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# Cobalancing Numbers: Another Way of Demonstrating Their Properties

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**Abstract**
In this study, previously obtained cobalancing numbers are considered from a different perspective, and the properties of the numbers are re-examined. The main purpose is to change the recurrence relation of cobalancing numbers and calculate some relations and properties in a more diverse and easier way. The reason that led us to this method is that the recurrence relation of cobalancing numbers has a second-order but non-homogeneous difference equations. Thus, it will be much easier to find the Binet formula, generating function, sum formulas, and many other relations with a sequence that is homogeneous and has a third-degree recurrence relation. Also some identities that have not been found before in the sequence are also included in this study.

**Keywords:** Binet formulas, Cobalancing numbers, Third-order recurrence relations
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## 1. Introduction

Number sequences have been studied and researched by hundreds of mathematicians for many years. While the authors in [1] work with a special equation, Diophantine equation,

$$1+2+3+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

on triangular numbers, obtained an interesting relation of the numbers $n$ in the solutions $(n,r)$, which they call balancing numbers, with square triangular numbers. The number $r$ in $(n,r)$ is called the balancer corresponding to $n$. In the following years, Behera and Panda continued their work rapidly and continued to find interesting features related to this new number sequence. Later on, the first article that bridges the gap between Fibonacci numbers and balancing numbers was made by Panda [2].

Behera and Panda [1] proved that the square of any balancing number is a triangular number. Subramaniam [3] is another mathematician who established a relationship between balancing numbers and triangular numbers. Panda and Ray [4], studied another Diophantine equation

$$1+2+3+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.2}$$

on triangular numbers and call $n$ a cobalancing number and $r$ the cobalancer corresponding to $n$. Cobalancing numbers relate to different triangular numbers that can be given as the product of two consecutive natural numbers or as the arithmetic mean of the squares of two approximately consecutive natural numbers [5]. Also, Liptai [5] mentioned his name in a study that produced important results regarding balancing numbers.

Then, another article that aroused interest in other new topics by giving the literature the relationship between balancing and cobalancing numbers was written by Panda [6].

There are many studies on the Fibonacci sequence, which is related to the golden ratio, and the Pell sequence, which is related to the silver ratio, and these articles contain a lot of information that paves the way for integer sequences. Behera and Panda [1], introduced the Diophantine equation (1.1), then they obtained the sequence $\{B_n\}_{n\in\mathbb{N}}$ of balancing numbers and give some interesting properties of this sequence. $\{B_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation of the second order, given by

$$B_{n+1} = 6B_n - B_{n-1} \quad , n \geq 1 \tag{1.3}$$

with initial terms $B_0 = 0$ and $B_1 = 1$, where $B_n$ denotes the $n-$th balancing number. Taking $a_1 = 1+\sqrt{2}$ and $a_2 = 1-\sqrt{2}$, the Binet formula for $B_n$ can be written as,

$$B_n = \frac{a_1^{2n} - a_2^{2n}}{4\sqrt{2}}.$$

The (ordinary) generating function of a sequence $\{x_n\}_{n\in\mathbb{N}^+}$ of real or complex numbers is given by

$$f(s) = \sum_{n=1}^{\infty} x_n s^n = x_1 s^1 + x_2 s^2 + x_3 s^3 + \cdots \tag{1.4}$$

[7]. The generating function for the sequence of balancing numbers $\{B_n\}_{n\in\mathbb{N}}$ ,is

$$g(s) = \frac{s}{1-6s+s^2}.$$

On the other hand, following Panda and Ray [4] a positive integer $n$ is a cobalancing number with cobalancer $r$, if it is the solution of the Diophantine equation (1.2). The sequence $\{b_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation of second-order given by

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad , n \geq 2 \tag{1.5}$$

with initial terms $b_1 = 0$ and $b_2 = 2$, where $b_n$ denotes the $n-$th cobalancing number. We will denote cobalancing numbers with $\left\{b_{(2),n}\right\}_{n\in\mathbb{N}}$ instead of $\{b_n\}_{n\in\mathbb{N}}$ to avoid confusion throughout the article. Because throughout the article, a sequence that has a third-order recurrence relation with the new sequence that will be found shortly will be discussed. Since a cobalancing number with a second-order recurrence relation is expressed here, this notation will be used. The Binet formula for cobalancing numbers is given

$$b_{(2),n} = \frac{a_1^{2n-1} - a_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}. \tag{1.6}$$

The generating function for the sequence of cobalancing numbers $\left\{b_{(2),n}\right\}_{n\in\mathbb{N}}$ ,is

$$g(s) = \frac{2s^2}{(1-s)(1-6s+s^2)}. \tag{1.7}$$

As can be seen, information was given about the sequences with quadratic recurrence relations. Note that the recurrence relation we wrote with (1.5) has a non-homogeneous structure. Naturally, it is quite difficult to work with this recurrence, and the results will be obtained more easily if it is transformed into a sequences with another third-order recurrence relation, which makes it easier to use without changing the sequence. So, let's get some ideas about sequence with third-order recurrence relations [8]. The best known of these property is the tribonacci number sequence. The tribonacci sequence is defined by for

