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A Case Study of the Relationship between Meaning and Formalism

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Abstract
The purpose of this study was to explore the sources of mathematical ideas in terms of the relationships between meaning and formalism and their role in the transition between elementary mathematics and advanced mathematics. The two participants were high school mathematics teachers, who vary in their levels of experience. Two forms of data were collected to obtain more in-depth data about the transformations within among mathematical ideas: a questionnaire including 14 open-ended mathematical tasks and semi-structured interviews. Results indicated that individuals had different ways in constructing mathematical ideas and that their mathematical ideas were derived from the transition between meaning and formalism.

Keywords: relations, meaning, formalism, advanced mathematics

1. Introduction

The purpose of this study was to explore the sources of mathematical ideas in terms of the relationships between meaning and formalism and their role in the transition between elementary and advanced mathematics concepts and procedures. In elementary mathematics individuals make sense of the objects by describing whereas in advanced mathematics individuals make sense of them by defining. In this accordance, sense-making in mathematics is the emergence for building mathematical ideas. It is the basis for stepping into creative mathematics and linking between elementary mathematics and advanced mathematics. The nature of mathematics is not just about formal definitions, symbols, theorems or proofs but it is also concerned about individuals' ideas that are grounded in their daily life experiences. Indeed, Nunéz (2000) indicated that in order to understand mathematical meaning both sociocultural and cognitive mechanisms should be studied. Overwhelmingly, we understand mathematics only with the regularity in the manipulations of symbols but do not think about the ideas or the meanings. Several research studies were conducted confirming that fact within the development of mathematical ideas of a focus group of students (Davis & Maher, 1990, 1997; Maher, 2002; Maher & Martino, 1996; Maher & Speiser, 1997) and the generation of meaning (Dörfler, 2000). In line with that Sierpinska (1994) mentioned the close relationship between mathematical meaning and understanding. Vygotsky (1978, 1986) examined how social forms of meaning influence individual cognition. Accordingly, meaning construction was refered to as internalization understood both as mastery of cultural tools and as appropriation (Wertsch, 1998).

The only research study was conducted by Dubinsky (2000), focus of which was the relations between meaning and formalism in mathematics and the transition between elementary and advanced mathematical concepts. He explored these two themes in 'convergence of a sequence' by arguing that formalism can be used to construct meaning. In the light of Dubinsky (2000), the present study intends to explore how an in-service mathematics teacher and a pre-service mathematics teacher connect meaning and formalism in their problem solutions of relations, equivalence relations, and functions through individual task-based interviews. Secondary aim of this study was to highlight mathematics educators' perceptions about constructing mathematical ideas and whether they are aware of the transition between meaning and formalism through individual semi-structured interviews. The research questions were:

1. Does meaning drive formalism or formalism drive meaning in abstract algebra concepts?
2. How does the transition between meaning and formalism occur in constructing mathematical ideas?
3. To what extent do individuals use the mechanisms that are the sources of mathematical ideas?
4. What mathematical ideas do individuals generate and how do they represent these ideas?

By putting forth the movements between meaning and formalism, this study would highlight whether individuals operate informally, whether they use mathematical definitions formally in terms of their making sense of mathematical knowledge, and explorations in their ways of representing abstract algebra concepts.

1.1. Meaning and Formalism

Dubinsky (2000) stated that as a result of thinking ideas arise in the mind of human beings in terms of cognitive development. And this forces an individual to make further explorations and transform the relevant knowledge to a new dimension through making connections with the prior knowledge. This progress takes place between various sources including meaning and formalism. Meaning includes (1) *The physical world*; to which we have access through our five senses. Our everyday functioning structure the meaning of mathematics along with our reasoning skills. (2) *Familiar experiences;* to which we relate in some way through memory and make sense out of a situation, (3) *Connections;* by which an individual give meaning to a new concept in terms of relating it to the other concepts. This aspect of meaning refers to mathematics' hierarchy. (4) *Calculations;* are concerned with the computations which are the core of mathematics that give meaning to the concepts. (5) *Mental images*; are the parts of a person's thinking. In order to build a mental image individuals manipulate their imagery and reconstruct their mathematical ideas. Starting from physical manipulations and symbol manipulations; the ground for new concepts are prepared by mental images and calculations in which formalism is rooted in meaning.

Formalism is the set of symbols that represent mathematical objects and operations which are the description of mathematics. Objects define the characters and words referring to number, sets, functions, booleans etc. and operations define both the standardized and structural (logical) operations. Formalism can be considered to be statistic since symbols are motionless on the place where they are presented on. Dynamism of meaning is embodied by formalism. Both terms are important individually and in combination with each other. And individuals should link among symbolic, visual, and verbal representations which needs the active construction of knowledge that will help them in broader contexts and in their future experiences at an advanced level of mathematics. Meaning as well as formalism involves subtle processes that provide to translate imagery, or definitions. Thus, these two notions involve the meaningful manipulation of both concepts and symbols.

2. Method

The focus of my research has been on the processes by which mathematics teachers create mathematical meaning in particular relation tasks rather than on the products of their teaching/learning relations. Therefore, the majority of my research has been conducted within the tradition of qualitative research in which an interpretive stance guides the analyses (Creswell, 1998).

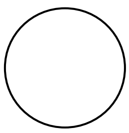
2.1. Sample

The two participants (one female and one male) were two high school mathematics teachers. The female teacher had five years' experience. She taught in a public high school. As a person, she was very interested in mathematics and wanted to spend time to improve mathematics education at her school. The male teacher, on the other hand, had three years' experience. He also taught in a public high school. As aperson, he was very interested in instructional technology and wanted to spend time to use media Both participants were graduated from Department of Mathematics in a public university. In order to become a mathematics teacher, they attended to a non-thesis master program in the Department of Secondary Science and Mathematics Education at a public university. The pseudonyms were used as Elif and Barış. Both teachers participated voluntarily.

