_n - 1)t} \le \frac{h(t)}{t}.$$

Therefore, the inequality $th'(t) - h(t) \le 0$ holds. Conversely, showing the second condition in Corollary 1.9 is sufficient for the $\mathbb{B}^{-1}$-convexity of h. Due to the inequality $th'(t) - h(t) \le 0$, we have $\left(\frac{h(t)}{t}\right)' \le 0$, that is, $\left(\frac{h(t)}{t}\right)$ is decreasing on W. Finally, we get the inequality

$$\frac{h(\lambda t)}{\lambda t} \le \frac{h(t)}{t} \Rightarrow h(\lambda t) \le \lambda h(t)$$

for all $\lambda t \in W$, where $t \in W$ and $\lambda \in [1,\infty)$. □

Theorem 2.2 *Let $g : [0,1] \to \mathbb{R}_{++}$ be a function such that the restriction of g to $(0,1]$ is $\mathbb{B}^{-1}$-convex. Then, $B_n^{\alpha,\beta}(g)$ is also $\mathbb{B}^{-1}$-convex on $(0,1]$ for all $n \in \mathbb{N}$ and $0 \le \alpha \le \beta$.*

Proof Let $t \in (0,1]$ and $n \in \mathbb{N}$. Because of the equation (2), $B_n^{\alpha,\beta}(g;t)$ is increasing:

$$B'^{\alpha,\beta}_n(g;t) = n \sum_{j=0}^{n-1} \rho_{n-1,j}(t) \left[g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right) \right]. \tag{2}$$

Also, considering the equation (2) and the following inequality

$$\begin{aligned} \frac{B_n^{\alpha,\beta}(g;t)}{t} &= t^{-1} \sum_{j=0}^{n} \rho_{n,j}(t) g\left(\frac{j+\alpha}{n+\beta}\right) \\ &\ge t^{-1} \sum_{j=1}^{n} \rho_{n,j}(t) g\left(\frac{j+\alpha}{n+\beta}\right) \\ &= \frac{n}{j+1} \sum_{j=0}^{n-1} \rho_{n-1,j}(t) g\left(\frac{j+1+\alpha}{n+\beta}\right), \end{aligned} \tag{3}$$

we obtain that:

$$B'^{\alpha,\beta}_n(g;t) - \frac{B^{\alpha,\beta}_n(g;t)}{t} \leq n\sum_{j=0}^{n-1}\rho_{n-1,j}(t)\left[\frac{j}{j+1}g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right)\right].$$

For $j = 0$,

$$0g\left(\frac{1}{n}\right) - g(0) \leq 0$$

and considering conditions in Corollary 1.9, we get the following inequality for $0 < j \leq n-1$:

$$\frac{j+1}{j}g\left(\frac{j+\alpha}{n+\beta}\right) \geq g\left(\frac{j+1}{j}\frac{j+\alpha}{n+\beta}\right) \geq g\left(\frac{j+\alpha+1}{n+\beta}\right)$$

and therefore,

$$\left[\frac{j}{j+1}g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right)\right] \leq 0. \tag{4}$$

Finally, since $\rho_{n-1,j}(t) \geq 0$ for all $t \in (0,1]$ and the ineqaulity (4) holds for each j $(0 \leq j \leq n-1)$, we obtain that

$$tB'^{\alpha,\beta}_n(g;t) - B^{\alpha,\beta}_n(g;t) \leq 0.$$

□

Theorem 2.3 *Let $g : [0,\infty) \to \mathbb{R}_{++}$ be a function such that the restriction of g to $(0,\infty)$ is $\mathbb{B}^{-1}$-convex, then $S_n(g)$ is also $\mathbb{B}^{-1}$-convex on $(0,\infty)$ for all $n \in \mathbb{N}$.*