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, n \geq 2$$

with initial conditions

$$T_0 = 0, T_1 = 1, T_2 = 1.$$

Tribonacci sequence is a well known generalization of the Fibonacci sequence. The roots of characteristic equation of Tribonacci numbers are $\alpha_1, \beta_1, \gamma_1$ for the $x^3 - x^2 - x - 1 = 0$. The Binet formula of Tribonacci sequence is given by

$$T_n = \frac{\alpha_1^{n+1}}{(\alpha_1 - \beta_1)(\alpha_1 - \gamma_1)} + \frac{\beta_1^{n+1}}{(\beta_1 - \alpha_1)(\beta_1 - \gamma_1)} + \frac{\alpha_1 \gamma_1^{n+1}}{(\gamma_1 - \alpha_1)(\gamma_1 - \beta_1)}$$

[9]. The tribonacci numbers can also be computed using the generating function

$$g(z) = \frac{z}{1 - z - z^2 - z^3}.$$

Now let's talk about a method used for the recurrence change we mentioned above. In the study [10], as a result of the process performed for the Leonardo sequence with a non-homogeneous second-order recurrence relation, a new sequence with a third-order recurrence relation is obtained. Let's obtain the third-order recurrence relation we target with a similar method.

Let's write $n = n - 1$ and $n = n$ in the equation (1.5)

$$\begin{aligned} b_{(2),n} &= 6b_{(2),n-1} - b_{(2),n-2} + 2 \\ b_{(2),n+1} &= 6b_{(2),n} - b_{(2),n-1} + 2. \end{aligned}$$

Now we wrote last above and subtract it side by side then the third-order recurrence relation to be obtained is as follows

$$b_{(3),n+1} = 7b_{(3),n} - 7b_{(3),n-1} + b_{(3),n-2} \tag{1.8}$$

for the initial conditions $b_{(3),0} = 0, b_{(3),1} = 0$ and $b_{(3),2} = 2$. The roots of characteristic equation $x^3 - 7x^2 + 7x - 1 = 0$ of cobalancing numbers are

$$\alpha = 3 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2} \text{ and } \gamma = 1. \tag{1.9}$$

In mathematics, a recurrence relation is an equation that defines a sequence recursively; each term of the sequence is defined as a function of the preceding terms [11].

## 2. Cobalancing Numbers and Some Properties

In this section, it is aimed to find identities regarding cobalancing numbers. The first of these is to obtain the generating function that was previously found for the sequence that has a second-order recurrence relation. The usual generating function $g(x)$ for the sequence (1.8) of real numbers is defined as:

$$g(x) = b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots.$$

Now let's write the $g(x)$ generating function in a different way from (1.7).

**Theorem 2.1.** *Let $b_{(3),n}$ can be the cobalancing number. The generating function $g(x)$, can be written as follows:*

$$g(x) = \frac{2x^2}{1 - 7x + 7x^2 - x^3}.$$

*Proof.* The generating function of the sequence (1.8) is

$$g(x) = b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots.$$

Let's multiply the function $g(x)$ by $7x$, $7x^2$ and $x^3$.

$$\begin{aligned} g(x) &= b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots \\ 7x.g(x) &= 7b_{(3),0}x + 7b_{(3),1}x^2 + 7b_{(3),2}x^3 + 7b_{(3),3}x^4 + \cdots + 7b_{(3),n}x^{n+1} + \cdots \\ 7x^2.g(x) &= 7b_{(3),0}x^2 + 7b_{(3),1}x^3 + 7b_{(3),2}x^4 + \cdots + 7b_{(3),n-1}x^{n+1} + \cdots \\ x^3.g(x) &= b_{(3),0}x^3 + b_{(3),1}x^4 + b_{(3),2}x^5 + b_{(3),3}x^3 + \cdots + b_{(3),n-2}x^{n+1} + \cdots. \end{aligned}$$

Let's take the necessary actions in the equations we have obtained and let's get the (2.1) equality.

$$\begin{aligned} & g(x)-7xg(x)+7x^2g(x)-x^3g(x) & (2.1)\\ = \; & b_{(3),0}+b_{(3),1}x+b_{(3),2}x^2-7b_{(3),0}x-7b_{(3),1}x^2+(b_{(3),3}-7b_{(3),2}+7b_{(3),1}-b_{(3),0})x^2+\cdots & (2.2)\\ & +(b_{(3),n+1}-7b_{(3),n}+7b_{(3),n-1}-b_{(3),n-2})x^{n+1}. \end{aligned}$$

Let's take the left side of the equation into the $g(x)$ bracket and write $b_{(3),0}=0, b_{(3),1}=0$ and $b_{(3),2}=2$ in the equality. In the last case, we find the $g(x)$ function written below.

$$g(x)(1-7x+7x^2-x^3)=b_{(3),0}+b_{(3),1}x+b_{(3),2}x^2-7b_{(3),0}x-7b_{(3),1}x^2+7b_{(3),0}x^2$$

then

$$g(x) \;=\; \frac{2x^2}{1-7x+7x^2-x^3}.$$

□

If the generating function is expressed in terms of the roots of the characteristic equation in (1.9), the following result is obtained.

