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Construction of the New Wave Solutions of Modified Camassa-Holm and Degasperis-Procesi Equations with Atangana's Conformable Derivative

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Atangana Konformal Türevli Modifiye Camassa-Holm ve Degasperis-Procesi Denklemlerinin Yeni Dalga Çözümlerinin Elde Edilmesi

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Öz

Bu çalışmada, akışkanlar mekaniği, hidrodinamik ve fiber optik alanlarında birçok fiziksel olayı tanımlamak için sıkça kullanılması nedeni ile zaman-kesirli modifiye Camassa-Holm (mCH) ve zaman-kesirli modifiye Degasperis-Procesi (mDP) denklemlerinin yeni tam çözümlerinin elde edilmesi amaçlanmıştır. Bu kesirli denklemler, Atangana konformal türevi göz önüne alınarak nonlineer adi diferansiyel denklemlere dönüştürülmüştür. Kesirli evolüsyon denklemlerinin istenen tam çözümlerini elde etmek için bu nonlineer adi diferansiyel denklemlere (m+1/G')-genişleme metodu uygulanmıştır. Hesaplamalar Mathematica yazılım sistemi ile gerçekleştirilmiştir. Ayrıca bu çalışmada sunulan çözümler literatürde zaman-kesirli CH ve DP denklemleri için elde edilen çözümler ile kıyaslanmış ve çözümlerin davranışları grafiksel olarak sunulmuştur.

Anahtar Kelimeler: *Camassa-Holm ve Degasperis-Procesi Denklemleri; (m+1/G')-Genişleme Metodu; Atangana Konformal Türevi.*

Abstract

In this study it is aimed to expose the new exact wave solutions of time-fractional modified Camassa-Holm (mCH) and time-fractional modified Degasperis-Procesi (mDP) equations due to being extensively used to delineate many physical phenomena in fluid mechanics, hydrodynamics and optical fibers. The aforementioned fractional equations are transformed into nonlinear ordinary differential equations (NLODE) considering the Atangana's conformable derivative (ACD). Then the (m+1/G')-expansion method is applied for these NLODEs to obtain the desired exact solutions of the fractional evolution equations. The evaluations are fulfilled through the software system Mathematica. Also the reported solutions in this manuscript are compared with the ones in the literature for the time-fractional CH and DP equations and the behaviors of the solutions are presented graphically.

Keywords: *Camassa-Holm and Degasperis-Procesi Equations; (m+1/G')-Expansion Method; Atangana's Conformable Derivative.*

1. Introduction

In recent years, many researchers are motivated by the applications of nonlinear partial differential equations (NLPDEs) with fractional derivatives (FD). In this context, a time delay fractional COVID-19 SEIR epidemic model is solved via Caputo FD (Kumar and Erturk 2023), the FDs based on the Mittag-Leffler kernels in the Liouville-Caputo concept has been regarded to investigate the conveyance of infectious diseases in a prey-predator system (Ghanbari 2023), a fractional order model for the transmission of Chlamydia is considered to explore the dynamics of the disease (Vellappandi et al. 2023), a time-fractional HIV/AIDS model is analyzed in the sense of Atangana-Baleanu Caputo derivative (Farman et al. 2023), a fractional order stochastic model based on the Lotka-Voltera system is presented (Ali and Khan 2023), fractional Maclaurin series is employed to solve various fractional differential equations that arise in physics and engineering (Alquran 2023), an iterative method by combining the generalized power series and artificial neural networks is proposed to solve certain fractal-fractional differential equations (Shloof 2023), a fractional order mathematical model governing meningitis is presented and the dynamics of the disease is studied (Peter et al. 2022), a new defined Liouville–Caputo fractional conformable derivative is considered through modeling some real world problems (Ozarslan et al. 2019).

The fractional Sturm-Liouville problems which have important applications in science, engineering and mathematics have analyzed by Ercan (2020; 2022) and Bas et al. (2021). The conformable Dirac system with separated boundary conditions are studied by Ercan and Bas (2021). These researches underline the efficiency of

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fractional calculus in modeling real world problems. The FD is one of the main notions in this area. Therefore several definitions for FDs such as Riemann-Liouville, Caputo and Grünwald-Letnikov derivatives are reported which are generalizations of classical derivative. On the other hand these FDs have some disadvantages in applications. For instance, Riemann-Liouville derivative of a constant does not result with zero or the Caputo derivative needs much requirement of regularity for differentiability (Atangana 2017).

Alternative FD definitions are presented to eliminate these drawbacks which have compatible properties with the traditional derivative. Within this framework, Khalil et al. (2014) have introduced a new conformable FD. Atangana (2015) has also defined a new local derivative by motivating the study of Khalil et al. (2014) and named it beta derivative or ACD.

In this paper, a physically significant equation called modified δ-equation in Eq. (1) is considered (Jawarneh 2023, Zhang et al. 2023, Fang et al. 2022, Ganji 2008, Wazwaz 2007):

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}-\frac{\partial}{\partial t}\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)+(\delta+1)v^2(x,t)\frac{\partial v(x,t)}{\partial x}$$
$$-\delta\frac{\partial v(x,t)}{\partial x}\frac{\partial^2 v(x,t)}{\partial x^2}-v(x,t)\frac{\partial^3 v(x,t)}{\partial x^3}=0. \quad (1)$$

In Eq. (1) setting $\delta=2$ and $\delta=3$ gives the fractional mCH and the mDP equations, respectively as follows,

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}-\frac{\partial}{\partial t}\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)+3v^2(x,t)\frac{\partial v(x,t)}{\partial x}$$
$$-2\frac{\partial v(x,t)}{\partial x}\frac{\partial^2 v(x,t)}{\partial x^2}-v(x,t)\frac{\partial^3 v(x,t)}{\partial x^3}=0, \quad (2)$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial}{\partial t}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)+4u^2(x,t)\frac{\partial u(x,t)}{\partial x}$$
$$-3\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}-u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}=0. \quad (3)$$

In Eq. (3) $u(x,t)$ notation is used instead of $v(x,t)$ to present the solutions of these equations distinguishably. Eq. (2) and Eq. (3) are able to characterize the nonlinear features of dispersive waves. Hence, this family of equations has been an issue for various studies. Veeresha and Prakasha (2020) have applied the q-homotopy analysis transform method; Zhang et al. (2023) used the Aboodh Adomian decomposition and homotopy perturbation transform methods; a method mixing the Elzaki transform, homotopy perturbation method and Adomian decomposition method has been applied by Alesemi (2023); Singh and Gupta (2022) have employed q-homotopy analysis generalized transform method and homotopy perturbation generalized transform method; Alquran et al. (2021) have exhibited new solutions of these equations via Kudryashov-expansion method and the sech-csch function method. Khatun and Akbar (2024) have applied the $(G'/G\,,1/G)$-expansion method for beta time-fractional mCH and mDP equations.

In this study an analytical method is proposed which is not applied before for Eqs. (2)-(3). This paper aims to obtain the new exact wave solutions of Eqs. (2)-(3) with ACD via $(m+1/G')$-expansion method. For this purpose the structure of the present research is as follows: In Section 2 the definition of ACD together with some properties and the steps of the proposed method are given. In Section 3 the $(m+1/G')$-expansion method is applied to the time-fractional mCH and mDP equations. In Section 4 the graphical results are presented and the obtained solutions are compared with the ones in the literature. Finally in Section 5 the conclusion is given.