Proof Let $t \in (0,\infty)$ and $n \in \mathbb{N}$. As a result of the following statements,

$$S'_n(g;t) = n\sum_{j=0}^{\infty} s_{n,j}(t)\left[g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \tag{5}$$

and

$$\begin{aligned}\frac{S_n(g;t)}{t} &= t^{-1}\sum_{j=0}^{\infty} s_{n,j}(t)g\left(\frac{j}{n}\right)\\ &\geq t^{-1}\sum_{j=1}^{\infty} s_{n,j}(t)g\left(\frac{j}{n}\right)\\ &= \frac{n}{j+1}\sum_{j=0}^{\infty} s_{n,j}(t)g\left(\frac{j+1}{n}\right),\end{aligned} \tag{6}$$

we see that $S_n(g;t)$ is increasing from (5) and we get the following inequality by using (5) and (6):

$$S_n'(g;t) - \frac{S_n(g;t)}{t} \leq n \sum_{j=0}^{\infty} s_{n,j}(t) \left[\frac{j}{j+1} g\left(\frac{j+1}{n} \right) - g\left(\frac{j}{n} \right) \right].$$

Due to $\mathbb{B}^{-1}$-convexity of g, we get the following inequalities: in case $j = 0$,

$$0g'\left(\frac{1}{n}\right) - g(0) \leq 0$$

and for each $j \in \mathbb{N}$

$$\frac{j+1}{j} g\left(\frac{j}{n}\right) \geq g\left(\frac{j+1}{j}\frac{j}{n}\right) = g\left(\frac{j+1}{n}\right).$$

Thus, from the above expressions, we obtain the following inequality for each $j \in \mathbb{N}_0$:

$$\left[\frac{j}{j+1} g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \leq 0. \tag{7}$$

Moreover, considering $s_{n,j}(t) \geq 0$ and (7), we obtain that

$$tS_n'(g;t) - S_n(g;t) \leq 0.$$

□

Theorem 2.4 *Let* $g : [0, \infty) \to \mathbb{R}_{++}$ *be a function which is* $\mathbb{B}^{-1}$*-convex on* $(0, \infty)$. *Then* $V_n(g)$ *is also* $\mathbb{B}^{-1}$*-convex on* $(0, \infty)$ *for all* $n \in \mathbb{N}$.

Proof The proof can be easily seen by using the following inequality and similar operations in Theorem 2.3.

$$\begin{aligned} V_n'(g;t) - \frac{V_n(g;t)}{t} &= \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \left[g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \\ &\quad - t^{-1} \sum_{j=0}^{\infty} v_{n,j}(t) g\left(\frac{j}{n}\right) \\ &\leq \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \left[g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \\ &\quad - t^{-1} \sum_{j=1}^{\infty} v_{n,j}(t) g\left(\frac{j}{n}\right) \\ &= \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \left[g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \end{aligned}$$

$$- \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \frac{1}{j+1} g\left(\frac{j+1}{n}\right)$$

$$= \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \left[\frac{j}{j+1} g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right].$$

□

3. Comparison of $\mathbb{B}^{-1}$-Convexity and $\mathbb{B}$-Concavity of Functions Defined on a Subset of $\mathbb{R}_{++}$

In this section, it is shown that there is no difference between $\mathbb{B}^{-1}$-convex functions and $\mathbb{B}$-concave functions on subsets of $\mathbb{R}_{++}$.

It is clear that, the statement in Teorem 1.4 can be expanded into any $\mathbb{B}$-convex set in $\mathbb{R}_{+}$ as follows:

Lemma 3.1 *For a $\mathbb{B}$-convex set $V \subset \mathbb{R}_{+}$, the function $h: V \to \mathbb{R}_{+}$ is $\mathbb{B}$-concave if and only if*

(i) h is increasing on V,

(ii) The inequality $h(\lambda t) \geq \lambda h(t)$ holds for all $\lambda t \in V$, where $t \in V$ and $\lambda \in [0,1]$.

If h is a differentiable function on V, then h is $\mathbb{B}$-concave function if and olny if h is increasing on V and the inequality $th'(t) - h(t) \leq 0$ holds for all $t \in V$.