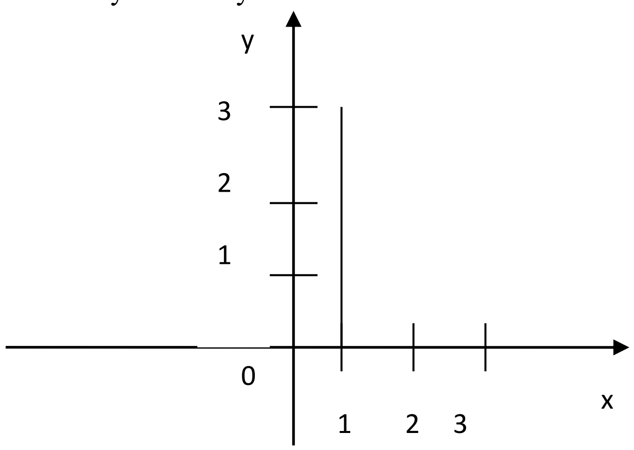
2.2. Instrument

Two forms of data were collected for triangulation. The questionnaire including 14 open-ended items was used to measure knowledge about relations. The teachers were asked additional questions about their conceptions about mathematics as a subject. Results are used in order to describe more in-depth, the valuable elements in the meaning and formalism for the two teachers with varying levels of experience. The items included in the questionnaire and the questions used in the semi-structured interviews were presented in the following sections, respectively.

2.2.1. The Questionnaire Items

1. What does 'number' mean to you?
2. What does 'natural numbers' mean to you?
3. 

   What does the figure given above mean to you?
4. What does (*a*,*b*) mean to you?
5. In which other mathematics subjects is the notion of (*a*,*b*) used?
6. The graph below illustrates a relation. How would you represent this relation symbolically?



7. What does ‘relation’ mean to you?
8. What does ‘equivalence relation’ mean to you?
9. How would you represent the axioms of equivalence relations visually? (graph, table etc.)
10. Let $X=\{1, 2, 3\}$ and the relation # be defined where 1#2, 2#1, 1#1, 2#2, but no other relations satisfied. Is this an equivalence relation?
11. Draw the graph of R= $\{(x, y)| x + y > 3\}$ which is defined on real numbers?
12. How would define the function concept moving on from the relation concept? (You can use any representation such as symbol, graph, table, etc. in your expression.).
13. Let R, be a relation defined on A=$\{1, 2, 3, 4\}$. Which of the equivalence relation axioms does R=$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (3,1), (3,3), (4,1)\}$satisfied?
14.

speed (*m/s*)	time (*s*)
16	4
32	8
64	12

Given above the table that presents a relation rule:

a) Please express this rule.
b) Does the rule that you formed express an equivalence relation? If so, why?

2.2.2. Semi-structured Interview Questions

1. What does mathematics mean to you?
2. What comprises mathematics?
3. How do individuals represent mathematics?
4. Can mathematics be considered without symbols?
5. Which one is difficult for you? Definitions or symbols?
6. How do you progress in your solution procedures? By definitions or symbols?
7. How often do you use concept definitions while problem solving?
8. What constructs mathematical meaning?
9. Do you face difficulties while you are making real life connections in mathematics?
10. In the times of Pythagoras and Euclid problems and problem solutions were represented only by words. What do you think about its advantages and disadvantages?
11. How could you express mathematical concepts?
12. What do you observe in students’ use of mental representations? How often do they use graphs or definitions?
13. According to the new curriculum definitions are removed from mathematics textbooks. What do you think about that?
14. What is the relationship between elementary mathematics and advanced mathematics in terms of using mental representations?
15. Is mathematics considered about making calculations?
16. What more is mathematics concerned about?
17. What do you think about mathematics’ relationship with other sciences?
18. We often use simple – informal – mathematics such as counting, measurement and estimation in our daily life and we learn some topics by relating them to their practices which we call ‘experiences’. Can we learn everything in mathematics by using such experiences?
19. Are there real life connections in advanced mathematics?
20. What would you suggest to understand mathematics better? Are materials a necessity?

2.3. Data Analysis

The data were analyzed in several ways. Firstly, the teachers individually solved 14 items and then, their solutions were discussed using think-aloud procedures. I video-recorded and then transcribed these task-based interviews. Henceforth, the data were composed of the written work of the participants and the transcribed videotape recordings. Transcriptions of the data involved the interactions of the individuals with the sources of mathematical ideas while they are doing mathematics.

The timing of the interviews and the items were contingent on the participants' responses. All the interviews were conducted individually with each participant at their own houses in order to make them feel confident in a familiar environment. Subsequently, semi-structured interviews were video-recorded and then transcribed. The initial purposes for these interviews were to inform ongoing mathematical meaning making and understand how teachers made sense of mathematics in terms of the transition between meaning and formalism.

I watched the recordings several times in order to get familiar to the participants and leave nothing missed behind. Along with Maher (2002), I tried to determine the significant leaps in the participants' cognitive processes including both the recordings and their written work. Thus, I related the critical events to the research items in a two folded lens. Following the purpose of distinguishing the sequences in the participants' flow of ideas I determined the transition between meaning and formalism in terms of the subtitles related to these two notions. Verbatim transcribed videotaped interviews along with the problem solutions were interpreted within meaning, formalism, the transition between them, and the mechanisms that serve to the development mathematical ideas. Coding the data occurred the subtitles composing meaning and formalism along with the mechanisms used to construct both meaning and formal mathematical thought (see Figure 1).

MEANING	FORMALISM

• The physical world
• Familiar experiences
• Connections
• Calculations
• Sense making
• Mental images
    ➢ Concept images
	• Notations, symbols
	• Definitions
	    ➢ Formal
	    ➢ Informal
	    ➢ Example
	    ➢ Picture
	• Embodiment-symbolism-formalism
	• Mental repraṡentations
	    ➢ Graphs
	    ➢ Tables
	    ➢ Diagrams

Figure 1. Subtitles that constitute meaning and formalism.