2. Materials and Methods

2.1. Basic definitions

In this subsection the basis about ACD is given that will be used in the background of the research.

Definition 1: Let $\psi:[0,\infty)\to\mathbb{R}$ be a function. Then, the β-derivative of ψ is defined as (Atangana 2015)

$${}_0^A D_t^\beta \psi(t)=\lim_{h\to 0}\frac{\psi\left(t+h\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-\psi(t)}{h}, \quad (4)$$

where $0<\beta\le 1$ and $\Gamma(.)$ is the gamma function.
Atangana (2015) has introduced the properties of this derivative as follows,

1. ${}_0^A D_t^\beta(a\psi(t)+b\Phi(t))=a\,{}_0^A D_t^\beta(\psi(t))+b\,{}_0^A D_t^\beta(b\Phi(t)),\forall a,b\in\mathbb{R},$
2. ${}_0^A D_t^\beta(c)=0,\forall c\in\mathbb{R},$
3. ${}_0^A D_t^\beta(\psi(t)\Phi(t))=\Phi(t)\,{}_0^A D_t^\beta(\psi(t))+\psi(t)\,{}_0^A D_t^\beta(\Phi(t)),$
4. ${}_0^A D_t^\beta\left(\frac{\psi(t)}{\Phi(t)}\right)=\frac{\Phi(t)\,{}_0^A D_t^\beta(\psi(t))-\psi(t)\,{}_0^A D_t^\beta(\Phi(t))}{(\Phi(t))^2},\Phi(t)\neq 0,$
5. ${}_0^A D_t^\beta(\psi(\Phi(t)))=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\Phi'(t)\psi'(\Phi(t)),$

where ψ,Φ are β differentiable functions. The proofs of these items can be found in (Atangana 2015).

2.2 $(m+1/G')$-expansion method

Assume a time fractional NLPDE as in Eq. (5),

$$F\left(u,u_x,D_t^\beta u,u_{xx},\dots\right)=0, 0<\beta\le 1, \quad (5)$$

where F is a polynomial of u. Eq. (5) is converted into a NLODE as in Eq. (7) with the transformation below,

$$u(x,t)=U(\eta),\eta=x-\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta, \quad (6)$$

then one gets,

$$G(U, U', U'', \dots) = 0, \quad (7)$$

where $' = \frac{d}{d\eta}$. The proposed method suggests a solution for Eq.(7) as in the form,

$$U(\eta) = \sum_{i=-M}^{M} a_i \left(m + \frac{1}{G'}\right)^i, \quad (8)$$

where $a_i, (i = 0, \pm 1, \pm 2, \dots \pm M)$, m are constants. M will be determined by regarding homogeneous balance between the nonlinear term and the highest order derivative. The function $G(\eta)$ in Eq. (8) satisfies the differential equation below,

$$G'' + (\lambda + m\mu)G' + \mu = 0. \quad (9)$$

The general solution of Eq. (9) can be found easily as in the form,

$$G = -\frac{\mu\eta}{\lambda + 2m\mu} - \frac{A_1}{\lambda + 2m\mu} e^{-(\lambda + 2m\mu)\eta} + A_2, \quad (10)$$

where A_1, A_2 are constants. Then the term $1/G'$ in Eq. (8) will be determined from Eq. (10).

The suggested solution in Eq. (8) is put into Eq. (7) by considering the term $1/G'$ and the coefficients of the term $\left(m + \frac{1}{G'}\right)^i$ are set to zero to obtain an algebraic equation system for the parameters $a_i (i = 0, \pm 1, \pm 2, \dots, M), \lambda, m, \mu, A_1$ and ω. Evaluating these parameters via Mathematica software, the exact wave solutions of Eq. (5) are revealed.

3. Applications of the $(m + 1/G')$-Expansion Method

In this section the $(m + 1/G')$-expansion method is applied to Eq. (2) and Eq. (3) to obtain the exact wave solutions.

3.1. Solutions of the time-fractional mCH equation

Employing the traveling wave transform in Eq. (6), the mCH equation in Eq. (2) becomes,

$$-\omega V' + \omega V''' + 3V^2 V' - 2V'V'' - VV''' = 0, \quad (11)$$

where $v(x, t) = V(\eta)$. Integrating Eq. (11) with zero integration constant Eq. (12) is obtained as follows,

$$-\omega V + \omega V'' + V^3 - VV'' - \frac{1}{2}(V')^2 = 0. \quad (12)$$

Balancing the terms VV'' and V^3 in Eq. (12) gives $M = 2$ which will be used in Eq. (8) to construct the solution of Eq. (12) as follows,

$$V(\eta) = \sum_{i=-2}^{2} a_i \left(m + \frac{1}{G'}\right)^i. \quad (13)$$

Putting Eq. (13) into Eq. (12) and equating the coefficients of the terms $\left(m + \frac{1}{G'}\right)^i$ to zero gives the equations below,

$\left(m + \frac{1}{G'}\right)^{-6}: -8m^2\lambda^2 a_{-2}^2 - 16m^3\lambda\mu a_{-2}^2 - 8m^4\mu^2 a_{-2}^2 + a_{-2}^3 = 0,$

$\left(m + \frac{1}{G'}\right)^{-5}: 14m\lambda^2 a_{-2}^2 + 14m^2\lambda\mu a_{-2}^2 - 10m^2\lambda^2 a_{-2}a_{-1} - 20m^3\lambda\mu a_{-2}a_{-1} - 10m^4\mu^2 a_{-2}a_{-1} + 3a_{-2}^2 a_{-1} = 0,$

$\left(m + \frac{1}{G'}\right)^{-4}: 6m^2 w\lambda^2 a_{-2} + 12m^3 w\lambda\mu a_{-2} + 6m^4 w\mu^2 a_{-2} - 6\lambda^2 a_{-2}^2 + 12m\lambda\mu a_{-2}^2 + 12m^2\mu^2 a_{-2}^2 + 17m\lambda^2 a_{-2}a_{-1} + 17m^2\lambda\mu a_{-2}a_{-1} - \frac{5}{2}m^2\lambda^2 a_{-1}^2 - 5m^3\lambda\mu a_{-1}^2 - \frac{5}{2}m^4\mu^2 a_{-1}^2 + 3a_{-2}a_{-1}^2 - 6m^2\lambda^2 a_{-2}a_0 - 12m^3\lambda\mu a_{-2}a_0 - 6m^4\mu^2 a_{-2}a_0 + 3a_{-2}^2 a_0 = 0,$

$\left(m + \frac{1}{G'}\right)^{-3}: -10mw\lambda^2 a_{-2} - 10m^2 w\lambda\mu a_{-2} - 10\lambda\mu a_{-2}^2 + 2m^2 w\lambda^2 a_{-1} + 4m^3 w\lambda\mu a_{-1} + 2m^4 w\mu^2 a_{-1} - 7\lambda^2 a_{-2}a_{-1} + 14m\lambda\mu a_{-2}a_{-1} + 14m^2\mu^2 a_{-2}a_{-1} + 4m\lambda^2 a_{-1}^2 + 4m^2\lambda\mu a_{-1}^2 + a_{-1}^3 + 10m\lambda^2 a_{-2}a_0 + 10m^2\lambda\mu a_{-2}a_0 - 2m^2\lambda^2 a_{-1}a_0 - 4m^3\lambda\mu a_{-1}a_0 - 2m^4\mu^2 a_{-1}a_0 + 6a_{-2}a_{-1}a_0 - 4m^2\lambda^2 a_{-2}a_1 - 8m^3\lambda\mu a_{-2}a_1 - 4m^4\mu^2 a_{-2}a_1 + 3a_{-2}^2 a_1 = 0,$