Proof The technique of the proof is the same as the proof of Theorem 1.4. □

Remark 3.2 *According to Remarks 1.2 and 1.7, $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are also $\mathbb{B}$-convex sets. Then, considering Lemma 2.1 and Lemma 3.1 , $\mathbb{B}^{-1}$-convexity and $\mathbb{B}$-concavity have the same inequality for differentiable functions defined on a $\mathbb{B}^{-1}$-convex set $W \subset \mathbb{R}_{++}$. Therefore, $\mathbb{B}^{-1}$-convexity of a differentiable function is equivalent to $\mathbb{B}$-concavity. Moreover, the following corollary shows that this is remain true even if a function is not differentiable.*

Corollary 3.3 *Let $W \subset \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex set. Then, for a function $h: W \to \mathbb{R}_{++}$, $\mathbb{B}^{-1}$-convexity of h is equivalent to $\mathbb{B}$-concavity of h.*

Proof The property of increasing is common for both convexities. Let h be a $\mathbb{B}^{-1}$-convex function. Then we have $h(\lambda t) \leq \lambda h(t)$ for all $\lambda t \in W$, where $t \in W$ and $\lambda \in [1, \infty]$. Given $\mu \in (0,1]$ and $t \in W$ with $\mu t \in W$, we obtain

$$h(t) = h\left(\mu \frac{1}{\mu} t\right) \leq \frac{1}{\mu} h(\mu t).$$

Conversely, let h be a $\mathbb{B}$-concave function. Then we have $h(\mu t) \geq \mu h(t)$ for all $\mu t \in W$, where $t \in W$ and $\mu \in [0,1]$. Given $\lambda \in [1,\infty)$ and $t \in W$ with $\lambda t \in W$, we obtain

$$h(t) = h\left(\lambda \frac{1}{\lambda} t\right) \geq \frac{1}{\lambda} h(\lambda t).$$

□

Based on the Corallary 3.3, we conclude the following corollaries:

Corollary 3.4 *If a function $h : (0,1) \to \mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex, then h is continuous on $(0,1)$.*

Proof The proof can be seen from Theorem 1.5. □

Corollary 3.5 *According to algebraic operations in Theorems 2.2, 2.3 and 2.4, it is clear that the theorems are also valid for $\mathbb{B}$-concavity property of functions.*

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Mustafa Uzun]: Collected the data, contributed to research method or evaluation of data, contributed to completing the research and solving the problem, wrote the manuscript (%50).
Author [Tuncay Tunç]: Thought and designed the research/problem, contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Adilov G., Yeşilce İ., *B^{-1}-convex sets and B^{-1}-measurable maps*, Numerical Functional Analysis and Optimization, 33(2), 131-141, 2012.

[2] Baskakov V.A., *An instance of a sequence of linear positive operators in the space of continuous functions*, Doklady Akademii Nauk SSSR, 113(2), 249-251, 1957.

[3] Briec W., Horvath C.D., *$\mathbb{B}$-convexity*, Optimization, 53(2), 103-127, 2004.

[4] Briec W., Liang Q.B., *On some semilattice structures for production technologies*, European Journal of Operational Research, 215(3), 740-749, 2011.

[5] Briec W., Yeşilce İ., *Ky-Fan inequality, Nash equilibria in some idempotent and harmonic convex structure*, Journal of Mathematical Analysis and Applications, 508(1), 1-25, 2022.

[6] Gal S.G., Shape-Preserving Approximation by Real and Complex Polynomials, Birkhauser-Boston, 2008.

[7] Kemali S., Yeşilce İ., Adilov G., *B-convexity, B^{-1}-convexity and their comparison*, Numerical Functional Analysis and Optimization, 36(2), 133-146, 2015.

[8] Meleşteu A.D., *About the B-concavity of functions with many variables*, Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica, 31(1), 199-206, 2023.

[9] Stancu D.D., *Asupra unei generilazari a polinoamelor lui Bernstein*, Studia Universitatis Babes-Bolyai, 12(2), 31-45, 1969.

[10] Szász O., *Generalization of S. Bernstein's polynomials to the infinite interval*, Journal of Research of the National Bureau of Standards, 45(3), 239-245, 1950.

[11] Tunç T., Uzun M., *On the preservation of $\mathbb{B}$-convexity and $\mathbb{B}$-concavity of functions by Bernstein Operators*, Journal of Contemporary Applied Mathematics, 10(2), 77-83, 2020.

[12] Uzun M., Tunç T., *$\mathbb{B}$-convexity and $\mathbb{B}$-concavity preserving property of two-dimensional Bernstein operators*, 7th Ifs and Contemporary Mathematics Conference, Türkiye, 1-7, 25-29 May 2021.

[13] Yeşilce İ., Adilov G., *Hermite-Hadamard inequalities for $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex functions*, International Journal of Nonlinear Analysis and Applications, 8(2), 225-233, 2017.