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ARAŞTIRMA MAKALESİ / **RESEARCH ARTICLE**

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İKİ LİNEER MODELDEKİ BLUE'LAR ARASINDAKİ İLİŞKİLER

ÖZ

Tam model olarak bilinen ve yalnızca kovaryans matrisleri farklı olan iki lineer model $\mathrm{A} = \{y, X\beta, V_{\mathrm{A}}\}$ ve $\mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}$ altında $X\beta$ vektörünün en iyi lineer yansız tahmin edicilerinin (BLUE'larının) tahmini ele alınmaktadır. Tam modeller altında $X\beta$ vektörünün BLUE'ları arasındaki eşitlikler ile ilgili bazı sonuçlar, Rao (1971) tarafından tanıtılmış olan Pandora's Box denklemi kullanılarak verilmektedir. Ayrıca $V_{\mathrm{A}} = I_n$ özel durumu da, elde edilen sonuçlar altında ele alınmaktadır.

Anahtar Kelimeler: BLUE; Bordered matris; Genel Gauss-Markov modeli; Matrislerin genelleştirilmiş tersi; dik izdüşüm

RELATIONS BETWEEN BLUEs IN TWO LINEAR MODELS

ABSTRACT

The estimation of the best linear unbiased estimators (BLUEs) of $X\beta$ in two linear models denoted by $\mathrm{A} = \{y, X\beta, V_{\mathrm{A}}\}$ and $\mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}$ which are known as full models and differ only in their covariance matrices are considered. Some results related to the equality between the BLUEs of $X\beta$ under the full models are given using the Pandora's Box equation introduced by Rao (1971). The special case $V_{\mathrm{A}} = I_n$ is also investigated under the basis of the results.

Keywords: BLUE; Bordered matrix; General Gauss-Markov model; Generalized inverse of matrices; Orthogonal projector

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1. INTRODUCTION

Throughout this paper, $R^{m\times n}$ stands for the set of all $m\times n$ real matrices. Let $A\in R^{m\times n}$ be any matrix. The symbols A', A^{-}, A^{+}, $\mathrm{C}(A)$, $\mathrm{C}(A)^{\perp}$, $\mathrm{N}(A)$ and $r(A)$ denote the transpose, a generalized inverse, the Moore-Penrose inverse, the column space, the orthogonal complement of the column space, the null space, and the rank of the matrix A, respectively. Notation $(A:B)$ stands for a partitioned matrix with $A\in R^{m\times n}$ and $B\in R^{m\times k}$ as submatrices. Furthermore, let P_A and Q_A stand for the two orthogonal projectors $P_A = AA^{+} = A(A'A)^{-}A'$ and $Q_A = I_m - P_A$. Notation $A^{\perp}$ stands for the any matrix satisfying $\mathrm{C}(A^{\perp}) = \mathrm{N}(A') = \mathrm{C}(A)^{\perp}$. In particular,

$$Q = I - P_X . \tag{1}$$

Consider the general Gauss-Markov model denoted by

$$\mathrm{M} = \{y, X\beta, V_{\mathrm{M}}\}, \tag{2}$$

where the vector $y\in R^{n\times 1}$ is an observable random vector with $E(y) = X\beta$ and $\mathrm{Cov}(y) = V_{\mathrm{M}}$, $X\in R^{n\times p}$ is a known matrix deficient in rank, $\beta\in R^{p\times 1}$ is a vector of unknown parameters, and $V_M\in R^{n\times n}$ is a known positive semi definite matrix.

In order to study on estimations under the model M, we assume that this model is consistent, that is,

$$y\in \mathrm{C}(X:V_{\mathrm{M}}) = \mathrm{C}(X:V_{\mathrm{M}}Q) \tag{3}$$

holds with probability 1, where Q is given in (1), see, e.g., Rao (1971, 1973).

2. SOME RESULTS on BLUEs

Consider the estimation of a linear parametric function $K\beta$, $K\in R^{k\times p}$, under the model M. A parametric function $K\beta$ is said to be estimable under the model M if and only if $\mathrm{C}(K')\subseteq \mathrm{C}(X')$. Assume that $K\beta$ is estimable under M. The best linear unbiased estimator (BLUE) for $K\beta$ under M, denoted by $\mathrm{BLUE}(K\beta)_{\mathrm{M}}$, is defined to be a linear statistic Gy such that $E(Gy) = K\beta$ and $\mathrm{Cov}(Gy)$ is minimal, in the Löwner sense, among all covariance matrices $\mathrm{Cov}(Fy)$ such that Fy is unbiased for $K\beta$. It is well known that Gy is $\mathrm{BLUE}(K\beta)_{\mathrm{M}}$ if and only if G satisfies the fundamental BLUE equation