$\left(m + \frac{1}{G'}\right)^{-2}: -wa_{-2} + 4w\lambda^2 a_{-2} - 8mw\lambda\mu a_{-2} - 8m^2 w\mu^2 a_{-2} - 4\mu^2 a_{-2}^2 - 3mw\lambda^2 a_{-1} - 3m^2 w\lambda\mu a_{-1} - 11\lambda\mu a_{-2}a_{-1} - \frac{3}{2}\lambda^2 a_{-1}^2 + 3m\lambda\mu a_{-1}^2 + 3m^2\mu^2 a_{-1}^2 - 4\lambda^2 a_{-2}a_0 + 8m\lambda\mu a_{-2}a_0 + 8m^2\mu^2 a_{-2}a_0 + 3m\lambda^2 a_{-1}a_0 + 3m^2\lambda\mu a_{-1}a_0 + 3a_{-1}^2 a_0 + 3a_{-2}a_0^2 + 7m\lambda^2 a_{-2}a_1 + 7m^2\lambda\mu a_{-2}a_1 - m^2\lambda^2 a_{-1}a_1 - 2m^3\lambda\mu a_{-1}a_1 - m^4\mu^2 a_{-1}a_1 + 6a_{-2}a_{-1}a_1 - 4m^2\lambda^2 a_{-2}a_2 - 8m^3\lambda\mu a_{-2}a_2 - 4m^4\mu^2 a_{-2}a_2 + 3a_{-2}^2 a_2 = 0,$

$\left(m + \frac{1}{G'}\right)^{-1}: 6w\lambda\mu a_{-2} - wa_{-1} + w\lambda^2 a_{-1} - 2mw\lambda\mu a_{-1} - 2m^2 w\mu^2 a_{-1} - 4\mu^2 a_{-2}a_{-1} - 2\lambda\mu a_{-1}^2 - 6\lambda\mu a_{-2}a_0 - \lambda^2 a_{-1}a_0 + 2m\lambda\mu a_{-1}a_0 + 2m^2\mu^2 a_{-1}a_0 + 3a_{-1}a_0^2 - 3\lambda^2 a_{-2}a_1 + 6m\lambda\mu a_{-2}a_1 + 6m^2\mu^2 a_{-2}a_1 + 2m\lambda^2 a_{-1}a_1 + 2m^2\lambda\mu a_{-1}a_1 + 3a_{-1}^2 a_1 + 6a_{-2}a_0 a_1 +$

$8m\lambda^2 a_{-2}a_2 + 8m^2\lambda\mu a_{-2}a_2 - 2m^2\lambda^2 a_{-1}a_2 - 4m^3\lambda\mu a_{-1}a_2 - 2m^4\mu^2 a_{-1}a_2 + 6a_{-2}a_{-1}a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^0 : 2w\mu^2 a_{-2} + w\lambda\mu a_{-1} - \frac{1}{2}\mu^2 a_{-1}^2 - wa_0 - 2\mu^2 a_{-2}a_0 - \lambda\mu a_{-1}a_0 + a_0^3 - mw\lambda^2 a_1 - m^2 w\lambda\mu a_1 - 5\lambda\mu a_{-2}a_1 - \lambda^2 a_{-1}a_1 + 2m\lambda\mu a_{-1}a_1 + 2m^2\mu^2 a_{-1}a_1 + m\lambda^2 a_0 a_1 + m^2\lambda\mu a_0 a_1 + 6a_{-1}a_0 a_1 - \frac{1}{2}m^2\lambda^2 a_1^2 - m^3\lambda\mu a_1^2 - \frac{1}{2}m^4\mu^2 a_1^2 + 3a_{-2}a_1^2 + 2m^2 w\lambda^2 a_2 + 4m^3 w\lambda\mu a_2 + 2m^4 w\mu^2 a_2 - 4\lambda^2 a_{-2}a_2 + 8m\lambda\mu a_{-2}a_2 + 8m^2\mu^2 a_{-2}a_2 + 5m\lambda^2 a_{-1}a_2 + 5m^2\lambda\mu a_{-1}a_2 + 3a_{-1}^2 a_2 - 2m^2\lambda^2 a_0 a_2 - 4m^3\lambda\mu a_0 a_2 - 2m^4\mu^2 a_0 a_2 + 6a_{-2}a_0 a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^1 : -wa_1 + w\lambda^2 a_1 - 2mw\lambda\mu a_1 - 2m^2 w\mu^2 a_1 - 2\mu^2 a_{-2}a_1 - 2\lambda\mu a_{-1}a_1 - \lambda^2 a_0 a_1 + 2m\lambda\mu a_0 a_1 + 2m^2\mu^2 a_0 a_1 + 3a_0^2 a_1 + 2m\lambda^2 a_1^2 + 2m^2\lambda\mu a_1^2 + 3a_{-1}a_1^2 - 6mw\lambda^2 a_2 - 6m^2 w\lambda\mu a_2 - 8\lambda\mu a_{-2}a_2 - 3\lambda^2 a_{-1}a_2 + 6m\lambda\mu a_{-1}a_2 + 6m^2\mu^2 a_{-1}a_2 + 6m\lambda^2 a_0 a_2 + 6m^2\lambda\mu a_0 a_2 + 6a_{-1}a_0 a_2 - 4m^2\lambda^2 a_1 a_2 - 8m^3\lambda\mu a_1 a_2 - 4m^4\mu^2 a_1 a_2 + 6a_{-2}a_1 a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^2 : 3\omega\lambda\mu a_1 - \mu^2 a_{-1}a_1 - 3\lambda\mu a_0 a_1 - \frac{3}{2}\lambda^2 a_1^2 + 3m\lambda\mu a_1^2 + 3m^2\mu^2 a_1^2 + 3a_0 a_1^2 - \omega a_2 + 4\omega\lambda^2 a_2 - 8m\omega\lambda\mu a_2 - 8m^2\omega\mu^2 a_2 - 4\mu^2 a_{-2}a_2 - 7\lambda\mu a_{-1}a_2 - 4\lambda^2 a_0 a_2 + 8m\lambda\mu a_0 a_2 + 8m^2\mu^2 a_0 a_2 + 3a_0^2 a_2 + 11m\lambda^2 a_1 a_2 + 11m^2\lambda\mu a_1 a_2 + 6a_{-1}a_1 a_2 - 4m^2\lambda^2 a_2^2 - 8m^3\lambda\mu a_2^2 - 4m^4\mu^2 a_2^2 + 3a_{-2}a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^3 : 2\omega\mu^2 a_1 - 2\mu^2 a_0 a_1 - 4\lambda\mu a_1^2 + a_1^3 + 10\omega\lambda\mu a_2 - 4\mu^2 a_{-1}a_2 - 10\lambda\mu a_0 a_2 - 7\lambda^2 a_1 a_2 + 14m\lambda\mu a_1 a_2 + 14m^2\mu^2 a_1 a_2 + 6a_0 a_1 a_2 + 10m\lambda^2 a_2^2 + 10m^2\lambda\mu a_2^2 + 3a_{-1}a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^4 : -(5/2)\mu^2 a_1^2 + 6\omega\mu^2 a_2 - 6\mu^2 a_0 a_2 - 17\lambda\mu a_1 a_2 + 3a_1^2 a_2 - 6\lambda^2 a_2^2 + 12m\lambda\mu a_2^2 + 12m^2\mu^2 a_2^2 + 3a_0 a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^5 : -10\mu^2 a_1 a_2 - 14\lambda\mu a_2^2 + 3a_1 a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^6 : -8\mu^2 a_2^2 + a_2^3 = 0.$

Solving this system of equations gives the cases and the corresponding solutions of Eq. (2) which are presented as in the following where η is stated as in Eq. (6).