**Corollary 2.2.** *Let $b_{(3),n}$ can be the cobalancing number. The generating function $g(x)$, can be written as follows:*

$$g(x)=\frac{2x^2}{1-7x+7x^2-x^3}=\frac{-\frac{1}{2}}{1-x}+\frac{\frac{5-\alpha}{2(\beta-\alpha)}}{1-\alpha x}+\frac{\frac{5-\beta}{2(\alpha-\beta)}}{1-\beta x}.$$

*Proof.* We can write the $g(x)$ function as the sum of rational numbers as follows,

$$\frac{2x^2}{(1-x)(1-6x+x^2)}=\frac{P}{1-x}+\frac{Q}{1-\alpha x}+\frac{R}{1-\beta x}.$$

Let's equalize the denominators and find the values $P$, $Q$ and $R$ using polynomial equality,

$$2x^2=P(1-\alpha x)(1-\beta x)+Q(1-x)(1-\beta x)+R(1-x)(1-\alpha x)$$

Let's write $x=1$ in the equation we found. $P=-\frac{1}{2}$ is found. We can write the equality we have found in the following way:

$$2x^2=(P+Q\beta+R\alpha)x^2+(-6P-Q(\beta+1)-R(\alpha+1))x+P+Q+R.$$

If we use the equality we have ended up with, we reach the following equations.

$$\begin{aligned} P+Q\beta+R\alpha &= 2,\\ -6P-Q(\beta+1)-R(\alpha+1) &= 0,\\ P+Q+R &= 0. \end{aligned}$$

Let's write $P=-\frac{1}{2}$ and let's solve the following system of equations

$$\begin{aligned} Q\beta+R\alpha &= 2+\frac{1}{2},\\ Q+R &= \frac{1}{2}. \end{aligned}$$

Finally, $Q=\frac{5-\alpha}{2(\beta-\alpha)}$ and $R=\frac{5-\beta}{2(\alpha-\beta)}$ is found, and the function $g(x)$ can be written as follows,

$$g(x)=\frac{-\frac{1}{2}}{1-x}+\frac{\frac{5-\alpha}{2(\beta-\alpha)}}{1-\alpha x}+\frac{\frac{5-\beta}{2(\alpha-\beta)}}{1-\beta x}.$$

□

The result given below for [8] was found in another way. We will find the Binet formula for the sequence $\left\{b_{(3),n}\right\}_{n\in\mathbb{N}}$.

**Theorem 2.3.** *Let $b_{(3),n}$ can be the cobalancing number. The Binet formula for the cobalancing number is as follows:*

$$b_{(3),n} = \frac{\alpha^{n-1}(\alpha^2-2\alpha+5)+\beta^{n-1}(\beta^2-2\beta+5)}{64} - \frac{1}{2}.$$

*Proof.* Let's write the following equation for the cobalancing sequence,

$$\begin{aligned} x^3-7x^2+7x-1 &= 0, \\ (x-1)(x^2-6x+1) &= 0. \end{aligned}$$

Let the roots of the equation we have arranged be (1.9), also

$$\begin{aligned} \alpha+\beta+\gamma &= 7, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Let's write $b_{(3),n} = A1^n + B\alpha^n + C\beta^n$ and let us obtain the following equations

$$\text{for } n = 0 \Rightarrow b_{(3),0} = A1^0 + B\alpha^0 + C\beta^0 \Rightarrow A+B+C = 0, \tag{2.3}$$

$$\text{for } n = 1 \Rightarrow b_{(3),1} = A1^1 + B\alpha^1 + C\beta^1 \Rightarrow A+B\alpha+C\beta = 0, \tag{2.4}$$

$$\text{for } n = 2 \Rightarrow b_{(3),0} = A1^2 + B\alpha^2 + C\beta^2 \Rightarrow A+B\alpha^2+C\beta^2 = 2\ . \tag{2.5}$$

Let's subtract (2.3) from (2.4), also subtract (2.3) from (2.5) and get the following system of equations

$$B(\alpha-1)+C(\beta-1) = 0 \tag{2.6}$$

$$B(\alpha^2-1)+C(\beta^2-1) = 2.$$

Let's multiply the (2.6) by $-(\alpha+1)$ and add the equations side by side.

$$\begin{aligned} C(\beta^2-1)-C(\beta-1)(\alpha+1) &= 2 \\ C(\beta-1)(\beta+1-\alpha-1) &= 2 \end{aligned}$$

then we find

$$C = \frac{2}{5\beta+\alpha-2}. \tag{2.7}$$

In the above equations, we have reached the (2.7) value by typing $\beta^2 = 6\beta-1$ and $\alpha.\beta = 1$. If $C = \frac{2}{5\beta+\alpha-2}$ is written in (2.6), $B = \frac{2}{5\alpha+\beta-2}$ is found. Let's write $B = \frac{2}{5\alpha+\beta-2}$ and $C = \frac{2}{5\beta+\alpha-2}$ in the (2.3). So

$$\begin{aligned} A+B+C &= 0 \\ A+\frac{2}{5\alpha+\beta-2}+\frac{2}{5\beta+\alpha-2} &= 0 \end{aligned}$$

then

$$A = -2\left(\frac{1}{5\alpha+\beta-2}+\frac{1}{5\beta+\alpha-2}\right). \tag{2.8}$$

