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ON G_2-MORPHIC MANIFOLDS AND G_2 STRUCTURES

ABSTRACT

In this paper, we consider two G_2-morphic 7-manifolds with G_2 structures and show that they belong to the same class of G_2structures. The converse may not be true, however.

Keywords: G_2Structures, G_2-Morphisms.

G_2-MORFİK MANİFOLDLAR VE G_2 YAPILARI

ÖZ

Bu çalışmada G_2yapısına sahip 7-boyutlu G_2-morfik manifoldlar ele alınmıştır. Herhangi iki manifoldun G_2-morfik olması durumunda, bu manifoldların G_2yapıların aynı sınıfında yer aldıkları gösterilmiştir. Ancak, önermenin tersi doğru olmayabilir.

Anahtar Kelimeler: G_2 Yapıları, G_2-Morfizmler.

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1. INTRODUCTION

Consider $\mathbb{R}^7$with its standard basis $\{e_1, \ldots, e_7\}$ and dual basis $\{e^1, \ldots, e^7\}$. The 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ is called the fundamental 3-form on $\mathbb{R}^7$. The group G_2 is defined as

$$G_2 := \{ f \in GL(7, \mathbb{R}) \mid f^* \varphi_0 = \varphi_0 \},$$

where $GL(7, \mathbb{R})$ is the group of isomorphisms of $\mathbb{R}^7$. The group G_2 is a compact, simple and simply connected 14-dimensional Lie group. A 7-dimensional manifold M is called a manifold with G_2structure if its structure group reduces to the group G_2. The existence of a G_2structure is equivalent to the existence of a 3-form on M which can be locally written as φ_0. This 3-form gives a Riemannian metric and a volume form on M (Bryant, 1987).

Manifolds having G_2 structures were classified by Fernandez and Gray in (Fernandez and Gray, 1982). There are 16 classes of G_2 structures depending on the space the covariant derivative of the fundamental 3-form φ belongs to. The defining relations for each of the 16 classes were given by Fernandez and Gray (Fernandez and Gray, 1982) and then an equivalent characterization was obtained by Cabrera using $d\varphi$ and $d * \varphi$ in (Cabrera, 1996). This characterization is given in the Table 1.

Note that $* d\varphi \wedge \varphi = - * d * \varphi \wedge * \varphi$, $\alpha = -\frac{1}{4} * (* d\varphi \wedge \varphi)$, $\beta = -\frac{1}{3} * (* d\varphi \wedge \varphi)$ and $f = \frac{1}{7} * (\varphi \wedge d\varphi)$ (Cabrera, 1996).

Let (M_1, φ_1) and (M_2, φ_2) be 7-manifolds with G_2structures. If there exists a diffeomorphism $F: M_1 \longrightarrow M_2$ such that $F^*(\varphi_2) = \varphi_1$, then F is called a G_2-morphism and (M_1, φ_1) and (M_2, φ_2) are said to be G_2-morphic (Cho, Salur and Todd, 2011).

Let (M_1, φ_1) and (M_2, φ_2) be G_2-morphic. Then $d\varphi_1 = 0$ iff $d\varphi_2 = 0$ since d commutes with pullback maps (Cho, Salur and Todd, 2011).

Table 1. Classification of G_2structures

$\mathcal{P}$	$d\varphi = 0$ and $d * \varphi = 0$
$\mathcal{W}_1$	$d\varphi = k * \varphi$ and $d * \varphi = 0$
$\mathcal{W}_2$	$d\varphi = 0$
$\mathcal{W}_3$	$d * \varphi = 0$ and $d\varphi \wedge \varphi = 0$
$\mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi$ and $d * \varphi = \beta \wedge * \varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_2$	$d\varphi = k * \varphi$ and $* d * \varphi \wedge * \varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$d * \varphi = 0$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$d\varphi \wedge \varphi = 0$ and $* d\varphi \wedge \varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi + f * \varphi$ and $d * \varphi = \beta \wedge * \varphi$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi$
$\mathcal{W}_3 \oplus \mathcal{W}_4$	$d\varphi \wedge \varphi = 0$ and $d * \varphi = \beta \wedge * \varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$* d\varphi \wedge \varphi = 0$ or $* d * \varphi \wedge * \varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi + f * \varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d * \varphi = \beta \wedge * \varphi$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d\varphi \wedge \varphi = 0$
$\mathcal{W}$	No relation

In the present paper, we show that G_2-morphisms do not only preserve closed G_2structures, but also other 15 classes too. We consider two G_2morphic manifolds and prove that both belong to the same class of G_2structures.