Case-1:

$a_{-2} = 2m^2(-1+\lambda)^2, a_{-1} = -4m(-1+\lambda)\lambda,$

$a_0 = 2(-1+\lambda^2), a_1 = 0, a_2 = 0,$

$\mu = -\frac{1+\lambda}{2m}, \omega = 2.$

The corresponding solution of Eq. (2) is

$$v_1 = -2 + \lambda^2 + \frac{2m^2(-1+\lambda)^2}{\left(m - \frac{1}{-A1e^{\eta} + \frac{1+\lambda}{2m}}\right)^2} - \frac{4m(-1+\lambda)\lambda}{m - \frac{1}{-A1e^{\eta} + \frac{1+\lambda}{2m}}}. \quad (14)$$

Case-2:

$a_{-2} = 0, a_{-1} = 0, a_0 = 2(-1+\lambda^2), \mu = -\frac{1+\lambda}{2m}, a_1 = -\frac{4\lambda(1+\lambda)}{m}, a_2 = \frac{2(1+\lambda)^2}{m^2}, \omega = 2.$

The corresponding solution of Eq. (2) is

$$v_2 = 2(-1+\lambda^2) - \frac{4\lambda(1+\lambda)\left(m - \frac{1}{-A1e^{\eta} + \frac{1+\lambda}{2m}}\right)}{m} + \frac{2(1+\lambda)^2\left(m - \frac{1}{-A1e^{\eta} + \frac{1+\lambda}{2m}}\right)^2}{m^2}. \quad (15)$$

Case-3:

$a_{-2} = 2m^2(-i+\lambda)^2, a_{-1} = -4m\lambda(-i+\lambda), a_0 = 1 + 2\lambda^2, a_1 = 0, a_2 = 0, \mu = -\frac{i+\lambda}{2m}, \omega = 1.$

The corresponding solution of Eq. (2) is

$$v_3 = 1 + 2\lambda^2 + \frac{2m^2(-i+\lambda)^2}{\left(m - \frac{i}{-iA1e^{i\eta} + \frac{i+\lambda}{2m}}\right)^2} - \frac{4m\lambda(-i+\lambda)}{m - \frac{i}{-iA1e^{i\eta} + \frac{i+\lambda}{2m}}}. \quad (16)$$

Case-4:

$a_{-2} = 0, a_{-1} = 0, a_0 = 1 + 2\lambda^2, a_1 = -\frac{4\lambda(i+\lambda)}{m}, a_2 = \frac{2(i+\lambda)^2}{m^2}, \mu = -\frac{i+\lambda}{2m}, \omega = 1.$

The corresponding solution of Eq. (2) is

$$v_4 = 1 + 2\lambda^2 - \frac{4\lambda(i+\lambda)\left(m - \frac{i}{-iA1e^{i\eta} + \frac{i+\lambda}{2m}}\right)}{m} + \frac{2(i+\lambda)^2\left(m - \frac{i}{-iA1e^{i\eta} + \frac{i+\lambda}{2m}}\right)^2}{m^2}. \quad (17)$$

Case-5:

$a_{-2} = 2m^2(i+\lambda)^2, a_{-1} = -4m\lambda(i+\lambda), a_0 = 1 + 2\lambda^2, a_1 = 0, a_2 = 0, \mu = -\frac{-i+\lambda}{2m}, \omega = 1.$

The corresponding solution of Eq. (2) is

$$v_5 = 1 + 2\lambda^2 + \frac{2m^2(i+\lambda)^2}{\left(m+\frac{i}{iA1e^{-i\eta}+\frac{-i+\lambda}{2m}}\right)^2} - \frac{4m\lambda(i+\lambda)}{m+\frac{i}{iA1e^{-i\eta}+\frac{-i+\lambda}{2m}}}. \quad (18)$$

Case-6:

$a_{-2} = 0, a_{-1} = 0, a_0 = 1 + 2\lambda^2, a_1 = -\frac{4\lambda(-i+\lambda)}{m}, a_2 = \frac{2(-i+\lambda)^2}{m^2}, \mu = -\frac{-i+\lambda}{2m}, \omega = 1.$

The corresponding solution of Eq. (2) is

$$v_6 = 1 + 2\lambda^2 - \frac{4\lambda(-i+\lambda)\left(m+\frac{i}{iA1e^{-i\eta}+\frac{-i+\lambda}{2m}}\right)}{m} + \frac{2(-i+\lambda)^2\left(m+\frac{i}{iA1e^{-i\eta}+\frac{-i+\lambda}{2m}}\right)^2}{m^2}. \quad (19)$$

Case-7:

$a_{-2} = 2m^2(1+\lambda)^2, a_{-1} = -4m\lambda(1+\lambda), a_0 = 2(-1+\lambda^2), a_1 = 0, a_2 = 0, \mu = -\frac{-1+\lambda}{2m}, \omega = 2.$

The corresponding solution of Eq. (2) is

$$v_7 = -\frac{4m\lambda(1+\lambda)}{m+\frac{1}{A1e^{-\eta}+\frac{-1+\lambda}{2m}}} + \frac{2m^2(1+\lambda)^2}{\left(m+\frac{1}{A1e^{-\eta}+\frac{-1+\lambda}{2m}}\right)^2} + 2(-1+\lambda^2). \quad (20)$$

Case-8:

$a_{-2} = 0, a_{-1} = 0, a_0 = 2(-1+\lambda^2), a_1 = -\frac{4(-1+\lambda)\lambda}{m}, a_2 = \frac{2(-1+\lambda)^2}{m^2}, \mu = -\frac{-1+\lambda}{2m}, \omega = 2.$

The corresponding solution of Eq. (2) is

$$v_8 = \frac{2\left(m+\frac{1}{A1e^{-\eta}+\frac{-1+\lambda}{2m}}\right)^2(-1+\lambda)^2}{m^2} - \frac{4\left(m+\frac{1}{A1e^{-\eta}+\frac{-1+\lambda}{2m}}\right)(-1+\lambda)\lambda}{m} + 2(-1+\lambda^2). \quad (21)$$

3.2. Solutions of the time-fractional mDP equation

In this subsection the second equation of the modified δ-equation in Eq. (3) is converted into the following NLODE by using the transform in Eq. (6),

$$-\omega U' + \omega U''' + 4U^2U' - 3U'U'' - UU''' = 0, \quad (22)$$

where $u(x,t) = U(\eta)$. Integrating Eq. (22) by setting the integration constant zero gives,

$$-\omega U + \omega U'' + \frac{4}{3}U^3 - UU'' - (U')^2 = 0. \quad (23)$$

The terms UU'' and U^3 are considered for the balancing principle to reach $M = 2$. Then the solution of Eq. (23) will be,

$$U(\eta) = \sum_{i=-2}^{2} a_i\left(m + \frac{1}{G'}\right)^i. \quad (24)$$

The algebraic equation system is obtained when Eq. (24) is substituted into Eq. (23) as follows,

