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WHEN IS THE INTERNAL CANCELLATION PROPERTY INHERITED BY FREE MODULES?

Figen TAKIL MUTLU [1]

ABSTRACT

In this paper we deal with the internal cancellation property for free modules. To this end, we deduce that the internal cancellation property is not Morita invariant. In contrast, it is shown that the direct sum of two copies of a right Ore domain has the internal cancellation property as a right module over itself.

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İÇ KISALTMA ÖZELLİĞİ SERBEST MODÜLLER TARAFINDAN NE ZAMAN KALITIMSALDIR?

ÖZ

Bu çalışmada serbest modüller için iç kısaltma özelliği incelenmiştir. Sonuç olarak, iç kısaltma özelliğinin Morita değişmez olmadığı sonucuna varılmıştır. Tersine, bir sağ Ore Bölgesinin kendisi ile dik toplamının, kendi üzerine bir sağ modül olarak, iç kısaltma özelliğine sahip olduğu gösterilmiştir.

Anahtar Kelimeler: İç kısaltma özelliği, Matris halkaları, Bir kürenin teğet demeti, Ads-Modül.

1. INTRODUCTION

Throughout this paper, all rings are associative with unity and R always denotes such a ring. Modules are unital and for an abelian group M, we use M_R to denote a right R-module.

A module M_R has the *internal cancellation property* if, for submodules K, K', N, N' of M whenever

$$K \oplus N = K' \oplus N', \quad N \cong N'$$

implies that $K \cong K'$. The motivation for the present study of this property was provided by the following result of (Khurana and Lam, 2005) (see, also (Ojanguren and Sridharan, 1971)): a full matrix ring over R has internal cancellation property if and only if finitely generated projective R-modules satisfy internal cancellation property. The internal cancellation property of rings and modules has been studied by many authors including (Bhatwadekar, 2003), (Brookfield, 1998), (Khurana and Lam, 2005) and (Ojanguren and Sridharan, 1971).

[1,] Anadolu University Department of Mathematics, Eskişehir, 26470 Turkey.
e-mail: figent@anadolu.edu.tr

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We will use $M_m(R)$, R^m and $udimM_R$ to denote the full m-by-m matrix ring over R, the direct sum of m copies of R_R for any positive integer m and the uniform (Goldie) dimension of an R-module M, respectively. For any unexplained terminology, we refer to (Lam, 1999) and (Mohamed and Muller, 1990). In this paper we investigate conditions on a ring R with the internal cancellation property for the following property to hold:

(P_k): Every free right R-module with a basis of cardinality k has the internal cancellation property.

Note that for $k=1$ P_1 is always true. We recall that from (Khurana and Lam, 2005) the class of right R-modules which satisfy the internal cancellation property is closed under direct summands for any ring R. We use matrix techniques in the study of the aforementioned property.

2. RESULTS

Let R be any ring, e an idempotent in R such that $R = ReR$ and S the subring eRe. Let M be a right R-module. Then, Me is a right S-module.

Lemma 2.1 *Let L be a submodule of M_R. Then, L is a direct summand of M_R if and only if Le is a direct summand of $(Me)_S$.*

Proof. Suppose L is a direct summand of M_R. Then, $M = L \oplus L'$ for some $L' \leq M_R$. Thus, $Me = Le + L'e$. But $Le \cap L'e \leq L \cap L' = 0$. Therefore, $Me = Le \oplus L'e$. Conversely, suppose that $Me = Le \oplus K$ for some $K \leq (Me)_S$. It is easy to check that $L \cap KR = 0$, and

$$\begin{aligned} M &= MeR \\ &= (Le+K)R \\ &= LeR + KR \\ &= LReR + KR \\ &= LR + KR = L \oplus KR \end{aligned}$$

i.e., L is a direct summand of M_R.

Theorem 2.2 *With the above notation, let M be a right R-module. Then, the right R-module M has the internal cancellation property if and only if the right S-module Me has the internal cancellation property.*

Proof. Clear by Lemma 2.1.

Corollary 2.3 *The ring R_R has the internal cancellation property if and only if the right eRe-module Re has the internal cancellation property.*

Proof. Immediate by Theorem 2.2.

Example 2.4 *Let T be any ring with identity, m a positive integer and R the ring $M_m(T)$ of all $m \times m$ matrices with entries in T. Let e_{11} denote the matrix in R with $(1,1)$ entry 1 and all other entries 0. It is clear that e_{11} is idempotent and $T \cong e_{11}Te_{11}$ and $R = Re_{11}R$.*

We note that following Corollary also follows from (Khurana and Lam, 2005, Proposition 3.6) which allows us to use matrix techniques in the study of the property (P_k).