I worked to ensure the trustworthiness of my findings and interpretations in a number of ways: Three types of triangulation (Bogdan & Biklen, 2006) lend credibility to the present study: (1) multiple techniques (e.g., interviews, written responses), (2) multiple sources (e.g., variety of tasks, participants), and (3) multiple investigators (e.g., another mathematics education researcher analyzed the written responses and transcriptions).

3. Findings

3.1. Task-Based Interviews

With respect to Tall's (2005) considerations covering embodiment, symbolism, and formalism as thought processes both Elif and Barış embodied the visual representations for the axioms of equivalence relations through symbolism and formalism in their responses to Item 9: How would you represent the axioms of equivalence relations visually?. Especially the formal definitions of the axioms were taken as starting points for enactive drawing of the graphs. Both Elif and Barış stated that what gives meaning to the visual representations is the definitions.

<58> E: Surely…When it is said to be reflexive, it has a definition. Actually, you start up from that.[…].[graphs] relate between definitions and symbols. Actually, the relation has a one-to-one correspondence, [graphs are a] different way of representation.

<57> I: What gives meaning to the reflexive, symmetric, transitive properties are these symbols [shows the definitions Barış gave above the graphs].
<58> B: Hı hı…
<59> I: While you were drawing the graph you utilized from these?
<60> B: Actually, what I drew is an example. Just an example. We can not say 'Aaa…This is reflexivity.' [showing the graph]. Because reflexivity's meaning is here [shows the definitions he gave above]. When there is a thing [definition] like that we can say 'Haa…This is reflexivity.'
<61> I: So these [graphs] do not facilitate us to understand the reflexivity, symmetry, transitivity axioms alone?
<62> B: No. [shows the definitions]. These do. […]

Tall (2005) underlined that formal approach focuses less on embodiment and more on the logical structure. In the sense of Tall, Elif used embodiment in order to support her solution in Item 12: How would define the function concept moving on from the relation concept? (You can use any representation such as symbol, graph, table, etc. in your expression.). In her response, again the formal definition of function corresponds to the symbols of (x, y) and (x, z) and to the embodied visualization of venn diagrams. The embodied interpretation of this visualization led Elif to a new development in which she stated 'Every function is a relation, but every relation may not be a function.'. She also indicated that the simple visualizations do not make sense of what really lies beneath the definition.

<93> E: Yes…As a matter of fact this [venn diagram] does not fully express the definition. It is just something to support the definition…A little more visual…Something that facilitates to keep in mind…What I mean, x leads to $y\beta x$. Otherwise, only this [venn diagram] does not express the definition. I mean, through this [venn diagram]…It is not something for the statement that 'a component has only one image'. But this [shows the definition of function] is one to one.

In essence, both Elif and Barış use mental representations as a mechanism to support formalism. But there is a difference; Barış experienced a sense of conflict with the concept definition of function. Although his concept image was right he failed to write down the second axiom of the function that '$(x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$'.

<107> I: Each component in a given set has one image…Wouldn't it have to be $x_1 = x_2$?
<108> B: [thinks] If $y_1 = y_2$ then…Ohh yeah, right…I had given the one for one-to-one correspondence…If $y_1 \in y_2$ [means to say $y_1 = y_2$]…Anyway, the images are equal.
<109> I: For example, if you showed it by drawing a set on the paper?
<110> B: Yes…I had written the inverse. If $x_1 = x_2$ then $y_1 = y_2$ should have been written. […]. The statement I had written is the definition of one-to-one correspondance.

He confused it with the definition of one-to-one function. I tried to make him relate his result to a diagram in order to appraise his concept image but he immediately recognised the correct definition.

Chin (2002) investigated that situation of the transition between the concepts of ‘relation’ and ‘equivalence’ relation and he stated that the notion of relation is embodied very differently from the notion of equivalence relation. Similar to Chin, I determined that the participants of the study do not see an equivalence relation as a subset of AxB whereas a relation from A to B has a natural representation as a subset of AxB. Even though they were able to write the formal definition of a relation in Item 7: What does ‘relation’ concept mean to you?, they do not state that an equivalence relation on A is represented as a subset of AxA in their responses to Item 8: What does ‘equivalence relation’ concept mean to you?. Supporting Chin and Tall’s (2001, 2002) findings none of the participants respond to the notion of equivalence relation by using the general notion of relation as a set of ordered pairs in their definitions of equivalence relation on a set A as a subset of AxA. But they both stated the in Item 7 that a relation is a subset of the cartesian product of A and B. Furthermore they supported their formal definitions by informal explanations in which they used their own language. In contrast to Elif, Barış forgot to indicate that the sets A and B must be different from an empty set. He identified the sets ‘Let A and B be two sets…’ while Elif gave the concrete symbolic notation as ‘Let $A, B \neq \Phi$ …’

In contrast to Barış, Elif gave a more satisfactory definition of equivalence relation in Item 8 and she supported it by symbolization of the axioms that displayed a complete formalism. Barış’s written response does not give a complete satisfaction to his embodiment of the definition of equivalence relation. His explicit knowledge of the equivalence relations was expressed implicitly as a formal definition. But according to the task-based interview I found out that he might have given a symbolic definition if the question was asked in a different way. He stated that shifting from meaning to formalism requires a transformation of a new language.

However they were both successful in showing the visual representations of the equivalence relation axioms explicitly in Item 9. Their responses to Item 9 showed that they did not have difficulties in visualizing the axioms of equivalence relations but they had some deficiencies in remembering a verbal definition in terms of a subset of a cartesian product. The responses to this item were consistent with Lakoff and Nunéz’s (2000) notion of embodied mathematics that gives a deeper sense of meaning. And their development of formal thinking was underpinned by the embodied concept image-definition-usage in the sense of Moore (1994).