$\left(m+\frac{1}{G'}\right)^{-6}: -10m^2\lambda^2a_{-2}^2 - 20m^3\lambda\mu a_{-2}^2 - 10m^4\mu^2a_{-2}^2 + \frac{4a_{-2}^3}{3} = 0,$

$\left(m+\frac{1}{G'}\right)^{-5}: 18m\lambda^2a_{-2}^2 + 18m^2\lambda\mu a_{-2}^2 - 12m^2\lambda^2a_{-2}a_{-1} - 24m^3\lambda\mu a_{-2}a_{-1} - 12m^4\mu^2a_{-2}a_{-1} + 4a_{-2}^2a_{-1} = 0,$

$\left(m+\frac{1}{G'}\right)^{-4}: 6m^2w\lambda^2a_{-2} + 12m^3w\lambda\mu a_{-2} + 6m^4w\mu^2a_{-2} - 8\lambda^2a_{-2}^2 + 16m\lambda\mu a_{-2}^2 + 16m^2\mu^2a_{-2}^2 + 21m\lambda^2a_{-2}a_{-1} + 21m^2\lambda\mu a_{-2}a_{-1} - 3m^2\lambda^2a_{-1}^2 - 6m^3\lambda\mu a_{-1}^2 - 3m^4\mu^2a_{-1}^2 + 4a_{-2}a_{-1}^2 - 6m^2\lambda^2a_{-2}a_0 - 12m^3\lambda\mu a_{-2}a_0 - 6m^4\mu^2a_{-2}a_0 + 4a_{-2}^2a_0 = 0,$

$\left(m+\frac{1}{G'}\right)^{-3}: -10mw\lambda^2a_{-2} - 10m^2w\lambda\mu a_{-2} - 14\lambda\mu a_{-2}^2 + 2m^2w\lambda^2a_{-1} + 4m^3w\lambda\mu a_{-1} + 2m^4w\mu^2a_{-1} - 9\lambda^2a_{-2}a_{-1} + 18m\lambda\mu a_{-2}a_{-1} + 18m^2\mu^2a_{-2}a_{-1} + 5m\lambda^2a_{-1}^2 + 5m^2\lambda\mu a_{-1}^2 + \frac{4a_{-1}^3}{3} + 10m\lambda^2a_{-2}a_0 + 10m^2\lambda\mu a_{-2}a_0 - 2m^2\lambda^2a_{-1}a_0 - 4m^3\lambda\mu a_{-1}a_0 - 2m^4\mu^2a_{-1}a_0 + 8a_{-2}a_{-1}a_0 - 2m^2\lambda^2a_{-2}a_1 - 4m^3\lambda\mu a_{-2}a_1 - 2m^4\mu^2a_{-2}a_1 + 4a_{-2}^2a_1 = 0,$

$\left(m+\frac{1}{G'}\right)^{-2}: -wa_{-2} + 4w\lambda^2a_{-2} - 8mw\lambda\mu a_{-2} - 8m^2w\mu^2a_{-2} - 6\mu^2a_{-2}^2 - 3mw\lambda^2a_{-1} - 3m^2w\lambda\mu a_{-1} - 15\lambda\mu a_{-2}a_{-1} - 2\lambda^2a_{-1}^2 + 4m\lambda\mu a_{-1}^2 + 4m^2\mu^2a_{-1}^2 - 4\lambda^2a_{-2}a_0 + 8m\lambda\mu a_{-2}a_0 + 8m^2\mu^2a_{-2}a_0 + 3m\lambda^2a_{-1}a_0 + 3m^2\lambda\mu a_{-1}a_0 + 4a_{-1}^2a_0 + 4a_{-2}a_0^2 + 3m\lambda^2a_{-2}a_1 + 3m^2\lambda\mu a_{-2}a_1 + 8a_{-2}a_{-1}a_1 + 4a_{-2}^2a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^{-1}: 6w\lambda\mu a_{-2} - wa_{-1} + w\lambda^2a_{-1} - 2mw\lambda\mu a_{-1} - 2m^2w\mu^2a_{-1} - 6\mu^2a_{-2}a_{-1} - 3\lambda\mu a_{-1}^2 - 6\lambda\mu a_{-2}a_0 - \lambda^2a_{-1}a_0 + 2m\lambda\mu a_{-1}a_0 + 2m^2\mu^2a_{-1}a_0 +$

$4a_{-1}a_0^2 - \lambda^2 a_{-2}a_1 + 2m\lambda\mu a_{-2}a_1 + 2m^2\mu^2 a_{-2}a_1 + 4a_{-1}^2 a_1 + 8a_{-2}a_0a_1 + 8a_{-2}a_{-1}a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^0: 2w\mu^2 a_{-2} + w\lambda\mu a_{-1} - \mu^2 a_{-1}^2 - wa_0 - 2\mu^2 a_{-2}a_0 - \lambda\mu a_{-1}a_0 + (4a_0^3)/3 - mw\lambda^2 a_1 - m^2 w\lambda\mu a_1 - \lambda\mu a_{-2}a_1 + m\lambda^2 a_0a_1 + m^2\lambda\mu a_0a_1 + 8a_{-1}a_0a_1 - m^2\lambda^2 a_1^2 - 2m^3\lambda\mu a_1^2 - m^4\mu^2 a_1^2 + 4a_{-2}a_1^2 + 2m^2 w\lambda^2 a_2 + 4m^3 w\lambda\mu a_2 + 2m^4 w\mu^2 a_2 + m\lambda^2 a_{-1}a_2 + m^2\lambda\mu a_{-1}a_2 + 4a_{-1}^2 a_2 - 2m^2\lambda^2 a_0a_2 - 4m^3\lambda\mu a_0a_2 - 2m^4\mu^2 a_0a_2 + 8a_{-2}a_0a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^1: -wa_1 + w\lambda^2 a_1 - 2mw\lambda\mu a_1 - 2m^2 w\mu^2 a_1 - \lambda^2 a_0a_1 + 2m\lambda\mu a_0a_1 + 2m^2\mu^2 a_0a_1 + 4a_0^2 a_1 + 3m\lambda^2 a_1^2 + 3m^2\lambda\mu a_1^2 + 4a_{-1}a_1^2 - 6mw\lambda^2 a_2 - 6m^2 w\lambda\mu a_2 - \lambda^2 a_{-1}a_2 + 2m\lambda\mu a_{-1}a_2 + 2m^2\mu^2 a_{-1}a_2 + 6m\lambda^2 a_0a_2 + 6m^2\lambda\mu a_0a_2 + 8a_{-1}a_0a_2 - 6m^2\lambda^2 a_1a_2 - 12m^3\lambda\mu a_1a_2 - 6m^4\mu^2 a_1a_2 + 8a_{-2}a_1a_2 = 0,$