$$Gy = \mathrm{BLUE}(X\beta)_{\mathrm{M}} \Leftrightarrow G(X:V_{\mathrm{M}}Q) = (X:0), \tag{4}$$

e.g., Rao (1971, p.4) and Baksalary (2004). This equation is always consistent, that is, $(X:0)(X:V_{\mathrm{M}}Q)^{+}(X:V_{\mathrm{M}}Q) = (X:0)$. In this case, the general solution to (4), denoted by G_0, can be written as $G_0 = (X:0)(X:V_{\mathrm{M}}Q)^{+} + UQ_{(X:V_{\mathrm{M}}Q)}$, where $U\in R^{n\times n}$ is arbitrary. The equation (4) has a unique solution if and only if $r(X:V_{\mathrm{M}}) = n$. Gy is unique with probability 1 if and only if M is consistent; see, e.g., Rao (1973, p.282) and Tian (2009a). When X has full column rank and V_M is positive definite, then the BLUE of $X\beta$ in the model M is expressible $\mathrm{BLUE}(X\beta)_{\mathrm{M}} = X(X'V_{\mathrm{M}}^{-1}X)^{-1}X'V_{\mathrm{M}}^{-1}y$. The expression $\mathrm{BLUE}(X\beta)_{\mathrm{M}}$ is not unique when X and V_M might be deficient in rank. For example, one representation for the BLUE of $X\beta$ in the model M is $\mathrm{BLUE}(X\beta)_{\mathrm{M}} = X(X'W_{\mathrm{M}}^{+}X)^{+}X'W_{\mathrm{M}}^{+}y$, where $W_{\mathrm{M}} = V_{\mathrm{M}} + k^2XX',\ k \neq 0$, satisfying $C(W_M) = C(X : V_M)$ For the properties of the matrix W_M , see, e.g., Baksalary and Mathew (1990).

In this study, we will use the following lemma to give some relations on the BLUEs under the full models. The fundamental BLUE equation in (4) can be rewritten in the form given in the following lemma (Rao, 1971; Magnus and Neudecker, 1988; Isotalo et al. 2008a, b):

Lemma 1: (Pandora's Box). Consider the model $\mathrm{M} = \{y, X\beta, V_{\mathrm{M}}\}$. $Gy = \mathrm{BLUE}(X\beta)_{\mathrm{M}}$ if and only if there exists a matrix L such that G is solution to

$$\begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix}\begin{pmatrix} G' \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ X' \end{pmatrix}. \tag{5}$$

Note that Lemma 1 follows from the fundamental BLUE equation given in (4) where the second submatrix equality $GV_{\mathrm{M}}Q = 0$ holds if and only if there exists a matrix $L \in R^{p\times n}$ such that $V_{\mathrm{M}}G' = -XL$. Combining this with the first submatrix equality $GX = X$ in (4) yields (5).

Let us denote

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} = \begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix}^{-}. \tag{6}$$

Rao (1971) calls (5) the Pandora's Box. It is shown that the problem of inference from a linear model can be completely solved once one has obtained a matrix C given in (6). The computation of the matrix C is like opening a Pandora's Box, giving all that is necessary for drawing inferences on β (Rao, 1971). Many useful results are obtained by considering the equations (5) and (6) concerning linear models by Rao (1971, 1972). For example, the following hold:

$$\mathrm{BLUE}(X\beta)_{\mathrm{M}} = XC_2'y = XC_3y, \tag{7}$$

$$XC_2'X = X \text{ and } XC_3X = X, \tag{8}$$

$$V_{\mathrm{M}}C_2X' = XC_2'V_{\mathrm{M}} = V_{\mathrm{M}}C_3'X' = XC_3V_{\mathrm{M}}. \tag{9}$$

The result (7) is obtained from solving the matrix equation in (5) with respect to the matrix G and the results (8) and (9) are established by considering the equation $\begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}\begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix} = \begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix}$, where $\begin{pmatrix} V_{\mathrm{M}} & X \\ X' & 0 \end{pmatrix}$ is a special type of the bordered matrices. As further references to this equation and the Pandora's Box, we may mention Rao (1973, p. 298-300), Hall Meyer (1975), Harville (1997) and Isotalo et al. (2008a).