If $\alpha.\beta = 1$ and $\alpha+\beta = 6$ are written in the (2.8), $A = -\frac{1}{2}$ is found. Let's write the values we found in their places in the equality $b_{(3),n} = A1^n + B\alpha^n + C\beta^n$ then,

$$\begin{aligned} b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^n+\frac{2}{5\beta+\alpha-2}\beta^n \\ &= -\frac{1}{2}+2\left(\frac{\alpha^n(5\beta+\alpha-2)+\beta^n(5\alpha+\beta-2)}{(5\alpha+\beta-2)(5\beta+\alpha-2)}\right). \end{aligned}$$

In the last case, the Binet formula is as follows:

$$b_{(3),n} = \frac{\alpha^{n-1}(\alpha^2-2\alpha+5)+\beta^{n-1}(\beta^2-2\beta+5)}{64} - \frac{1}{2}.$$

□

Catalan, Cassini and d'Ocagne identities are given in [12], now let's talk about their proof in a different way. It is the Catalan identity in an identity that we can express with cobalancing numbers. We will show the correctness of this identity in the following theorem.

**Theorem 2.4.** *Let $b_{(3),n}$ can be the cobalancing number. The Catalan identity for the cobalancing number is as follows:*

$$b_{(3),n+k}b_{(3),n-k}-b^2_{(3),n}=\frac{1}{32}(\alpha^k-\beta^k)^2-\frac{\alpha^{n-k}(\alpha^k-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-k}(\beta^k-1)^2}{5\beta+\alpha-2}.$$

*Proof.* Let's write $b_{(3),n+k},\ b_{(3),n-k},\ b_{(3),n}$ then

$$\begin{aligned} b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^n+\frac{2}{5\beta+\alpha-2}\beta^n, \\ b_{(3),n+k} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+k}+\frac{2}{5\beta+\alpha-2}\beta^{n+k}, \\ b_{(3),n-k} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n-k}+\frac{2}{5\beta+\alpha-2}\beta^{n-k}. \end{aligned}$$

Now $B=\frac{2}{5\alpha+\beta-2}$ and $C=\frac{2}{5\beta+\alpha-2}$ also we have $\alpha+\beta=6$ and $\alpha.\beta=1$.

$$b_{(3),n+k}b_{(3),n-k} = \frac{1}{4}-\frac{1}{2}B\alpha^{n-k}-\frac{1}{2}C\beta^{n-k}-\frac{1}{2}B\alpha^{n+k}-BB\alpha^{2n}+BC\alpha^{n+k}\beta^{n-k}-\frac{1}{2}C\beta^{n+k}+BC\alpha^{n-k}\beta^{n+k}+CC\beta^{2n}$$

and

$$b_{(3),n}b_{(3),n} = \frac{1}{4}-\frac{1}{2}B\alpha^n-\frac{1}{2}C\beta^n-\frac{1}{2}B\alpha^n+BB\alpha^{2n}+BC\alpha^n\beta^n-\frac{1}{2}C\beta^n+BC\alpha^n\beta^n+CC\beta^{2n}.$$

Let's subtract the equations written above from side to side

$$\begin{aligned} b_{(3),n+k}b_{(3),n-k}-b^2_{(3),n} &= BC\left(\left(\frac{\alpha}{\beta}\right)^k-2+\left(\frac{\beta}{\alpha}\right)^k\right)-\frac{B}{2}(\alpha^{n-k}-2\alpha^n+\alpha^{n+k})-\frac{C}{2}(\beta^{n-k}-2\beta^n+\beta^{n+k}) \\ &= BC\left(\frac{\alpha^k}{\beta^k}-2+\frac{\beta^k}{\alpha^k}\right)-\frac{B}{2}\alpha^n\left(\frac{\alpha^k}{1}-2+\frac{1}{\alpha^k}\right)-\frac{C}{2}\beta^n\left(\beta^k-2+\frac{1}{\beta^k}\right) \\ &= BC(\alpha^k-\beta^k)^2-\frac{1}{2}\frac{2}{5\alpha+\beta-2}\alpha^{n-k}(\alpha^k-1)^2-\frac{1}{2}\frac{2}{5\beta+\alpha-2}\beta^{n-k}(\beta^k-1)^2. \end{aligned}$$

Let's show that $B.C=\frac{1}{32}$ for $\alpha=3+2\sqrt{2}$ and $\beta=3-2\sqrt{2}$, we have $\alpha+\beta=6$ and $\alpha.\beta=1$,

$$\begin{aligned} B.C &= \frac{2}{5\alpha+\beta-2}\frac{2}{5\beta+\alpha-2} \\ &= \frac{4}{26\alpha\beta+5[(\alpha+\beta)^2-2\alpha\beta]-12(\alpha+\beta)+4} \\ &= \frac{1}{32}. \end{aligned}$$

The final state of equality is as follows:

$$b_{(3),n+k}b_{(3),n-k}-b^2_{(3),n}=\frac{1}{32}(\alpha^k-\beta^k)^2-\frac{\alpha^{n-k}(\alpha^k-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-k}(\beta^k-1)^2}{5\beta+\alpha-2}.$$

□

One of the other identities for cobalancing numbers are

$$b_{(3),n-1}b_{(3),n+1}-b^2_{(3),n}=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

We will show the correctness of this identity in the following theorem. This identity is called the Cassini identity.