2. RELATIONS BETWEEN G_2-MORPHIC MANIFOLDS

Take two G_2-morphic manifolds (M_1, φ_1) and (M_2, φ_2). Then there exists a diffeomorphism $F: M_1 \longrightarrow M_2$ such that $F^*(\varphi_2) = \varphi_1$. Let g_1 and g_2 denote the Riemannian metrics and Ω_1, Ω_2 the volume forms determined by φ_1 and φ_2 respectively. Then for $x, y \in TM_1$ and $x', y' \in TM_2$, followings hold (Bryant, 1987) :

$$(x \lrcorner \varphi_1) \wedge (y \lrcorner \varphi_1) \wedge \varphi_1 = g_1(x, y)\Omega_1,$$

$$(x' \lrcorner \varphi_2) \wedge (y' \lrcorner \varphi_2) \wedge \varphi_2 = g_2(x', y')\Omega_2,$$

where the symbol "$\lrcorner$" denotes the contraction of the 3-form φ. Now

$$\begin{aligned} g_1(x, y)\Omega_1 &= (x \lrcorner \varphi_1) \wedge (y \lrcorner \varphi_1) \wedge \varphi_1 \\ &= (x \lrcorner\, F^*\varphi_2) \wedge (y \lrcorner\, F^*\varphi_2) \wedge\, F^*\varphi_2 \\ &= \big(F^*(F_*(x) \lrcorner \varphi_2)\big) \wedge \big(F^*(F_*(y) \lrcorner \varphi_2)\big) \wedge\, F^*\varphi_2 \\ &= F^*\{(F_*(x) \lrcorner \varphi_2) \wedge\, (F_*(y) \lrcorner \varphi_2) \wedge \varphi_2\} \\ &= F^*\{g_2\big(F_*(x), F_*(y)\big)\Omega_2\} \\ &= g_2\big(F_*(x), F_*(y)\big)F^*\Omega_2. \end{aligned}$$

Take a local orthonormal frame $\{e_1, \dots, e_7\}$ on an open neighbourhood of q\∈ M_1. Then

$$\begin{aligned} g_1(x, y) &= g_1(x, y)\Omega_1(e_1, \dots, e_7) \\ &= g_2\big(F_*(x), F_*(y)\big)F^*\Omega_2(e_1, \dots, e_7) \\ &= g_2\big(F_*(x), F_*(y)\big)\Omega_2(F_*(e_1), \dots, F_*(e_7)) \\ &= g_2\big(F_*(x), F_*(y)\big)\left(det g_2\left(F_*(e_i), F_*(e_j)\right)\right)^{1/2}. \end{aligned}$$

Thus if (M_1, φ_1) and (M_2, φ_2) are G_2-morphic, then

$$kF^*g_2 = g_1$$

where $k = \left(det g_2\left(F_*(e_i), F_*(e_j)\right)\right)^{1/2}$.

Let P_1 and P_2 denote the two-fold vector cross products determined by φ_1 and φ_2, respectively. Then

$$\begin{aligned} g_2(P_2(F_*(x), F_*(y)), F_*(z)) &= \varphi_2(F_*(x), F_*(y), F_*(z)) \\ &= \varphi_1(x, y, z) \\ &= g_1(P_1(x, y), z) \\ &= g_2(kF_*(P_1(x, y)), F_*(z)). \end{aligned}$$

for all $x, y \in T_q M_1$. Since F_* is an isomorphism and g_2 is non-degenerate, we obtain:

$$F^* P_2 = kF_* \circ P_1.$$

Note that the isomorphism $F_*: TM_1 \longrightarrow TM_2$ induces an isomorphism

$$F^* : \Lambda^p (TM_2)^* \longrightarrow \Lambda^p (TM_1)^*.$$

Now we extend the metrics g_1 and g_2 to spaces $\Lambda^p (TM_1)^*$ and $\Lambda^p (TM_2)^*$, respectively.