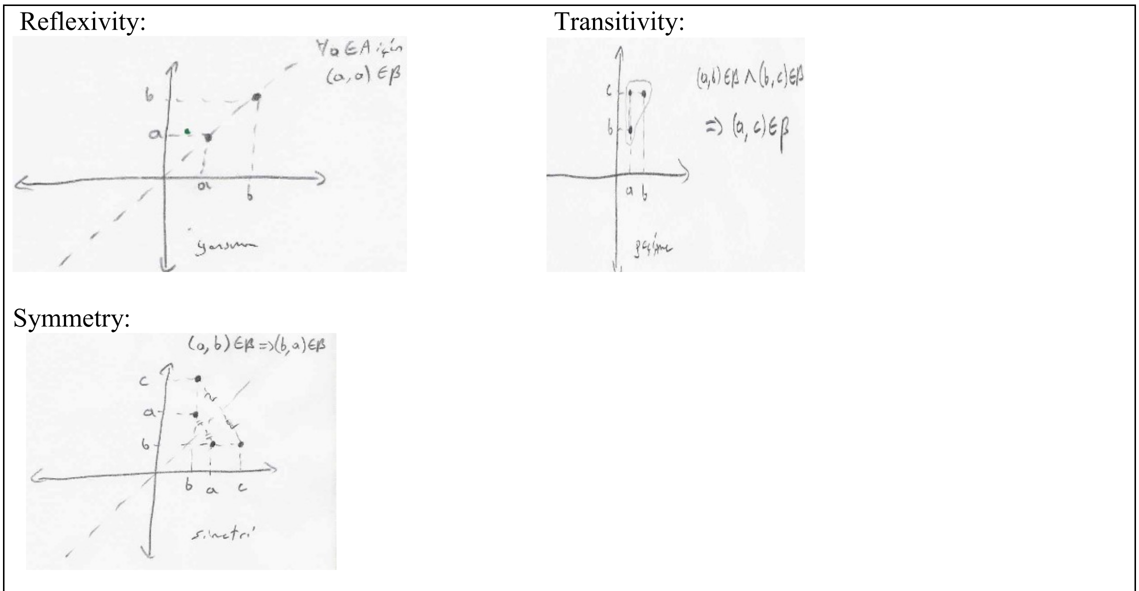
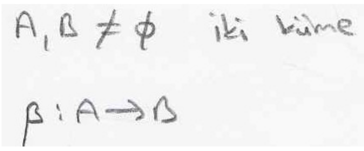


Figure 2. Barış’s response to Item 9.

Furthermore, it can be concluded that individuals embody the different kinds of relations such as functions and equivalence relations in different ways. Elif and Barış access the meaning of these concepts with similar notations deriving from the formalism that a function $f : A \to B$ lives in AxB while equivalence relations live in the set of A.



A, B ≠ ∅ two sets

Figure 3. Elif's notation in Item 12.

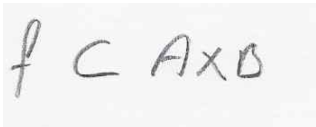


Figure 4. Barış's notation in Item 12.

Skemp (1977) indicated that what we actually progress in equivalence relations is the matching procedure by which the parent set is sorted to its subsets. Participants were asked to find which rules of equivalence relation is held on the given set in Item 13 and they responded it by following a matching procedure which can be considered as making formal classifications according to a particular rule. They matched the ordered pairs from the given set according to the equivalence relations' rules. Barış made sense of his matching procedure by giving the definition of equivalence relation in terms of analogical thinking. He assimilated the axioms by giving examples as if there are two people related to each other under a certain rule.
<131> B: Equivalence relation is a relation which shows that two components are equivalent. So initially, it has to hold the reflexivity. First he/she should be equivalent to himself/herself. Then it has to hold symmetry. If I am equivalent to someone than he/she has to be equivalent to me. Mutual…Everything is mutual…[laughs] It has to hold the transitivity too. A component is equivalent to another component. This other component is equivalent to another component. These three…There must be an equivalence between them. The first one must be equivalent to the last.

In contrast to Barış, Elif only showed why symmetry was held by writing down all the ordered pairs. She indicated that in real life symmetry is used in the same nature as the way it is in mathematics. Thus, grounding metaphors serve as a mechanism in facilitating the understanding of symmetry for Elif. Apart from that participants were aware of the fact that relations topic is strongly connected to daily life. They make sense of relations by grounding metaphors.
<44> E: When you attempt to make a list, first the name and then the surname. This is the rule. […] Relating… As its name implies... I mean, there will be a rule. To correspond them.
<46> B: $\leq, \geq$. We often use in our daily life. $3<5$ for example…
In this sense metaphors facilitate mathematical understanding of reflexivity, symmetry, and transitivity which need more algebraic subtleties.

Apart from the formal definitions of equivalence relations individuals have informal knowledge that allows some intuition to their meaning. Prior experiences of ordered pairs, sets and furthermore the subject of logic provide to give a formal basis for these concepts. In this sense they serve as concept images that facilitate the participants' problem solving. In other words it can be concluded that as a mechanism linking metaphors serve as a bridge between the concept images and the mathematical ideas. I think, meaning alone covers the deeper sense making in mathematics because of its underlying facts: connections, calculations, and mental representations. These facts nestle in the participants' mathematical explorations which was exactly determined within Item 14. Given a table participants were asked to find the relation rule. Barış embodied the table including the variables of speed and time by constructing ordered

pairs at the first step. According to the episode it is understood that he tried to find a pattern in order to extract meaning from a formal representation.
<114> B: I first looked at the table. I thought there will be ordered pairs like that [shows the entire row]. I tried to find a relationship between them but I couldn't. I first thought that it may be the multiple of 4 but it was not. So initially I wrote the ordered pairs.
On the other hand, Elif linked the ordered pairs to physics as a grounding metaphor $(x, y)=(t,V)$ but she indicated the ordered pairs at choice b). She preferred to express the relation rule directly (t = time and V = speed).

a) $$(x,y) = (\underset{\downarrow \atop t}{\text{zaman}}, \underset{\downarrow \atop V}{\text{Hız}}) = (t, V)$$
$$\beta = \{(x,y) \mid x = 4k,\ 1 \leq k \leq 3 \ \wedge \ y = 8.2^k,\ k \in \mathbb{Z}\}$$

Figure 5. Elif's representation of the relation between time and speed.