**Theorem 2.5.** *Let $b_{(3),n}$ can be the cobalancing number. The Cassini identity for the cobalancing number is as follows:*

$$b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

*Proof.* Let's use the Binet formula for the cobalancing numbers and write the following equations

$$\begin{aligned} b_{(3),n-1} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n-1}+\frac{2}{5\beta+\alpha-2}\beta^{n-1}, \\ &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+1}+\frac{2}{5\beta+\alpha-2}\beta^{n+1}. \end{aligned}$$

Let's write $B=\frac{2}{5\alpha+\beta-2}$ and $C=\frac{2}{5\beta+\alpha-2}$ in the above equations, let's edit the $b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2$ equation as follows:

$$b_{(3),n-1}b_{(3),n+1} = \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{n-1}+B^2\alpha^{2n}+BC(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)-\frac{C}{2}\beta^{n-1}+BC(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)+C^2\beta^{2n},$$

then

$$b_{(3),n}b_{(3),n} = \frac{1}{4}-\frac{B}{2}\alpha^n-\frac{C}{2}\beta^n-\frac{B}{2}\alpha^n+B^2\alpha^{2n}+BC(\alpha\beta)^n-\frac{C}{2}\beta^n+BC(\alpha\beta)^n+C^2\beta^{2n}.$$

Let's subtract the equations written above from side to side and get the expression $b_{n-1}b_{n+1}-b_n^2$,

$$\begin{aligned} b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2 &= -\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{n-1}+BC\left(\frac{\beta}{\alpha}\right)-\frac{C}{2}\beta^{n-1}+BC\left(\frac{\alpha}{\beta}\right)+B\alpha^n+C\beta^n-2BC \\ &= -\frac{B}{2}\alpha^n\left(\alpha+\frac{1}{\alpha}-2\right)-\frac{C}{2}\beta^n\left(\beta+\frac{1}{\beta}-2\right)+BC\left(\frac{\beta}{\alpha}-2+\frac{\alpha}{\beta}\right) \\ &= \frac{1}{32}(\alpha-\beta)^2-\frac{1}{2}\frac{2}{5\alpha+\beta-2}\alpha^{n-1}(\alpha-1)^2-\frac{1}{2}\frac{2}{5\beta+\alpha-2}\beta^{n-1}(\beta-1)^2. \end{aligned}$$

The final state of equality is as follows

$$b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

□

There are many identities for cobalancing numbers. Another one is d'Ocagne's identity. This equality will be proved in the following theorem.

**Theorem 2.6.** *Let $b_{(3),n}$ can be the cobalancing number. The d'Ocagne's identity for the cobalancing number is as follows:*

$$b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}=\frac{(\alpha-1)(\alpha^m-\alpha^n)}{5\alpha+\beta-2}+\frac{(\beta-1)(\beta^m-\beta^n)}{5\beta+\alpha-2}+\frac{(\alpha-\beta)(\alpha^n\beta^m-\alpha^m\beta^n)}{32}.$$

*Proof.* Helping by (1.8) sequence

$$b_{(3),m}=-\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^m+\frac{2}{5\beta+\alpha-2}\beta^m$$

and

$$b_{(3),n+1}=-\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+1}+\frac{2}{5\beta+\alpha-2}\beta^{n+1}$$

is written. Let's write $\frac{2}{5\alpha+\beta-2}=B$ and $\frac{2}{5\beta+\alpha-2}=C$ before multiplying the expressions $b_{(3),m}$ and $b_{(3),n+1}$.

$$b_{(3),m}b_{(3),n+1} = \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^m+B^2\alpha^{m+n+1}+BC\alpha^m\beta^{n+1}-\frac{C}{2}\beta^m+BC\alpha^{n+1}\beta^m+C^2\beta^{m+n+1}.$$

Also from the following equations

$$\begin{aligned} b_{(3),m+1} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{m+1}+\frac{2}{5\beta+\alpha-2}\beta^{m+1} \\ b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n}+\frac{2}{5\beta+\alpha-2}\beta^{n}, \end{aligned}$$

we get

$$\begin{aligned} b_{(3),m+1}b_{(3),n} &= \frac{1}{4}-\frac{B}{2}\alpha^{n}-\frac{C}{2}\beta^{n}-\frac{B}{2}\alpha^{m+1}+B^2\alpha^{m+n+1} \\ &\quad +BC\alpha^{m+1}\beta^{n}-\frac{C}{2}\beta^{m+1}+BC\alpha^{n}\beta^{m+1}+C^2\beta^{m+n+1}. \end{aligned}$$

Let's show that $BC=\frac{1}{32}$ before finding the $b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}$ difference. For $\alpha=3+2\sqrt{2}$ and $\beta=3-2\sqrt{2}$, we have $\alpha+\beta=6$ and $\alpha.\beta=1$, so

$$\begin{aligned} B.C &= \frac{2}{5\alpha+\beta-2}\frac{2}{5\beta+\alpha-2} \\ &= \frac{4}{26\alpha\beta+5[(\alpha+\beta)^2-2\alpha\beta]-12(\alpha+\beta)+4} \\ &= \frac{1}{32}. \end{aligned}$$