Let α and β be 1-forms on M_2. In this case $F^*\alpha$ and $F^*\beta$ are 1-forms on M_1. Let $\sharp$ denote the metric dual of a given 1-form or a vector field. Take a local orthonormal frame $\{e_1, \dots, e_7\}$ on an open neighbourhood of a point q of M_1. Then for any vector $x \in T_{F(q)} M_2$, we have

$$\begin{aligned} \alpha(x) &= g_2(x, \alpha^\sharp) \\ &= g_2(F_*((F_*)^{-1}(x)), F_*((F_*)^{-1}(\alpha^\sharp))) \\ &= k^{-1} g_1((F_*)^{-1}(x), (F_*)^{-1}(\alpha^\sharp)) \\ &= k^{-1} \left((F_*)^{-1}(\alpha^\sharp)\right)^\sharp ((F_*)^{-1}(x)) \end{aligned}$$

and thus

$$(F^*\alpha)^\sharp = k^{-1} (F_*)^{-1} (\alpha^\sharp).$$

This gives

$$\begin{aligned} g_1(F^*\alpha, F^*\beta) &= g_1((F^*\alpha)^\sharp, (F^*\beta)^\sharp) \\ &= k^{-2} g_1((F_*)^{-1}(\alpha^\sharp), (F_*)^{-1}(\beta^\sharp)) \\ &= k^{-2} k g_2(\alpha^\sharp, \beta^\sharp) \\ &= k^{-1} g_2(\alpha, \beta), \end{aligned}$$

that implies

$$g_2(\alpha, \beta) = k g_1(F^*\alpha, F^*\beta)$$

for any 1-forms α, β on M_2.

Now let $\alpha = \alpha_1 \wedge \dots \wedge \alpha_p$ and $\beta = \beta_1 \wedge \dots \wedge \beta_p$ be p-forms on M_2. Then

$F^*(\alpha) = F^*(\alpha_1) \wedge \dots \wedge F^*(\alpha_p)$ and $F^*(\beta) = F^*(\beta_1) \wedge \dots \wedge F^*(\beta_p)$ are p-forms on M_1. Then

$$\begin{aligned} g_1(F^*(\alpha), F^*(\beta)) &= g_1(F^*(\alpha_1) \wedge \dots \wedge F^*(\alpha_p), F^*(\beta_1) \wedge \dots \wedge F^*(\beta_p)) \\ &= det g_1(F^*(\alpha_i), F^*(\beta_j)) = k^{-p} det g_2(\alpha_i, \beta_j) \\ &= k^{-p} g_2(\alpha_1 \wedge \dots \wedge \alpha_p, \beta_1 \wedge \dots \wedge \beta_p) = k^{-p} g_2(\alpha, \beta), \end{aligned}$$

which gives,

$$g_2(\alpha, \beta) = k^p g_1(F^*\alpha, F^*\beta)$$

for p-forms α, β on M_2. Now since $k = \left(detg_2\left(F_*(e_i), F_*(e_j)\right)\right)^{1/2}$, we have

$$
\begin{aligned}
k^2 &= detg_2(F_*(e_i), F_*(e_j)) \\
&= detg_2((F_*(e_i))^{\#}, (F_*(e_j))^{\#}) \\
&= det(k^{-1} g_1(e_i, e_j)) \\
&= k^{-7} g_1(e_1 \wedge \dots \wedge e_7, e_1 \wedge \dots \wedge e_7) = k^{-7}.
\end{aligned}
$$

Thus we obtain $k^9 = 1$ which is possible only if $k = 1$.