$\left(m+\frac{1}{G'}\right)^2: 3w\lambda\mu a_1 - 3\lambda\mu a_0a_1 - 2\lambda^2 a_1^2 + 4m\lambda\mu a_1^2 + 4m^2\mu^2 a_1^2 + 4a_0a_1^2 - wa_2 + 4w\lambda^2 a_2 - 8mw\lambda\mu a_2 - 8m^2 w\mu^2 a_2 - 3\lambda\mu a_{-1}a_2 - 4\lambda^2 a_0a_2 + 8m\lambda\mu a_0a_2 + 8m^2\mu^2 a_0a_2 + 4a_0^2 a_2 + 15m\lambda^2 a_1a_2 + 15m^2\lambda\mu a_1a_2 + 8a_{-1}a_1a_2 - 6m^2\lambda^2 a_2^2 - 12m^3\lambda\mu a_2^2 - 6m^4\mu^2 a_2^2 + 4a_{-2}a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^3: 2w\mu^2 a_1 - 2\mu^2 a_0a_1 - 5\lambda\mu a_1^2 + (4a_1^3)/3 + 10w\lambda\mu a_2 - 2\mu^2 a_{-1}a_2 - 10\lambda\mu a_0a_2 - 9\lambda^2 a_1a_2 + 18m\lambda\mu a_1a_2 + 18m^2\mu^2 a_1a_2 + 8a_0a_1a_2 + 14m\lambda^2 a_2^2 + 14m^2\lambda\mu a_2^2 + 4a_{-1}a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^4: -3\mu^2 a_1^2 + 6w\mu^2 a_2 - 6\mu^2 a_0a_2 - 21\lambda\mu a_1a_2 + 4a_1^2 a_2 - 8\lambda^2 a_2^2 + 16m\lambda\mu a_2^2 + 16m^2\mu^2 a_2^2 + 4a_0a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^5: -12\mu^2 a_1a_2 - 18\lambda\mu a_2^2 + 4a_1a_2^2 = 0,$

$\left(m+\frac{1}{G'}\right)^6: -10\mu^2 a_2^2 + (4a_2^3)/3 = 0.$

The parameters $m, \mu, \omega, a_{-2}, a_{-1}, a_0, a_1, a_2, \lambda$ are evaluated by solving this system with the assist of Mathematica. The following results including different cases and the corresponding solutions are obtained where $\eta = x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$.

Case-1:

$a_{-2} = \frac{3}{32}m^2\gamma_1, a_{-1} = -\frac{3}{8}m\lambda\left(\sqrt{-125-5i\sqrt{15}} + 10\lambda\right), a_0 = \frac{1}{32}\left(39 - i\sqrt{15} + 60\lambda^2\right), a_1 = 0, a_2 = 0, \mu = -\frac{\sigma_1+\lambda}{2m}, \omega = \frac{1}{8}\left(11 + 3i\sqrt{15}\right),$

where $\sigma_1 = -\lambda + \frac{1}{10}(-\sqrt{-125-5i\sqrt{15}} + 10\lambda)$ and $\gamma_1 = -25 - i\sqrt{15} + 4\lambda\left(\sqrt{-125-5i\sqrt{15}} + 5\lambda\right)$.

The corresponding solution of Eq. (3) is

$$u_1 = \frac{1}{32}(39 - i\sqrt{15} + 60\lambda^2) + \frac{3m^2\gamma_1}{32\left(m - \frac{\sigma_1}{+\frac{\sigma_1+\lambda}{2m} - \mathrm{A1}e^{\eta\sigma_1}\sigma_1}\right)^2} - \frac{30m\lambda(\sigma_2+\lambda)}{8m - \frac{8\sigma_1}{+\frac{\sigma_1+\lambda}{2m} - \mathrm{A1}e^{\eta\sigma_1}\sigma_1}}, \quad (25)$$

where $\sigma_2 = \lambda + \frac{1}{10}(-\sqrt{-125-5i\sqrt{15}} - 10\lambda)$.

Case-2:

$a_{-2} = 0, a_{-1} = 0, a_0 = \frac{1}{32}\left(39 - i\sqrt{15} + 60\lambda^2\right), a_1 = \frac{15(-\sigma_1-\lambda)}{4m}, a_2 = \frac{3\gamma_2}{32m^2}, \mu = \frac{-\sigma_1-\lambda}{2m}, \omega = \frac{1}{8}(11 + 3i\sqrt{15}),$

where $\gamma_2 = -25 - i\sqrt{15} - 4(\sqrt{-125-5i\sqrt{15}} - 5\lambda)\lambda$.

The corresponding solution of Eq. (3) is

$$u_2 = \frac{1}{32}(39 - i\sqrt{15} + 60\lambda^2 + \frac{-30(\sigma_1+\lambda)\lambda\left(m - \frac{\sigma_1}{+\frac{\sigma_1+\lambda}{2m} - \mathrm{A1}e^{\eta\sigma_1}\sigma_1}\right)}{8m} + \frac{3\gamma_2\left(m - \frac{\sigma_1}{+\frac{\sigma_1+\lambda}{2m} - \mathrm{A1}e^{\eta\sigma_1}\sigma_1}\right)^2}{32m^2}. \quad (26)$$

Case-3:

$a_{-2} = \frac{3}{32}m^2\gamma_1, a_{-1} = \frac{15}{4}m(-\sigma_1 - \lambda)\lambda, a_0 = \frac{1}{32}\left(39 - i\sqrt{15} + 60\lambda^2\right), a_1 = 0, a_2 = 0, \mu = \frac{\sigma_2-\lambda}{2m}, \omega = \frac{1}{8}(11 + 3i\sqrt{15}).$

The corresponding solution of Eq. (3) is

$$u_3 = \frac{1}{32}\left(39 - i\sqrt{15} + 60\lambda^2\right) + \frac{3m^2\gamma_2}{32\left(m + \frac{\sigma_2}{\mathrm{A1}e^{\eta\sigma_2}\sigma_2 + \frac{\lambda-\sigma_2}{2m}}\right)^2} - \frac{30m(\lambda+\sigma_1)\lambda}{8m + \frac{8(\lambda-\sigma_2)}{\mathrm{A1}e^{\eta\sigma_2}\sigma_2 + \frac{\lambda-\sigma_2}{2m}}}. \quad (27)$$

Case-4:

$a_{-2} = 0, a_{-1} = 0, a_0 = \frac{1}{32}\left(39 - i\sqrt{15} + 60\lambda^2\right), a_1 = -\frac{15\lambda(\lambda-\sigma_2)}{4m}, a_2 = \frac{3\gamma_1}{32m^2}, \mu = \frac{\sigma_2-\lambda}{2m}, \omega = \frac{1}{8}(11 + 3i\sqrt{15}).$

The corresponding solution of Eq. (3) is

$$u_4 = \frac{1}{32}(39 - i\sqrt{15} + 60\lambda^2) - \frac{30\lambda(\lambda-\sigma_2)(m+\frac{\sigma_2}{A1e^{-\eta\sigma_2}\sigma_2+\frac{\lambda-\sigma_2}{2m}})}{8m} + \frac{3\gamma_1}{32m^2}\left(m + \frac{\sigma_2}{A1e^{-\eta\sigma_2}\sigma_2+\frac{\lambda-\sigma_2}{2m}}\right)^2. \quad (28)$$

Case-5:

$a_{-2} = \frac{3}{32}m^2\gamma_3, a_{-1} = \frac{15}{4}m(\sigma_3 - \lambda)\lambda, a_0 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2), a_1 = 0, a_2 = 0, \mu = \frac{\sigma_4-\lambda}{20m}, \omega = \frac{1}{8}(11 - 3i\sqrt{15})$,

where $\sigma_3 = \frac{1}{10}\left(\sqrt{-125 + 5i\sqrt{15}} - 10\lambda\right) + \lambda$, $\sigma_4 = \frac{1}{10}\left(-\sqrt{-125 + 5i\sqrt{15})} - 10\lambda\right) + \lambda$ and $\gamma_3 = -25 + i\sqrt{15} - 4\left(\sqrt{5i(25i + \sqrt{15})} - 5\lambda\right)\lambda$.