Then

$$\begin{aligned} b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n} &= \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{m}+C^2\alpha^{m+n+1}+BC\alpha^{m}\beta^{n+1}-\frac{C}{2}\beta^{m}+BC\alpha^{n+1}\beta^{m}+C^2\beta^{m+n+1} \\ &\quad -\frac{1}{4}+\frac{B}{2}\alpha^{n}+\frac{C}{2}\beta^{n}+\frac{B}{2}\alpha^{m+1}-B^2\alpha^{m+n+1}-BC\alpha^{m+1}\beta^{n}+\frac{C}{2}\beta^{m+1}-BC\alpha^{n}\beta^{m+1}-C^2\beta^{m+n+1} \\ &= \frac{B}{2}(\alpha-1)(\alpha^{m}-\alpha^{n})+\frac{C}{2}(\beta-1)(\beta^{m}-\beta^{n})+BC(\alpha-\beta)(\alpha^{n}\beta^{m}-\alpha^{m}\beta^{n}) \\ &= \frac{1}{2}\frac{2}{5\alpha+\beta-2}(\alpha-1)(\alpha^{m}-\alpha^{n})+\frac{1}{2}\frac{2}{5\beta+\alpha-2}(\beta-1)(\beta^{m}-\beta^{n}) \\ &\quad +\frac{1}{32}(\alpha-\beta)(\alpha^{n}\beta^{m}-\alpha^{m}\beta^{n}). \end{aligned}$$

The final state of equality is as follows.

$$b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}=\frac{(\alpha-1)(\alpha^{m}-\alpha^{n})}{5\alpha+\beta-2}+\frac{(\beta-1)(\beta^{m}-\beta^{n})}{5\beta+\alpha-2}+\frac{(\alpha-\beta)(\alpha^{n}\beta^{m}-\alpha^{m}\beta^{n})}{32}.$$

□

## 3. Sum of Cobalancing Numbers

By performing various operations, we can show the cobalancing numbers in the form of a different summation formulas. Sum formulas of cobalancing numbers with positive subscripts are obtained in the following three theorem. The results regarding the sum formulas can be obtained from the following theorems.

**Theorem 3.1.** *Let $b_{(3),n}$ can be the cobalancing number. The following difference equation is valid:*

$$(x^2+7)W_1-(1+7x^2)W_2-14x=x^{n}(b_{(3),n+3}-7b_{(3),n+2})+x^{n-1}(b_{(3),n+2}-7b_{(3),n+1})$$

*where*

$$W_1=\sum_{k=0}^{n-2}x^{k}.b_{(3),k+2} \text{ and } W_2=\sum_{k=0}^{n-2}x^{k}.b_{(3),k+3}.$$

*Proof.* Let's leave the term $b_{(3),n-2}$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$\begin{aligned}
n &= 2 \Rightarrow x^0 b_{(3),0} = x^0 b_{(3),3} - x^0 7b_{(3),2} + x^0 7b_{(3),1} \\
n &= 3 \Rightarrow x^1 b_{(3),1} = x^1 b_{(3),4} - x^1 7b_{(3),3} + x^1 7b_{(3),2} \\
&\vdots \\
n &= n \Rightarrow x^{n-2} b_{(3),n-2} = x^{n-2} b_{(3),n+1} - x^{n-2} 7b_{(3),n} + x^{n-2} 7b_{(3),n-1} \\
n &= n+1 \Rightarrow x^{n-1} b_{(3),n-1} = x^{n-1} b_{(3),n+2} - x^{n-1} 7b_{(3),n+1} + x^{n-1} 7b_{(3),n} \\
n &= n+2 \Rightarrow x^{n} b_{(3),n} = x^{n} b_{(3),n+3} - x^{n} 7b_{(3),n+2} + x^{n} 7b_{(3),n+1}.
\end{aligned}$$

Let's add the equations from side to side

$$\begin{aligned}
& b_{(3),0} + xb_{(3),1} + (x^2+7)\sum_{k=0}^{n-2} x^k b_{(3),k+2} + x^{n-1} 7b_{(3),n+1} + x^n 7b_{(3),n+2} \\
= \; & 14x + (1+7x^2)\sum_{k=0}^{n-2} x^k b_{(3),k+3} + x^{n-1} b_{(3),n+2} + x^n b_{(3),n+3}.
\end{aligned}$$

If the equations $b_{(3),0} = 0, b_{(3),1} = 0, b_{(3),2} = 2$ are written in their places, the following equality is obtained

$$\begin{aligned}
& (x^2+7)\sum_{k=0}^{n-2} x^k.b_{(3),k+2} - (1+7x^2)\sum_{k=0}^{n-2} x^k.b_{(3),k+3} - 14x \\
= \; & x^n(b_{(3),n+3} - 7b_{(3),n+2}) + x^{n-1}(b_{(3),n+2} - 7b_{(3),n+1}).
\end{aligned}$$

□

Now let the result be given that the even and odd terms of sums are in the same difference equation.