Therefore if (M_1, φ_1) and (M_2, φ_2) are G_2-morphic, then

$$F^* g_2 = g_1, \tag{1}$$

$$F^* P_2 = F_* \circ P_1 \tag{2}$$

Conversely, if (1) and (2) hold, then (M_1, φ_1) and (M_2, φ_2) are G_2-morphic .

Let $*_1$ and $*_2$ denote Hodge-star operators determined by metrics g_1 and g_2, respectively. If α and β are p and 7-p forms on M_2, then $F^*\alpha$ and $F^*\beta$ are p and 7-p forms on M_1. Take a local orthonormal frame $\{e_1, \dots, e_7\}$ on an open neighbourhood of a point q of M_1. We compute the following:

$$
\begin{aligned}
g_2(*_2\, \alpha, \beta) &= g_2(\alpha \wedge \beta, \Omega_2) \\
&= g_1(F^*(\alpha \wedge \beta), F^*\Omega_2) \\
&= g_1(F^*\alpha \wedge F^*\beta, m\Omega_1) \\
&= m\, g_1(*_1\, F^*\alpha, F^*\beta) \\
&= m\, g_1(F^*((F^*)^{-1}\, (*_1\, F^*\alpha)), F^*((F^*)^{-1}(F^*\beta))) \\
&= m\, g_2((F^*)^{-1}(*_1\, F^*\alpha), \beta)
\end{aligned}
$$

where $m = (F^*\Omega_2)(e_1, \dots, e_7)$. Therefore

$$F^*(*_2\, \alpha) = m *_1 F^*\alpha.$$

Now we use the formula $* \varphi(w, x, y, z) = \frac{1}{3} g(w, \mathfrak{S}_{xyz} P(P(x, y), z))$ given in (Fernandez and Gray, 1982) for the Hodge-star $*$ of the fundamental 3-form φ of a manifold with G_2 structure. There exist $w, x, y, z \in T_q M_1$ such that $F_*(w) = w', F_*(x) = x', F_*(y) = y', F_*(z) = z'$ for any $w', x', y', z' \in T_{F(q)} M_2$. Thus

$$*_2\ \varphi_2(w',x',y',z') =*_2\ \varphi_2\big(F_*\ (w),F_*(x),F_*(y),F_*(z)\big)$$

$$= \frac{1}{3}g_2(F_*(w),\mathfrak{S}_{xyz}P_2(P_2(F_*(x),F_*(y)),F_*(z)))$$

$$= \frac{1}{3}g_2(F_*(w),\mathfrak{S}_{xyz}P_2(F_*(P_1(x,y)),F_*(z)))$$

$$= \frac{1}{3}g_2(F_*(w),\mathfrak{S}_{xyz}F_*(P_1(P_1(x,y),z))$$

$$= \frac{1}{3}g_1(w,\mathfrak{S}_{xyz}P_1(P_1(x,y),z))$$

$$=*_1\ \varphi_1(w,x,y,z),$$

i. e. we get

$$F^*(*_2\ \varphi_2) =*_1\ \varphi_1.$$

Put $\alpha = \varphi_2$ in the equation $F^*(*_2\ \alpha) = m *_1 F^*\alpha$, to observe that $m = 1$. Hence we get the equation

$$F^*(*_2\ \alpha) =*_1 F^*\alpha \quad (3)$$

for any p-form α on M_2.

3. G_2-MORPHIC MANIFOLDS AND CLASSES OF G_2STRUCTURES

Let (M_1,φ_1) and (M_2,φ_2) be G_2-morphic manifolds. There exists a diffeomorphism $F: M_1 \longrightarrow M_2$ such that $F^*(\varphi_2) = \varphi_1$. Assume $\mathcal{U}$ is one of the sixteen classes of G_2 structures. In this section, we show that $M_1 \in \mathcal{U}$ iff $M_2 \in \mathcal{U}$. We use the characterization of Cabrera in (Cabrera, 1996). We investigate each class separately. We use relations (1), (2) and (3) we found in the previous section.