<119> E: […]. Directly speed-time. I represented them by V and t. […] And as the rule, I thought as an equation.
[…]
<125> I: What did you think while you were constructing that rule? Did you think about a definition or …?
<126> E: No, I thought about the relations between the numbers. I mean speed and time…These are just the names. In the end they [numbers] might have been representing something else. But in here they are representing them [speed and time]. But I thought about the numbers…About the relationships between them…

b) $$Z = \{4, 8, 12\} \qquad H = \{16, 32, 48\}$$
$$\beta = \{(4,16), (8,32), (12,48)\}$$

Figure 6. Elif's representation of the relation between time and speed with reference to numbers.

The common procedure that Elif and Barış applied was that of trying to find a pattern among the numbers given in the table. And it was interesting to find out that they achieved in a similar manner of mathematical thinking while many individuals have conflict in stating when there is no components that put up the relation rule; the rule is held. This was obtained from their statements about why the transitivity rule holds.
<132> E: It is transitive because… I mean in order to be transitive it has to hold if (x, y) and (y, z) are components of β then (x, z) has to be the component of β. But in here, since it [the relation] has no elements such as (x, y) and (y, z) I don't have to look for the other one. In the end I couldn't find something that put up the rule.
<121> B: There is (16,4) but there is no component that begins with 4. By the same way there is (32,8) but no component that starts with 8. There is (64,12) but no component that begins with 12. Since the necessity is not held, the rule [transivity] is held.

As Dubinsky (2000) mentioned individuals usually use meaning alone in their thinking. Similar to his statement I found out that the participants think in their mind but do not attempt to express them in the mathematical language. His statement appeared in Barış's response to Item 8. He directly gave the answer that 'the relation is said to be an equivalence relation that holds the rules of reflexivity, symmetry, and transitivity.' He did not use the more formal definition of equivalence relation which includes symbolic notations in each expression of the axioms rather he wrote down an informal/outline definition. In contrast, Elif's response revealed that her thinking process occurs in a harmony with the definition and symbols. And this thinking process according to mathematical meaning might be related to individuals' preferences in using mathematical language.

<54> B: In order to write it down symbolically we have to translate all the meanings to a [new] language. That needs a long work. Instead of that the terms are ready, we write them…On the other way, we have to transform them from Turkish that we speak to another language.

<52> E: […] It is more shorter. That saves from both the place and time. I mean, as a matter of fact, that [shows the symbols] expresses everything. It is harder for me to write…[…] Or to point [the notations of] 'for every' or 'some' is easier with the symbols for me. To tell what you mean…

That displays the fact that Barış has the meaning of the axioms in his mind but prefer to express them informally. As a matter of fact it can not be disagreed that if we were asked what an equivalence relation is majority of us would all give the rote response as Barış. I think this is because of the advanced manner of formalism. On the contrary Elif's representation refers to definition-based formal mathematics (Chin and Tall, 2000) in terms of which individuals encounter cognitive difficulties (Sierpinska, 1992). In line with that according to Chin and Tall's (2000) classification of the definitions I analyzed items 1, 2, 3, 4, 7, 8, and 12 to see if the participants gave operable definition or not. Actually, the aim was to explore how individuals make sense of basic mathematical concepts in terms of the nature of their mathematical ideas. On the other hand their transitions between meaning and formalism were investigated. Definitions were classified as:

Formal/detailed: giving an 'essentially correct' formal definition in full detail,
Informal/outline: either an informal verbal description, or 'reflexive, symmetric, transitive',
Example: giving a single specific or general example,
Picture: using visual imagery in drawing

With respect to that table the nature of mathematical ideas has individual differences. But both participants' meanings were constructed in more formal mathematics. The difference was that Elif supported her formal definitions along with examples and pictures in items 2, 3, and 12 while Barış supported his formal definition in Item 7 by the informal definition.

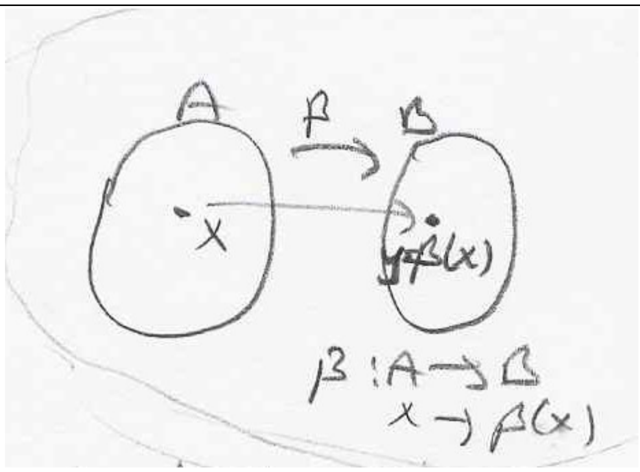


Figure 7. Elif's venn diagram in Item 12.

In contrast to Chin and Tall's (2000) study I found out that the participants used more formal/detailed definitions in which formalism drove meaning but I again link that difference to their experiences in mathematics. Because, individuals deduce properties from definitions by relating their mathematical ideas.
Their responses to the items asking what specific mathematical concepts mean to them shed light to understand the interactions within meaning and formalism. In the sense of Pinto and Tall (1999) they extracted meaning from the definitions and gave meaning to the concepts. That may be related to APOS theory approach of Dubinsky (2000) in which he underlined the relationship between the process and concept. In this sense Elif's response to Item 2 including the set of natural numbers and to Item 12 including the venn diagram which was given above can be considered as examples.