The corresponding solution of Eq. (3) is

$$u_5 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2) + \frac{3m^2\gamma_3}{32(m+\frac{\sigma_4}{A1e^{-\eta\sigma_4}\sigma_4+\frac{\lambda-\sigma_4}{2m}})^2} + \frac{15m(\sigma_3-\lambda)\lambda}{4(m+\frac{\sigma_4}{A1e^{-\eta\sigma_4}\sigma_4+\frac{\lambda-\sigma_4}{2m}})}, \quad (29)$$

Case-6:

$a_{-2} = 0, a_{-1} = 0, a_0 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2), a_1 = -\frac{15\lambda(\lambda-\sigma_4)}{4m}, a_2 = \frac{3\gamma_4}{32m^2}, \mu = \frac{\sigma_4-\lambda}{20m}, \omega = \frac{1}{8}(11 - 3i\sqrt{15})$,

where $\gamma_4 = -25 + i\sqrt{15} + 4\lambda\left(\sqrt{5i(25i + \sqrt{15})} + 5\lambda\right)$.

The corresponding solution of Eq. (3) is

$$u_6 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2) - \frac{30\lambda(\lambda-\sigma_4)(m+\frac{\sigma_4}{A1e^{-\eta\sigma_4}\sigma_4+\frac{\lambda-\sigma_4}{2m}})}{8m} + \frac{3\gamma_4}{32m^2}\left(m + \frac{\sigma_4}{A1e^{-\eta\sigma_4}\sigma_4+\frac{\lambda-\sigma_4}{2m}}\right)^2. \quad (30)$$

Case-7:

$a_{-2} = \frac{3}{32}m^2\gamma_4, a_{-1} = \frac{15}{4}m\lambda(\sigma_4 - \lambda), a_0 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2), a_1 = 0, a_2 = 0, \mu = \frac{\sigma_3-\lambda}{2m}, \omega = \frac{1}{8}(11 - 3i\sqrt{15})$.

The corresponding solution of Eq. (3) is

$$u_7 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2) + \frac{3m^2\gamma_4}{32\left(m+\frac{\sigma_3}{-\frac{\sigma_3-\lambda}{2m}+A1e^{-\eta\sigma_3}\sigma_3}\right)^2} - \frac{30m\lambda(\lambda-\sigma_4)}{8m+\frac{8\sigma_3}{-\frac{\sigma_3-\lambda}{2m}+A1e^{-\eta\sigma_3}\sigma_3}}. \quad (31)$$

Case-8:

$a_{-2} = 0, a_{-1} = 0, a_0 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2), a_1 = \frac{15(\sigma_3-\lambda)\lambda}{4m}, a_2 = \frac{3\gamma_3}{32m^2}, \mu = \frac{\sigma_3-\lambda}{2m}, \omega = \frac{1}{8}(11 - 3i\sqrt{15})$.

The corresponding solution of Eq. (3) is

$$u_8 = \frac{1}{32}(39 + i\sqrt{15} + 60\lambda^2) + \frac{30(\sigma_3-\lambda)\lambda(m+\frac{\sigma_3}{-\frac{\sigma_3-\lambda}{2m}+A1e^{-\eta\sigma_3}\sigma_3})}{8m} + \frac{3\gamma_3}{32m^2}\left(m + \frac{\sigma_3}{-\frac{\sigma_3-\lambda}{2m}+A1e^{-\eta\sigma_3}\sigma_3}\right)^2. \quad (32)$$

Case-9:

$a_{-2} = \frac{15}{8}m^2(-1 + \lambda)^2, a_{-1} = -\frac{15}{4}m(-1 + \lambda)\lambda, a_0 = \frac{15}{8}(-1 + \lambda^2), a_1 = 0, a_2 = 0, \mu = -\frac{1+\lambda}{2m}, \omega = \frac{5}{2}$.

The corresponding solution of Eq. (3) is

$$u_9 = \frac{15}{8}(-1 + \lambda^2) + \frac{15m^2(-1+\lambda)^2}{8\left(m-\frac{1}{-A1e^{\eta}+\frac{1+\lambda}{2m}}\right)^2} - \frac{15m(-1+\lambda)\lambda}{4m-\frac{4}{-A1e^{\eta}+\frac{1+\lambda}{2m}}}. \quad (33)$$

Case-10:

$a_{-2} = 0, a_{-1} = 0, a_0 = \frac{15}{8}(-1 + \lambda^2), a_1 = -\frac{15\lambda(1+\lambda)}{4m}, a_2 = \frac{15(1+\lambda)^2}{8m^2}, \mu = -\frac{1+\lambda}{2m}, \omega = \frac{5}{2}$.

The corresponding solution of Eq. (3) is

$$u_{10} = \frac{15}{8}(-1 + \lambda^2) - \frac{15\lambda(1+\lambda)(m-\frac{1}{-A1e^{\eta}+\frac{1+\lambda}{2m}})}{4m} + \frac{15(1+\lambda)^2(m-\frac{1}{-A1e^{\eta}+\frac{1+\lambda}{2m}})^2}{8m^2}. \quad (34)$$

Case-11:

$a_{-2} = \frac{15}{8}m^2(1 + \lambda)^2, a_{-1} = -\frac{15}{4}m\lambda(1 + \lambda), a_0 = \frac{15}{8}(-1 + \lambda^2), a_1 = 0, a_2 = 0, \mu = -\frac{-1+\lambda}{2m}, \omega = \frac{5}{2}$.

The corresponding solution of Eq. (3) is

$$u_{11}=-\frac{15m\lambda(1+\lambda)}{4\left(m+\frac{1}{\mathrm{A1}e^{-\eta}+\frac{-1+\lambda}{2m}}\right)}+\frac{15m^2(1+\lambda)^2}{8\left(m+\frac{1}{\mathrm{A1}e^{-\eta}+\frac{-1+\lambda}{2m}}\right)^2}+\frac{15(-1+\lambda^2)}{8}. \quad (35)$$

Case-12:

$a_{-2}=0, a_{-1}=0, a_0=\frac{15}{8}(-1+\lambda^2), a_1=-\frac{15(-1+\lambda)\lambda}{4m}, a_2=\frac{15(-1+\lambda)^2}{8m^2}, \mu=-\frac{-1+\lambda}{2m}, \omega=\frac{5}{2}$.