The class $\mathcal{P}$: Let $M_1 \in \mathcal{P}$. Then $d\varphi_1 = 0$ and $d *_1 \varphi_1 = 0$. Thus

$$0 = d\varphi_1 = dF^*\ \varphi_2 \ = F^*d\varphi_2$$

and $0 = d *_1 \varphi_1 = d(F^*(*_2\ \varphi_2)) = F^*(d *_2 \varphi_2)$. Since F^* is an isomorphism, we get $d\varphi_2 = 0$ and $d *_2 \varphi_2 = 0$.

The class $\mathcal{W}_1$: Let $M_1 \in \mathcal{W}_1$. Then $d\varphi_1 = t *_1 \varphi_1$ and $d *_1 \varphi_1 = 0$ for $d\varphi_1 \neq 0$. It is enough to show the first condition. For $t \neq 0$ we have $d\varphi_1 = t *_1 \varphi_1$, which is equivalent to

$d(F^*\varphi_2) = t(F^*(*_2\ \varphi_2))$. Since d commutes with pullback maps, we have $F^*(d\varphi_2) = F^*(t *_2 \varphi_2)$. This implies $d\varphi_2 = t *_2 \varphi_2$ since F^* is one-to-one. Since $d\varphi_1 \neq 0$ and F^* is an isomorphism, we get $d\varphi_2 \neq 0$. Thus $M_2 \notin \mathcal{P}$.

The class $\mathcal{W}_2$: Let $M_1 \in \mathcal{W}_2$. Then $d\varphi_1 = 0$ for $d *_1 \varphi_1 \neq 0$. Here it is enough to see that M_2 can not belong to the class $\mathcal{P}$. Assume that $M_2 \in \mathcal{P}$. Then $d *_2 \varphi_2 = 0$ which implies that $d((F^*)^{-1} *_1 \varphi_1) = (F^*)^{-1}(d *_1 \varphi_1)$. Thus $d *_1 \varphi_1 = 0$ which is a contradiction. Thus $M_2 \notin \mathcal{P}$.

The class $\mathcal{W}_3$: Let $M_1 \in \mathcal{W}_3$. Then $d *_1 \varphi_1 = 0$ and $d\varphi_1 \wedge \varphi_1 = 0$ for $d\varphi_1 \neq 0$. It is enough to see the second condition. Since

$$0 = d\varphi_1 \wedge \varphi_1 = d(F^*\varphi_2) \wedge (F^*\varphi_2) = F^*(d\varphi_2) \wedge (F^*\varphi_2) = F^*(d\varphi_2 \wedge \varphi_2)$$

and F^* is an isomorphism, we get $d\varphi_2 \wedge \varphi_2 = 0$. We can use the arguments we used while showing $M_2 \notin \mathcal{P}$ in the previous class here to see that $M_2 \notin \mathcal{P}$ similarly.

The class $\mathcal{W}_4$: Let $M_1 \in \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1$ and $d *_1 \varphi_1 = \beta \wedge *_1 \varphi_1$ for $\alpha, \beta \neq 0$. The condition $d\varphi_1 = \alpha \wedge \varphi_1$ is equivalent to $d(F^*\varphi_2) = \alpha \wedge F^*\varphi_2$. We can write this equation as $F^*(d\varphi_2) = F^*((F^*)^{-1}\alpha) \wedge F^*\varphi_2 = F^*((F^*)^{-1}\alpha \wedge \varphi_2)$ and since F^* is an isomorphism, we have $d\varphi_2 = (F^*)^{-1}\alpha \wedge \varphi_2$. In addition, $d *_1 \varphi_1 = \beta \wedge *_1 \varphi_1$ means $d(F^* *_2 \varphi_2) = \beta \wedge F^* *_2 \varphi_2$. Hence

$$F^*(d *_2 \varphi_2) = F^*((F^*)^{-1}\beta) \wedge F^* *_2 \varphi_2 = F^*((F^*)^{-1}\beta \wedge *_2 \varphi_2)$$

and similarly we get

$d *_2 \varphi_2 = (F^*)^{-1}\beta \wedge *_2 \varphi_2$. Note also that $d\varphi_2 \neq 0$ and $d *_2 \varphi_2 \neq 0$. Thus $M_2 \in \mathcal{W}_4$.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2$. Then $d\varphi_1 = t *_1 \varphi_1$ and