The process on the concepts enables individuals a deeper understanding about mathematical meanings deriving from formal definitions. In contrast to Moore's (1994) statement that concept image and definition are distinct entities I determined that individuals use their concept images and the definitions highly connected to each other. Basically, Elif and Barış stated that they had benefitted from their concept images most of which were constructed during their university instruction. For instance in Item 11 they drew the graph of the given relation according to the concept image of the notion of ‘ $\rangle$ ’.
<87> I: Why did you use a dashed line?
<88> B: There is no equality. It is given x plus y is greater than three. […] That is why it is dashed.
<69> I: From what did you benefit from while using this dashed line?
<70> E: From the sign of ‘greater’. I mean since there is no equality three is not included. If you pay attention the interiors are empty, I took them too.
<71> I: So the notion of ‘greater’ has two meanings?
[…]
<74> E: As a way of representation…As the exposition exists…Yes. One of it is the representation of the points. If they are not included they are empty. As a matter of fact this [shows the dashed line] is also composed of the points but it is different visually.
Thus the episodes given above state that the meaning of the notion of ‘ $\rangle$ ’ drove to formal representation as a graph.
In my opinion another dimension of the transition between meaning and formalism occurs in this situation. Some people use their knowledge of concepts in a more flexible and imaginative manner. Acccording to Chin and Tall (2000) even they do not identify the reason whether the rule holds or not, they may be able to directly link their thought processes to the properties. For instance in his response to Item 13 Barış, he directly stated that the symmetry holds without identifying the reason.

On the other hand, I found out that the notion of equivalence relations is easy to visualize and easy to remember as a verbal definition but may be difficult to represent by symbols according to the individual differences.
Tall (1995) characterized two distinctions within individuals' experiences with the objects with respect to their cognitive growth. He stated that the visual objects are our direct perceptions of the world outside or our personal constructions of what we see in the world outside. For instance, later in geometry an object such as ‘point’ takes on a more abstract meaning that it is not ‘a dot on the pencil mark’ but rather an abstract concept that ‘has position but no size’. In line with (Tall (1995), both Elif's and Barış's responses to items 1 and 2 confirmed that as the cognitive growth develops objects shift to an abstract level which brings out the use of conceptual knowledge along with relational understanding. In other words, the ‘number’ concept moves from a ‘count by finger’ process to a ‘number theory’ which needs abstractions

in the mind. Participants of this study gave nearly identical answers in which they identified the number concept as a component of a system.
<5> I: Didn't you think of showing as 3 or 5?
<6> B: They are concrete...[...] They don't make sense of what the number concept is. They are examples of a number. I mean 3, 5...

In addition to Barış, Elif illustrated the set of natural numbers and furthermore she defined what the components of this set mean as ' 0 refers to the empty set, 1 refers to the set with one component and so on'. In this sense, Elif's shift in a basic mathematical concept along with the individul's cognitive growth fits Tall's (1995) considerations mentioned above.
And Elif also stated that she has lost her elementary mathematical thinking. In Item 3 given a circle figure participants were asked 'What does the figure given above mean to you?'. In the discussion session of this item Elif indicated clearly the shift in her thinking style where she had defined a circle as 'the set of all points at a fixed distance':
<24> E: I mean...I think I have lost the way of ordinary thinking...When it is said to be a circle I have begun to see its center and the radius...But I also indicated that it is a circle. [laughs]

When Skemp's (1976) theory of instrumental and relational understanding is considered it seemed evident that relational understanding should be considered in terms of explicit knowledge. In other words implicit knowledge of concepts transforms to explicit knowledge as long as the relational understanding is achieved. Because explicit knowledge of mathematics covers more than the formulas rather it includes the mathematical language that goes beyond the representation of the statements. Thus it is composed of the relations and reasons that is the ability to explain why while relating the particular ideas and procedures within mathematics. In this sense Barış's response to the same item revealed that individuals may not represent their explicit knowledge of a concept, even though they have the relational understanding of it. Instead they give superficial answers.
<16> I: The set of all points at a fixed distance…[As soon as I began to identify this formal definition, he immediately completed my speech]
<17> B: The set of all points on a plane at a fixed distance…
<18> I: Didn't it cross your mind?
<19> B: No.
<20> I: When you see a figure as a visualization, although you don't give the formal definition what gives the circle concept its meaning is that figure, to you?
<21> B: To the circle concept?...No. because this circle is a figure. We can not write a definition just by looking at it. That's just an example. […]
Consistent with Dubinsky (1992) study in his written response, at first Barış did not transfer his understanding of circle in natural language to a mathematical context but later on, it was understood from the discussion session that despite his superficial written response he had the explicit knowledge of the circle concept but he represented it implicitly. He wasn't able to think about anything other than his remembered experience with the circle concept but in contrast he was able to address the point when I asked him about it. He even more added the term 'plane' to my circle definition. In this sense he used mathematical formalism as a language to express his understanding.

In line with Vygotsky (1978) and Skemp's suggestions for connections, I determined that both participants gave meaning to concepts by relating them to other concepts. For instance the notion of ordered pairs even in its simplest manner served them in their problem solutions and was connected to relations, equivalence relations, and functions. Also they had the ability to make connections within these concepts. Apart from that I investigated how they make use of the ordered pairs within mathematics in Item 5: In which other mathematics subjects, are ordered pairs used?. They gave the similar responses but Elif's exemplifications were comprised of wider subjects beginning from the cartesian product ending with vectors. Both Elif and Barış extended their responses by giving examples.

<31> E: In terms of complex numbers while representing them as $a+ib$, that is (a, b) or in calculus while representing it trigonometrically… For instance the radius and the angle θ…I mean we always use them [ordered pairs].

<27> B: [in complex numbers] The real part, the imaginary part. The first component [of the complex number] is the real part and the second component is the imaginary part. [in functions] The first one is the domain and the second is range. [in relations] the first one is the component of the first set and the second one is the component of the second set.

The core of mathematical progress is the calculations in which individuals operate with symbols and notations. Hence, in order to make sense in mathematics the one needs both calculations and connections. In this sense, participants used their visualization skills and verbal skills in their problem-solving performances in terms of calculations and connections. In line with that sense making was identified to be an essential necessity in each step of the problem solving process. The transition between meaning and formalism can not be considered without the ability of sense making.