Corollary 2.5 *With the above notation, $R = M_m(T)$ has the internal cancellation property if and only if the free right T-module T^m has the internal cancellation property.*

Proof. Since $Re_{11} = Te_{11} + Te_{21} + ... + Te_{n1} \cong T^n$ as a T-module, Re_{11} has the internal cancellation property by Theorem 2.2.

Corollary 2.6 *With the above notation, if the ring R has the internal cancellation property, then, so does the ring $S = eRe$.*

Proof. $Re = eRe \oplus (1-e)Re$ as right S-module. By Corollary 2.3, Re has the internal cancellation property. Since $(eRe)_S$ is a direct summand of $(Re)_S$, the desired result is obtained.

Let P be a ring theoretic property. Then P is said to be *Morita invariant* if and only if the following holds:

Whenever a ring R has P, so do $M_m(R)$ for every $m \geq 2$ and eRe for all $e^2 = e \in R$ such that $R = ReR$.

The following two results together with Corollary 2.5 show that the internal cancellation property is not Morita invariant, and commutativity, no nonzero divisors, ACC on ideals are not sufficient conditions for property (P_k) to hold when $k \geq 3$ is an odd integer.

Theorem 2.7 *Let $\mathbb{R}$ be the real field and n any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbb{R}[x_1,...,x_n]$ in indeterminates $x_1,...,x_n$ over $\mathbb{R}$. Let R be the ring S/Ss where $s = x_1^2 + x_2^2 + ... + x_n^2 - 1$. Let P be the R-module with generators $s_1,...,s_n$ and relation $\sum_{i=1}^{n} x_i s_i = 0$. Then, the free R-module $P \oplus R$ does not satisfy the internal cancellation property.*

Proof. Let $M_R = P \oplus R$. Then, it is clear that M is a free R-module. Note that P is an indecomposable R-module from (Swan, 1962, Theorem 3). It follows that M does not have the internal cancellation property.

Corollary 2.8 *Let $\mathbb{R}$ be the real field and n any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbb{R}[x_1,...,x_n]$ in indeterminates $x_1,...,x_n$ over $\mathbb{R}$. Let R be the ring S/Ss where $s = x_1^2 + x_2^2 + ... + x_n^2 - 1$. Then the free R-module $M = \bigoplus_{i=1}^{n} R$ does not satisfy the internal cancellation property.*

Proof. Let $\varphi : M \to R$ be the homomorphism defined by

$\varphi(a_1 + Ss, a_2 + Ss, ..., a_n + Ss) = a_1x_1 + a_2x_2 + ... + a_nx_n$

for all a_i in S, $1 \le i \le n$. Clearly, φ is an epimorphism and hence its kernel K is a direct summand of M, i.e., $M = K \oplus K'$ for some submodule $K' \cong R$. Note that K is the R-module of regular sections of the tangent bundle of the $(n-1)$-sphere S^{n-1}. Since the Euler characteristic $\chi(S^{n-1}) \neq 0$, it follows that $(n-1)$-sphere cannot have a nonvanishing regular section of its tangent bundle (see (Bredon, 1993, Corollary VI. 13.3)). Thus, K is an indecomposable and non-free module. It follows that M_R does not satisfy the internal cancellation property.

Next example is taken from (Swan, 1996) which provides that property (P_k) does not hold for $k = 2$. For details and more general form of this type of examples see (Ojanguren and Sridharan, 1971) and (Swan, 1996).

Example 2.9 *Let D be a noncommutative division ring and let $R = D[x, y]$ with $a, b \in R$ such that $c = ab - ba$ is a unit in the center of D. Let $M_R = R \oplus R$ and $\varphi : M \to R$ be a homomorphism defined by*

$$\varphi(\lambda, \mu) = \lambda(x + a) + \mu(y + b).$$

*Then φ is an epimorphism. Thus $M \cong R \oplus P$ where P is the kernel of φ. From (*Swan, 1996*), P is not free. Hence M does not satisfy the internal cancellation property.*

Suprisingly, our next result shows that the right Ore condition and no nonzero divisors clearly ensure a condition for the property (P_k) when $k = 2$.