$(*_1 d *_1 \varphi_1) \wedge *_1 \varphi_1 = 0$ for $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. It is enough to show the following:

$$\begin{aligned} 0 &= (*_1 d *_1 \varphi_1) \wedge *_1 \varphi_1 \\ &= *_1 d(F^* *_2 \varphi_2) \wedge F^*(*_2 \varphi_2) \\ &= *_1 F^*(d *_2 \varphi_2) \wedge F^*(*_2 \varphi_2) \\ &= F^* *_2 (d *_2 \varphi_2) \wedge F^*(*_2 \varphi_2) \\ &= F^*((*_2 d *_2 \varphi_2) \wedge *_2 \varphi_2). \end{aligned}$$

This gives $(*_2 d *_2 \varphi_2) \wedge *_2 \varphi_2 = 0$. Similar to previous classes we can see that M_2 can not belong to subclasses of $\mathcal{W}_1 \oplus \mathcal{W}_2$.

The class $\mathcal{W}_1 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_3$. Then $d *_1 \varphi_1 = 0$ for $d\varphi_1 \neq 0$. Here it is enough to show that M_2 can not be an element of subclasses of $\mathcal{W}_1 \oplus \mathcal{W}_3$ and this can be seen again by using that F^* is an isomorphism.

The class $\mathcal{W}_2 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_3$. Then $d\varphi_1 \wedge \varphi_1 = 0$ and $(*_1 d\varphi_1) \wedge \varphi_1 = 0$ for $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. It is enough to see followings:

$$\begin{aligned} 0 &= (*_1 d \varphi_1) \wedge \varphi_1 \\ &= *_1 d(F^*\varphi_2) \wedge F^*\varphi_2 \\ &= *_1 F^*(d\varphi_2) \wedge F^*\varphi_2 \\ &= F^*(*_2 d\varphi_2) \wedge F^*\varphi_2 \\ &= F^*((*_2 d\varphi_2) \wedge \varphi_2). \end{aligned}$$

This implies $(*_2 d\varphi_2) \wedge \varphi_2 = 0$. We can also show similarly that M_2 can not be in the subclasses.

The class $\mathcal{W}_1 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1 + f *_1 \varphi_1$ and

$d *_1 \varphi_1 = \beta \wedge *_1 \varphi_1$ for a nonzero function f, $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. We should see the following:

$$d\varphi_1 = \alpha \wedge \varphi_1 + f *_1 \varphi_1,$$

$$d(F^*\varphi_2) = \alpha \wedge F^*\varphi_2 + fF^*(*_2 \varphi_2),$$

$$F^*(d\varphi_2) = F^*((F^*)^{-1}\alpha) \wedge F^*\varphi_2 + fF^*(*_2 \varphi_2)$$

$$= F^*((F^*)^{-1}\alpha \wedge \varphi_2 + f \circ F^{-1} *_2 \varphi_2),$$

which implies $d\varphi_2 = (F^*)^{-1}\alpha \wedge \varphi_2 + f \circ F^{-1} *_2 \varphi_2$. We can also show similarly that M_2 can not be in the subclasses.

The class $\mathcal{W}_2 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1$ for $d\varphi_1 \neq 0$. We showed that the condition $d\varphi_1 = \alpha \wedge \varphi_1$ is equivalent to the condition $d\varphi_2 = (F^*)^{-1}\alpha \wedge \varphi_2$. We can see that M_2 can not belong to the subclasses similar to previous classes.