The corresponding solution of Eq. (3) is

$$u_{12}=\frac{15\left(m+\frac{1}{\mathrm{A1}e^{-\eta}+\frac{-1+\lambda}{2m}}\right)^2(-1+\lambda)^2}{8m^2}-\frac{15\left(m+\frac{1}{\mathrm{A1}e^{-\eta}+\frac{-1+\lambda}{2m}}\right)(-1+\lambda)\lambda}{4m}+\frac{15}{8}(-1+\lambda^2). \quad (36)$$

4. Results and Discussions

The new traveling wave solutions of time-fractional mCH and mDP equations are offered above and the results are supported with 3D and 2D plots. The graphical delineation of the solutions in Eq. (14), Eq. (16), Eq. (20), Eq. (28), Eq. (33) and Eq. (34) is presented in Figs. 1-8 for some values of the parameters.

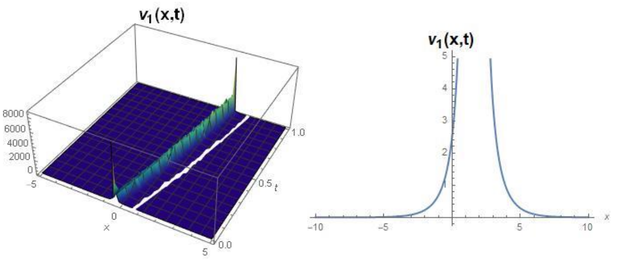

Figure 1. 3D and 2D graphs of Eq. (14) for $m=1, \lambda=2, \beta=0.5, \mathrm{A1}=15$ ($t=1$ for 2D plot).

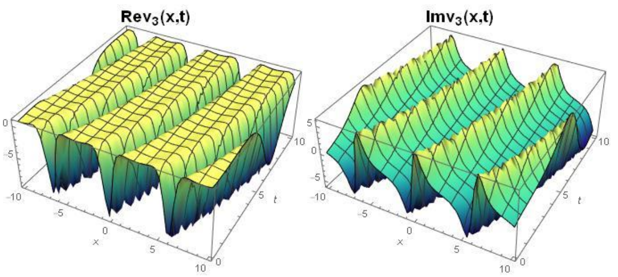

Figure 2. 3D graphs of real and imaginary parts of Eq. (16) for $m=1, \lambda=3, \beta=0.5, \mathrm{A1}=4$.

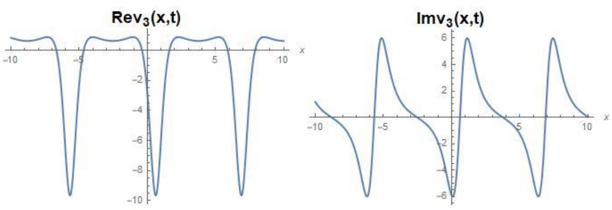

Figure 3. 2D graphs of real and imaginary parts of Eq. (16) for $m=1, \lambda=3, \beta=0.5, \mathrm{A1}=4, t=1$.

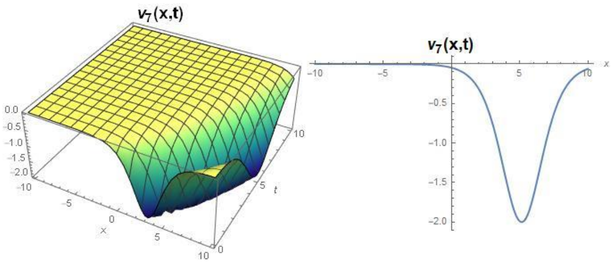

Figure 4. 3D and 2D graphs of Eq. (20) for $m=0.2, \lambda=2.5, \mathrm{A1}=10, \beta=0.5$ ($t=1$ for 2D plot).

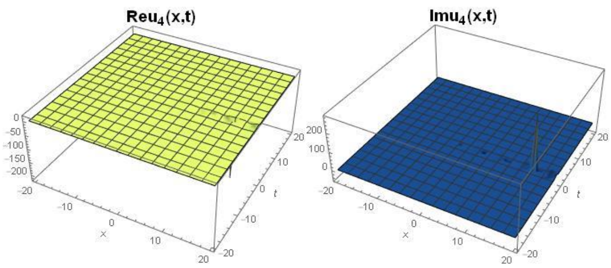

Figure 5. 3D graphs of real and imaginary parts of Eq. (28) for $\lambda=0.8, m=1, \mathrm{A1}=0.6, \beta=0.5$.

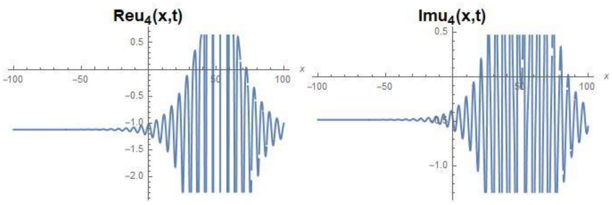

Figure 6. 2D graphs of real and imaginary parts of Eq. (28) for $\lambda=0.8, m=1, \mathrm{A1}=0.6, \beta=0.5, t=1$.

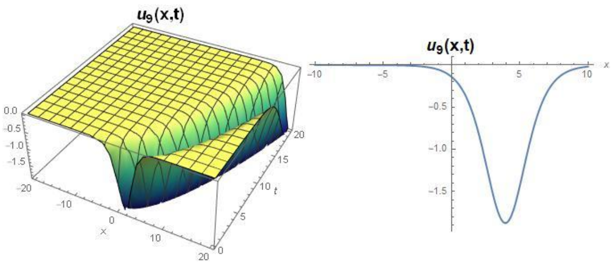

Figure 7. 3D and 2D graphs of Eq. (33) for $m=1, \lambda=0.01, \mathrm{A1}=5, \beta=0.5$ ($t=1$ for 2D plot).

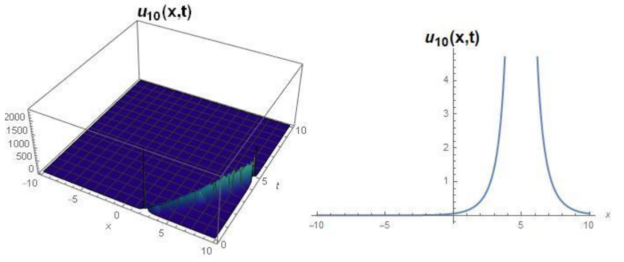

Figure 8. 3D and 2D graphs of Eq. (34) for $m = 4.7, \lambda = 0.3, \mathrm{A1} = 0.5, \beta = 0.5$ ($t = 1$ for 2D plot).

It is observed that the solutions reported in this study in Eqs. (14-36) have different structures when compared with the solutions obtained in Wazwaz (2006), Wazwaz (2007), Khatun and Akbar (2024). The dependability and the effectiveness of the $(m + 1/G')$-expansion method are highlighted together with these figures.

5. Conclusion

In conclusion, the new exact wave solutions of the beta time-fractional mCH and mDP equations are evaluated by using the $(m + 1/G')$-expansion method together with Mathematica software. This method has served as a powerful and an adaptable method to handle in analyzing the nonlinear wave propagation. The exact solutions reported in the present paper have yielded important perceptions for the dynamics of the nonlinear wave propagation. The results may be used to forecast many phenomena such as fluid mechanics, hydrodynamics and optical fibers. For further studies, this method can be used for many other NLPDEs to prove the new exact wave solutions.

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement

Author-1: Conceptualization, Methodology/Study design, Software, Validation, Formal analysis, Investigation, Resources, Visualization, Supervision, Writing – original draft, Writing – review and editing.

Declaration of Competing Interest

The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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