Theorem 2.10 *If R is a right Ore domain, then, $(R \oplus R)_R$ has the internal cancellation property.*

Proof. Let D be the division ring which is the classical right ring of quotients of R. Routine calculations show that any nontrivial idempotent of $M_2(D)$ is primitive and has one of the following forms where a, d, $f \in D$ with $a - a^2 \neq 0$ and $d \neq 0$:

$$e_1 = \begin{bmatrix} 1 & f \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ f & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & f \\ 0 & 1 \end{bmatrix},$$
$$e_4 = \begin{bmatrix} 1 & 0 \\ f & 0 \end{bmatrix}, e_5 = \begin{bmatrix} a & (1-a)d \\ d^{-1}a & d^{-1}(1-a)d \end{bmatrix}$$

It can be verified that

$$\begin{aligned} M_2(R) &= e_1M_2(R) \oplus e_2M_2(R) \\ &= e_1M_2(R) \oplus e_3M_2(R) \\ &= e_2M_2(R) \oplus e_4M_2(R) \\ &= e_1M_2(R) \oplus e_5M_2(R) \\ &= e_2M_2(R) \oplus e_5M_2(R). \end{aligned}$$

Then, it can be checked that $M_2(R)$ has the internal cancellation property as a right module over itself. By Corollary 2.5, $(R \oplus R)_R$ has the internal cancellation property.

Following (Burgess and Raphael, 1992), a module M is called *ads-module* if $M = A \oplus B$ and C is any complement of A in M, then, $M = A \oplus C$. Equivalently, whenever $M = X \oplus Y$, then, X is Y-injective (see (Burgess and Raphael, 1992, Proposition 1.1)). Recall that any quasi-continuous module is both ads-module and has the internal cancellation property. One might expect whether there is any implication between ads condition and the internal cancellation property. However, Examples 2.12 and 2.13 (or 2.14) below eliminate these possibilities. First, let $E(M_R)$ stand for the injective hull of a right R-module M. Then, we have the following observation.

Proposition 2.11 *Let R_R be a uniform ring and M be a right R-module which does not have the internal cancellation property. Then, the right R-module $F = M \oplus E(R_R)$ is an ads-module which does not have the internal cancellation property.*

Proof. First note that clearly F does not have the internal cancellation property. Since $E(R_R)$ is indecomposable and is X-injective for any submodule X of M, F_R is an ads-module by (Burgess and Raphael, 1992, Proposition 1.1).

Combining Corollary 2.8 together with Proposition 2.11 we have the following example.

Example 2.12 *Let R and M be as in Corollary 2.8. Then, by Proposition 2.11, $F_R = M \oplus E(R_R)$ is an ads-module but does not have the internal cancellation property.*

Example 2.13 *Let M_R and K be as in the proof of Corollary 2.8. Let $F_R = K \oplus K$. Then it is clear that F_R has the internal cancellation property. Note that K_R has $udimK_R \geq 2$. Now assume that K is K-injective, i.e., F_R is an ads-module. It follows that F_R is a continuous module. By (Mohamed and Muller, 1990, Proposition 2.5.), K_R is a uniform module. Thus, $udimK_R = 1$, which is a contradiction.*

Example 2.14 *Let R be any non-injective right Ore domain (e.g., $\mathbb{Z}$). Let $M_R = (R \oplus R)_R$. Since R is not R-injective, M_R is not an ads-module. By Theorem 2.10, has the internal cancellation property.*

Remark 2.15 *A free $\mathbb{Z}$-module M is quasi-continuous if and only if M has rank 1, by (Smith and Tercan, 1992, Example 9). Thus, Example 2.14 for the case $R = \mathbb{Z}$ provides a non quasi-continuous module which have the internal cancellation property by Theorem 2.10.*

REFERENCES

Bhatwadekar, S.M. (2003). A cancellation for projective modules over affine algebras over C_1-fields. *J. Pure and App. Algebra* 183**,** 17-26.

Bredon, G.E. (1993). *Topology and Geometry.* Springer Verlag, New York.

Brookfield, G. (1998). Direct Sum Cancellation of Noetherian Modules. *J. Algebra* 200, 207-224.

Burgess, W.D. and Raphael, R. (1992). On modules with The Absolute Direct Summand Property. *Proceedings of the Ohio State University Conference,* 137-148.

Khurana, D. and Lam, T.Y. (2005). Rings with Internal Cancellation. *J. Algebra* 284, 203-235.

Lam, T.Y. (1999). *Lectures on Modules and Rings.* Springer Verlag, New York.

Mohamed, S.H. and Muller, B.J. (1990). *Continuous and Discrete Modules.* Cambridge University Press.

Ojanguren, M. and Sridharan, R. (1971). Cancellation of Azumaya Algebras. *J. Algebra* 18, 501-505.

Smith, P.F. and Tercan, A. (1992). Continuous and quasi-continuous modules. *Houston J. Math.* 18, 339-348.

Swan, R. G. (1962). Vector bundles and projective modules. *Trans. Amer. Math. Soc.* 105, 264-277.

Swan, R. G. (1996). Quaternionic Bundles on Algebraic Spheres. *Rocky Mountain J. Math.* 262, 773-791.