The class $\mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d\varphi_1 \wedge \varphi_1 = 0$ and $d *_1 \varphi_1 = \beta \wedge *_1 \varphi_1$ for $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. We saw that this is only possible if $d\varphi_2 \wedge \varphi_2 = 0$ and
$d *_2 \varphi_2 = (F^*)^{-1}\beta \wedge *_2 \varphi_2$. We can eliminate the subclasses of $\mathcal{W}_3 \oplus \mathcal{W}_4$ similarly.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Then $*_1 d\varphi_1 = 0$ or

$*_1 d *_1 \varphi_1 \wedge *_1 \varphi_1 = 0$ for $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. If $*_1 d\varphi_1 = 0$, then

$0 = *_1 d\varphi_1 = *_1 d(F^*\varphi_2) = *_1 F^* d\varphi_2 = F^*(*_2 d\varphi_2)$,

so we get $*_2 d\varphi_2 = 0$. If $*_1 d *_1 \varphi_1 \wedge *_1 \varphi_1 = 0$, then

$$0 = *_1 d *_1 \varphi_1 \wedge *_1 \varphi_1$$

$$= *_1 d\big(F^*(*_2 \varphi_2)\big) \wedge F^*(*_2 \varphi_2)$$

$$= *_1 F^* d(*_2 \varphi_2) \wedge F^*(*_2 \varphi_2)$$

$$= F^* *_2 d(*_2 \varphi_2) \wedge F^*(*_2 \varphi_2)$$

$$= F^*(*_2 d(*_2 \varphi_2) \wedge *_2 \varphi_2)$$

and this implies $*_2 d(*_2 \varphi_2) \wedge *_2 \varphi_2 = 0$ since F^* is an isomorphism. We can see that M_2 can not belong to the subclasses similarly.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Then

$d\varphi_1 = \alpha \wedge \varphi_1 + f *_1 \varphi_1$

for a non-zero function f and $d\varphi_1 \neq 0$. It is enough to see that M_2 is not an element of a subclass. This can be done by using that F^* is an isomorphism.

The class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d *_1 \varphi_1 = \beta \wedge *_1 \varphi_1$ for $d\varphi_1 \neq 0$ and $d *_1 \varphi_1 \neq 0$. It is enough to see that M_2 is not an element of a subclass. This can be done by using that F^* is an isomorphism.

The class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d\varphi_1 \wedge \varphi_1 = 0$ for $d\varphi_1 \neq 0$. It is enough to see that M_2 is not an element of a subclass. This can be done by using that F^* is an isomorphism.

The class $\mathcal{W}$: Let $M_1 \in \mathcal{W}$. Then M_2 is in the same class, too. All subclasses can be eliminated by using that F^* is an isomorphism.

4. CONCLUSION

We observe that for two G_2 -morphic manifolds (M_1, φ_1) and (M_2, φ_2), if M_1 belongs to $\mathcal{U}$, then M_2 is in the same class too. We can easily see that being G_2-morphic is an equivalence relation. Thus $M_1 \in \mathcal{U}$ iff $M_2 \in \mathcal{U}$. That is, the classes of G_2 structures are preserved under G_2-morphisms. The converse may not be true. That is, if there are two diffeomorphic manifolds which are in the same class of G_2 structures, they need not to be G_2-morphic. An example of two diffeomorphic manifolds which are not G_2morphic are the Aloff-Wallach spaces. These are spaces $M_{k,l} := SU(3)/U(1)_{k,l}$ for $k \neq \pm l$, where $U(1)_{k,l}$ is the subgroup of $SU(3)$ generated by the elements of the form $e^{diag(ik,il,i(-k-l))}$. Note that a diffeomorphism $G: M_1 \longrightarrow M_2$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a homothety iff $G^* g_2 = c g_1$ for some nonzero constant c (O'Neill, 1983). It is known that $M_{k,l}$ admits two non-homothetic G_2structures in $\mathcal{W}_1$ (Cabrera, Monar and Swann, 1996) and since these G_2structures are not homothetic, they are not G_2morphic. Hence, the classification of manifolds with structure group G_2according to G_2morphisms is finer than the classification of Fernandez and Gray in (Fernandez and Gray, 1982).

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