Viewed together, results showed that developing mathematical ideas on abstract algebra needs creative thinking processes. In this sense individuals have different ways in constructing mathematical ideas and representing their mathematical knowledge in terms of their cognitive growth in mathematical thinking.

3.2. Semi-structured Interviews

The primary source of the interviews was to understand the participants' perceptions about mathematics and the sources mathematical ideas. I was also interested in their use of meaning and formalism within their mathematical experiences.

Coding the data occured the similar subtitles that constitute meaning and formalism which underlined the entity of a universal mathematical language in the sense of which the mathematical ideas are built on. Their perceptions are consistent with Dubinsky's (2000) statements.

The results showed that mathematics mean the source of the understanding of the external world which is built on rules and symbols.

<4> E: Set of rules …Universal set of rules.

[…]

<7> I: With what are these rules are represented?

<8> E: Represented by symbols. [laughs] Mathematics is a universal language which we represent by symbols.

<2> B: […] The science that formulize the functionality of the universe.

<3> I: What mathematics is composed of?

<4> B: […] Primarily the axioms.[…] Axioms are the things that we assume with respect to the basis of the universe. The points that do not conflict with this universe. Then it [mathematics] goes on through this universe.

<5> I: With what are these axioms and theorems are represented?

<6> B: We represent them by symbols. I mean mathematics has a universal language. Within this language we express it.

<7> I: Can mathematics be considered without symbols?

<8> B: Can be because the place that Mathematics is inspired of is the universe itself, the nature. The nature models [the mathematics]. And there aren't any symbols there.

In contrast to Barış, Elif stated that without symbols mathematics would be too difficult to represent. And she underlined that symbols and the language construct the mathematical culture. Along with the symbols their responses reveal that the use of definitions changes according to the problem context but in common they indicated that symbols are used to represent the meanings. But in order to understand what is needed in the solution requires the knowledge of a definition. In general, they agreed that mathematic's meaning lies beneath the symbols and notations which needs advanced mathematical thinking.
Barış's response was interesting in the sense of his consideration of mathematical meaning is that, mathematics' meaning is concerned about its being not contradictory and he links his statement to the mathematical hierarchy.

<21> B: It may not be concrete. It seems to me too mathematical that a system built on the prior statements with no conflicts within eachother.[…]
According to their observations they stated that the students are lack of formal thinking ability including visualization skills.
<26> E: […] The one who uses the graphs has to understand the subject very well. I mean, students who are above a certain level can deal with the graphs but of course the deficiencies of our education system must be taken into consideration. Since it is based on memorization children feel restricted to memorize the definitions. […] What they are used to is memorizing; graphs are difficult for them because without understanding they cannot progress on graphs.

On the contrary, Barış believed that students are not aware of the effective use of mathematical language in which formal definitions are given with symbols.
In terms of meaning and formalism, they put forward the transition between elementary mathematics and advanced mathematics with respect to the transition between definitions and symbols. They stated that mathematical meaning derives from concept images at elementary level and leads to symbols at advanced level. Hence, it is important to construct appropriate meanings in elementary mathematics because further mathematics is built on the concept images they construct at that level. In this sense it can be concluded that formalism serves more to advanced mathematics. Thus, another aspect of mathematics arises that in some cases individuals cannot learn by experiencing the external world. Participants of this study, agreed in this sense that formalism requires more than routine procedures to make sense in mathematics.

<50> I: We often use simple mathematics such as counting, measurement, and estimation in our daily life and we learn some subjects by relating them to their practices which we call 'experiences'. To you, can we learn everything in mathematics by these experiences?
<51> E: No. It may be problematic to learn advanced mathematics by our experiences. I mean, counting… In some problem solutions may be…It must be seen by eyes in order to be learnt by doing. As a matter of fact, derivative has an important place in our lives…The limit concept at the same time…But there is no situation that we can learn by doing. […] Hence, it is difficult in some cases.

<58> B: Hımm… We cannot. Because as I said before from a moment on in order to experience something we must be doing something upper than that. For example; in order to experience logarithm or derivative or integral we need something more advanced.

However they do not deny the importance of real life connections in constructing advanced mathematical ideas. In this sense, we cannot progress on formal mathematics by just experiencing it but we use our familiar experiences with the physical world that underline meaning to understand formal mathematics.
Elif, suggested constructivist approaches to develop mathematical thinking.

<57> I: What would you suggest to understand mathematics better?
<58> E: I'd say learn by living, I mean the student himself/herself should progress on it [mathematics]. The rules or whatever…In order to internalize a particular subject he/she should struggle himself/herself. The teachers may be guiders. Actually the one learns by himself/herself. The people and settings that facilitate this [learning] can be provided. But everybody internalize differently. […]
<59>: I: Material?
<60>: E: Practice-oriented…Of course materials…I mean, the ones that attract their attention and provide to keep them [learnings] in mind. […] Real life connections, computers…If it could be integrated computers would be very useful. I mean, it facilitates so much things.

<62>: B: So as to be understandable…Make it [mathematics] as concrete as possible […].

The episodes revealed that mathematics' abstract nature can be embodied by using mechanisms such as mental representations, and technology. The interviews characterize once again, that mathematical ideas are derived from the transition between meaning and formalism. Hence, mathematics should be considered as a whole within the concepts and symbols and the awareness of the effective use of these notions which provide individuals to develop sophisticated mathematical ideas.

4. Discussion and Conclusion

Prior research studies were rich with insights about constructing mathematical meaning, visualization, symbolization, actions and processes on concepts and thought processes applied on formal mathematics. The methods employed in these studies, however, also had limitations. The studies were carried out in student-centered settings and with small sample sizes. But the analyses generated rich insights. A little attention was paid to the transition between meaning and formalism in the development of mathematical ideas. In this study, I built on prior research by including an in-service and a pre-service teacher. I designed the tasks based on equivalence relations along with basic mathematical concepts in order to determine the relationship between elementary mathematics and advanced mathematics. I had a focus on equivalence relations because it is known as one of the topics that individuals have problems in understanding. According to Halmos (1987) equivalence relations are one of the basic building blocks out of which all mathematical thought is constructed. In line with that, in a more psychological language Skemp (1977) defined an equivalence relation as one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics. Despite the fact that equivalence relations are one of the most fundamental ideas of mathematics, little attention was paid as a research subject. I think this subject is the first step in individual's understanding and usage of formal concepts in terms of both school mathematics and abstract algebra. Thus, understanding of of their difficulties in this topic sheds light into further difficulties in the understanding of formal mathematics.