**Lemma 3.2.** *Let $b_{(3),n}$ can be the cobalancing number. The following difference equation is valid:*

$$(7+x)W_3 - (1+7x)W_4 = x^n(b_{(3),2n+1} + (7+x)b_{(3),2n})$$

*where*

$$W_3 = \sum_{k=1}^{n} x^k b_{(3),2k} \text{ and } W_4 = \sum_{k=1}^{n-1} x^k b_{(3),2k+1}.$$

*Proof.* Let's leave the term $7b_n$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$7b_{(3),n} = b_{(3),n+1} + 7b_{(3),n-1} - b_{(3),n-2}.$$

$$\begin{aligned}
n &= 2 \Rightarrow 7x^1 b_{(3),2} = x^1 b_{(3),3} + 7x^1 b_{(3),1} - x^1 b_{(3),0} \\
n &= 4 \Rightarrow 7x^2 b_{(3),4} = x^2 b_{(3),5} + 7x^2 b_{(3),3} - x^2 b_{(3),2} \\
&\vdots \\
n &= 2n-2 \Rightarrow 7x^{n-1} b_{(3),2n-2} = x^{n-1} b_{(3),2n-1} + 7x^{n-1} b_{(3),2n-3} - x^{n-1} b_{(3),2n-4} \\
n &= 2n \Rightarrow 7x^{n} b_{(3),2n} = x^{n} b_{(3),2n+1} + 7x^{n} b_{(3),2n-1} - x^{n} b_{(3),2n-2}.
\end{aligned}$$

Let's add the equations from side to side, then

$$xb_{(3),0} + (7+x)\sum_{k=1}^{n-1} x^k.b_{(3),2k} = (1+7x)\sum_{k=1}^{n-1} x^k.b_{(3),2k+1}.$$

If the equations $b_0 = 0$, are written in their places, the following equality is obtained

$$x^n b_{(3),2n+1} = (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1}.$$

Let's add and subtract the term $x^n b_{2n}$ in the sum symbol located on the right side of the equation

$$x^n b_{(3),2n+1} = (7+x)\sum_{k=1}^{n} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} - (7+x)(x^n b_{(3),2n}).$$

In the last case, the equality is most regularly as follows:

$$(7+x)\sum_{k=1}^{n} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} = x^n (b_{(3),2n+1} + (7+x) b_{(3),2n}).$$

□

**Lemma 3.3.** *Let $b_{(3),n}$ can be the cobalancing number. If $x \neq 0$, following equation is valid:*

$$(7+x)W_5 - (\frac{1}{x}+7)W_6 = x^n b_{(3),2n+2} - (\frac{1}{x}+7)2x$$

*where*

$$W_5 = \sum_{k=1}^{n-1} x^k b_{(3),2k+1} \ \text{and}\ W_6 = \sum_{k=1}^{n} x^k b_{(3),2k}.$$

*Proof.* Let's leave the term $7b_{(3),n}$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$7b_{(3),n} = b_{(3),n+1} + 7b_{(3),n-1} - b_{(3),n-2}.$$

$$\begin{aligned}
n &= 3 \Rightarrow 7x^1 b_{(3),3} = x^1 b_{(3),4} + 7x^1 b_{(3),2} - x^1 b_{(3),1} \\
n &= 5 \Rightarrow 7x^2 b_{(3),5} = x^2 b_{(3),6} + 7x^2 b_{(3),4} - x^2 b_{(3),3} \\
&\vdots \\
n &= 2n-1 \Rightarrow 7x^{n-1} b_{(3),2n-1} = x^{n-1} b_{(3),2n} + 7x^{n-1} b_{(3),2n-2} - x^{n-1} b_{(3),2n-3} \\
n &= 2n+1 \Rightarrow 7x^n b_{(3),2n+1} = x^n b_{(3),2n+2} + 7x^n b_{(3),2n} - x^n b_{(3),2n-1}.
\end{aligned}$$

Let's multiply and divide the first terms to the right of the equal sign by $x$

$$\begin{aligned}
n &= 3 \Rightarrow 7x^1 b_{(3),3} = \frac{x^2 b_4}{x} + 7x^1 b_{(3),2} - x^1 b_{(3),1} \\
n &= 5 \Rightarrow 7x^2 b_{(3),5} = \frac{x^3 b_{(3),6}}{x} + 7x^2 b_{(3),4} - x^2 b_{(3),3} \\
&\vdots \\
n &= 2n-1 \Rightarrow 7x^{n-1} b_{(3),2n-1} = \frac{x^n b_{(3),2n}}{x} + 7x^{n-1} b_{(3),2n-2} - x^{n-1} b_{(3),2n-3} \\
n &= 2n+1 \Rightarrow 7x^n b_{(3),2n+1} = x^n b_{(3),2n+2} + 7x^n b_{(3),2n} - x^n b_{(3),2n-1}.
\end{aligned}$$

Let's add the equations side by side

$$xb_{(3),1} + (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} = x^n b_{(3),2n+2} + (\frac{1}{x}+7)\sum_{k=2}^{n} x^k b_{(3),2k}.$$

In the last case, let's add the $xb_{(3),2}$ term to the sum term on the right side in the equality we obtained

$$xb_{(3),1} + (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} = x^n b_{(3),2n+2} + (\frac{1}{x}+7)\sum_{k=1}^{n} x^k b_{(3),2k} - (\frac{1}{x}+7)xb_{(3),2}$$

then

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-xb_{(3),1}-(\frac{1}{x}+7)xb_{(3),2}.$$

If it is written instead of $b_{(3),1}=0$ and $b_{(3),2}=2$, the most regular form of equality is as follows:

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-14x-2.$$

□

The results to be found now will be found with the help of the theorems given above.

**Theorem 3.4.** *Let $b_{(3),n}$ can be the cobalancing number. The following equation is valid:*

$$\sum_{k=1}^{n}x^k b_{(3),2k}=\frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(x+7x^2) \end{array}\right\}}{x^3-35x^2+35x-1} \tag{3.1}$$

*and*

$$\sum_{k=1}^{n-1}x^k b_{(3),2k+1}=\frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(7x+x^2) \end{array}\right\}}{x^3-35x^2+35x-1}. \tag{3.2}$$

*Proof.* The following equations have been proved in the previous lemmas

$$(7+x)\sum_{k=1}^{n}x^k b_{(3),2k}-(1+7x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}) \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})$$

and

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-14x-2.$$

Let $A_1=\sum_{k=1}^{n}x^k b_{(3),2k}$ and $B_1=\sum_{k=1}^{n-1}x^k b_{(3),2k+1}$. Let's arrange the above equations in the following way and get a system of equations

$$(7+x)A_1-(1+7x)B_1 \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n}) \tag{3.3}$$

$$-(\frac{1}{x}+7)A_1+(7+x)B_1 \quad = \quad x^n b_{(3),2n+2}-14x-2. \tag{3.4}$$

Let's multiply the first equation by $\frac{1}{x}+7$ and multiply the second equation by $7+x$, then add the equations side by side

$$(\frac{1}{x}+7)(7+x)A_1-(\frac{1}{x}+7)(1+7x)B_1 \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(\frac{1}{x}+7)$$

$$-(\frac{1}{x}+7)(7+x)A_1+(7+x)(7+x)B_1 \quad = \quad (x^n b_{(3),2n+2}-14x-2)(7+x).$$

If the equations are added side by side, the following equality is found

$$B_1=\frac{x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x)+(x^n b_{(3),2n+2}-14x-2)(7x+x^2)}{x^3-35x^2+35x-1}.$$

Let's multiply (3.3) by $(7+x)$ and multiply (3.4) by $(1+7x)$, then add the equations side by side

$$(7+x)(7+x)A_1-(7+x)(1+7x)B_1 \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(7+x)$$

$$-(\frac{1}{x}+7)(1+7x)A_1+(7+x)(1+7x)B_1 \quad = \quad (x^n b_{(3),2n+2}-14x-2)(1+7x).$$

When the above equations are added side by side, the equal of the expression $A_1$ will be as follows

$$A_1=\frac{x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x)+(x^nb_{(3),2n+2}-14x-2)(x+7x^2)}{x^3-35x^2+35x-1}.$$

In the last case, the following equations are correct

$$\begin{aligned}\sum_{k=1}^{n}x^kb_{(3),2k} &= \frac{\left\{\begin{array}{c}x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x)\\ +(x^nb_{(3),2n+2}-14x-2)(x+7x^2)\end{array}\right\}}{x^3-35x^2+35x-1}\\ \sum_{k=1}^{n-1}x^kb_{(3),2k+1} &= \frac{\left\{\begin{array}{c}x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x)\\ +(x^nb_{(3),2n+2}-14x-2)(7x+x^2)\end{array}\right\}}{x^3-35x^2+35x-1}.\end{aligned}$$

□

The following theorem is special cases of the summation formulas that we have found.

**Theorem 3.5.** *Let $b_{(3),n}$ can be the cobalancing number. $n\geq 0$, we have the following sum formulas:*

1. $\sum\limits_{k=1}^{n-1}(-1)^kb_{(3),2k+1}=\frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}+b_{(3),2n+2})+1$
2. $\sum\limits_{k=1}^{n}(-1)^kb_{(3),2k}=\frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}-b_{(3),2n+2})-1$
3. $\sum\limits_{k=0}^{n-2}(-1)^kb_{(3),k+2}=\frac{(-1)^n}{16}(b_{(3),n+3}-8b_{(3),n+2}+15b_{(3),n+1})+\frac{1}{8}$

*Proof.* 1) Let's write $x=-1$ in the (3.2) equation. Then

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^kb_{(3),2k+1} &= \frac{(-1)^n(b_{(3),2n+1}+6b_{(3),2n})(-6)+((-1)^nb_{(3),2n+2}+14-2)(-6)}{-1-35-35-1}\\ &= \frac{(-1)^n(-6)(b_{(3),2n+1}+6b_{(3),2n})+(-1)^n(-6)b_{(3),2n+2}}{-72}+\frac{-72}{-72}\\ &= \frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}+b_{(3),2n+2})+1\end{aligned}$$

As in other cases, it is proven in a similar way. □

## 4. Conclusions

In this study, cobalancing numbers, which are an integer sequence with a non-homogeneous second-order recurrence relation, are transformed into a sequence with a homogeneous third-order recurrence relation, thus providing ease of operation. Some of the results found are the generating function, Binet formula, specially defined Catalan, Cassini and d'Ocagne identities and some sum formulas.

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