It must be taken into account that elementary mathematics is producing individuals less ready to study mathematics at an advanced level. Thus they do meaningless manipulation of symbols because of their lack of the ability to give meaning to these symbols. The transition from elementary mathematics to advanced mathematics is a sequence rather than a jump hence on the way from meaning to formalism effective use of the mechanisms as the sources of mathematical ideas should be taken into consideration. Being sophisticated in mathematics requires the appropriate use of both mental and logical skills along with the reasoning ability. I think primarily we should seek for the ways to encourage individuals to think in a mathematical way that facilitates them to engage in formal mathematical content.

The analyses revealed the development of mathematical ideas of the individuals and the solution procedures they follow in terms of meaning and formalism. But the generalizability of the findings may be limited according to the small sample size. My intention in this study was to explore how individuals construct their mathematical ideas and make sense of mathematics. By viewing meaning and formalism mutually inform one another, I reveal that mathematics is strongly concerned about these notions. I have addressed new information and ideas under meaning and formalism. For instance; I put the concept image under meaning and definitions under formalism. I explored the cognitive growth which is driven by embodiment, symbolism, formalism framework in understanding the concept definitions by making sense in the concept images.

My findings showed that individuals use thought processes of embodiment, symbolism, and formalism in their visual representations. In line with that they use their embodiment skills in order to support their problem solutions in which they give both the formal and the informal definitions of a concept. New developments on definitions occur as they progress on embodiment and symbolism. In some cases individuals may not represent the concept definition in spite of having the right concept image. This reveals the conflict in sense making in the formal definition deriving from meaning. The point in here is the strong connection between the concept image and concept definition with respect to implicit and explicit knowledge.

Individuals embody the different kinds of relations such as functions and equivalence relations in different ways. Hence they use different notations and symbols. Since prior research has focused on students some contradictions occured according to the difference between the expert's and novice's mathematical thinking. I found out that experts are more flexible in giving meaning to a concept and extracting meaning from a concept with respect to their experiences with mathematics along with their cognitive growth in advanced mathematical thinking .
Analyses of their computations show that individuals use the mechanisms effectively, mostly the mental representations, visualizations, symbolizations and metaphors as the sources of mathematical ideas. However the deficiency of individuals still remains as their tendency to think in their mind and do not attempt to express their statements through a mathematical language. Thus, they use meaning alone in their thinking.

The classification of definitions highlights the transition between formalism and meaning along with the mechanisms mentioned above. The responses indicate that individuals have a wide viewpoint in terms of the concept definitions that they support their formal definitions by informal definitions. Thus, individuals deduce properties from definitions by relating their mathematical ideas and their processes on the concepts enable individuals a deeper understanding about mathematical meanings deriving from formal definitions. On the other hand the internalization of mathematical ideas depends on the cognitive growth in mathematical thinking. As the context moves to a more advanced level the progress on mathematics requires effective abstraction and reasoning skills. In this sense meaning and formalism go side-by-side in mathematics.

When considered from a lens of students understanding is the acquisition of reasoning skills in terms of using mathematical meanings by using the appropriate language. Mathematics requires the ownership of apparent logic but students' verbal or mental actions are generally lack of meaning which shows the low level of understanding. Critical mathematics education includes mathematical knowledge as a whole that is related to understanding, thinking and meaning of concepts. In my opinion students who are able to energize critical thinking, analytical thinking, reasoning skills and the ability to communicate both with the real life and other disciplines which are helpful in understanding what really underlines 'doing' mathematics. In this communication language is grounded in mathematical sense-making. Students' thinking becomes structured by social interaction and their developing ideas connect to mathematical language such as symbols and graphs. Thus, as long as teachers pay attention on mathematics' language that is rooted in meaning and formalism, the development of mathematical understanding of students will enhance. In order to understand the mathematical thinking involved in doing and learning mathematics the language of it should be underlined initially because the language of mathematics doesn't have the same fluency as the natural language. It is composed of numbers and symbols which are more abstract.

Thus, mathematical meaning along with the intuition and ideas is rooted in our both physical and cognitive experiences. What is learned should be explored in terms of how it is learned and why it is learned in this routine? In terms of this question further research may investigate the cognitive growth in mathematical thinking both within school mathematics and university by focusing on complex mathematical tasks using words, symbols, and diagrams. Furthermore, apart from the students researchers can focus on teacher education in order to gain insight about their mathematical and pedagogical understandings. In my opinion the awareness of the development of teachers' mathematical and pedagogical knowledge will provide bringing up students who use the language of mathematics effectively. Such studies can shed light to understand what goes on in an expert's mind different from a novice's. So that the transition between the elementary and advanced mathematics can be analyzed in detail.

This study suggests that teachers' awareness of the transition between meaning and formalism and the mechanisms that they are used within the construction of mathematical ideas, will provide effective teaching and learning environments both for teachers and students. Although this study illustrates only two persons' use of meaning and formalism in their thought processes, more research is needed with larger samples and upper level mathematics subjects. And researchers who are interested on elementary mathematics may imbed Bruner's knowledge representation steps as enactive-iconic-symbolic to the lens of meaning and formalism and explore the development of mathematical ideas in this connection.
I believe that the focus on mathematics educators and mathematicians will shed light to understand the transformation between elementary mathematics and advanced mathematics. Developing a cadre of a mathematics community including both educators and mathematicians who understand the complexity of constructing sophisticated mathematical ideas and how to integrate these ideas to problem solving processes will be sensitive to the needs of mathematics education.

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