The main purpose of this paper is to give some results related to the BLUEs of $X\beta$ under two linear models, which differ only in their covariance matrices. Such a topic has been considered by various authors; see, e.g., Mitra and Moore (1973), Rao (1973), Mathew and Bhimasankaram (1983), Baksalary and Mathew (1986), Tian (2009b), Haslett and Puntanen (2010), Hauke, Markiewicz, and Puntanen (2012). In particular, some results related to the BLUEs of estimable parametric functions under the Gauss-Markov model and its misspecified model are given by Tian (2009b) using matrix rank method. Haslett and Puntanen (2010) give a necessary and sufficient condition for the equality between the BLUEs of subparameters under two models which differ only in their covariance matrices utilize fundamental BLUE equation given in (4). Hauke, Markiewicz and Puntanen (2012) consider two models having different covariance matrices and present an upper bound for the Euclidean norm of the difference between their BLUEs. In this study, differing from the existing studies, the results obtained by using a generalized inverse of the symmetric matrix given in (6), which is also a special type of the bordered matrices, are based on Pandora's Box equation given in Lemma 1.

3. TWO LINEAR MODELS WITH DIFFERENT COVARIANCE MATRICES

Consider two linear models

$$\mathrm{A} = \{y, X\beta, V_{\mathrm{A}}\} \text{ and } \mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}, \tag{10}$$

which differ only in their covariance matrices. According to (7), the BLUE of $X\beta$ under the model A can be expressed as $\mathrm{BLUE}(X\beta)_{\mathrm{A}} = XA_2'y$, where $A_2 \in R^{n\times p}$ is the upper right corner of the partitioned matrix

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & -A_4 \end{pmatrix} = \begin{pmatrix} V_{\mathrm{A}} & X \\ X' & 0 \end{pmatrix}^{-}, \tag{11}$$

which is obtained from applying Lemma 1 to the model A .

Let us assume that every representation of $\mathrm{BLUE}(X\beta)$ under A continues to be BLUE of $X\beta$ in B or, in short,

$$\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\}, \tag{12}$$

where the notation $\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\}$ denote the set of all BLUEs of $X\beta$ under the model A . The notation in (12) means that if the matrix G satisfies $G(X : V_{\mathrm{A}}Q) = (X : 0)$, then it is also satisfied $G(X : V_{\mathrm{B}}Q) = (X : 0)$. The equality in (12) means that the classes of the BLUEs of $X\beta$ under A and B are identical.

The main results of this study are given in Theorem 1 and Theorem 2. It is collected some properties related to the condition (12).

Theorem 1: Consider the full models $\mathrm{A} = \{y, X\beta, V_{\mathrm{A}}\}$ and $\mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}$. Then the following statements are equivalent:

(i) $\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\}$,

(ii) $\mathrm{C}(A_2'V_{\mathrm{B}}Q) \subseteq \mathrm{C}(A_2'V_{\mathrm{A}}Q)$,

where A_2 is as given in (11).

Proof. $\mathrm{BLUE}(X\beta)_{\mathrm{A}} = XA_2'y$ can be written from (7). Then, $XA_2'(X : V_{\mathrm{A}}Q) = (X : 0)$. Assume that the condition in (i) holds. It is concluded that $XA_2'V_{\mathrm{B}}Q = 0$ holds for the matrix XA_2'. It is known that a necessary and sufficient condition for $K'z = 0$ to entail $L'z = 0$ for every z is that $\mathrm{C}(L) \subseteq \mathrm{C}(K)$, see, e.g., Baksalary and Mathew (1986). In view of this statement, the proof is completed.□

Corollary 1: Using the above notation, the following statements are equivalent:

(i) $\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} = \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\}$,

(ii) $\mathrm{C}(A_2'V_{\mathrm{B}}Q) = \mathrm{C}(A_2'V_{\mathrm{A}}Q)$.

Theorem 2: Consider the full models $\mathrm{A} = \{y, X\beta, V_{\mathrm{A}}\}$ and $\mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}$. If the statement $\mathrm{C}(V_{\mathrm{B}}A_2X') \subseteq \mathrm{C}(V_{\mathrm{A}}A_2X')$ holds, then the statement $\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\}$, where A_2 is as given in (11).

Proof. Since $\mathrm{BLUE}(X\beta)_{\mathrm{A}}$ can be written as $XA_2'y$, $XA_2'X = X$ and $XA_2'V_{\mathrm{A}}Q = 0$. Let $\mathrm{C}(A_2'V_{\mathrm{B}}Q) = \mathrm{C}(A_2'V_{\mathrm{A}}Q)$. Hence, $A_2'V_{\mathrm{B}}Q = A_2'V_{\mathrm{A}}QK$ can be written for some K. Thus, $XA_2'V_{\mathrm{B}}Q = XA_2'V_{\mathrm{A}}QK = 0$ is obtained and then the proof is completed.□

Remark 1: Considering (7), Theorem 1 and Theorem 2 can be expressed as follows, respectively,

$$\{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\} \Leftrightarrow \mathrm{C}(A_3V_{\mathrm{B}}Q) \subseteq \mathrm{C}(A_3V_{\mathrm{A}}Q)$$

and

$$\mathrm{C}(V_{\mathrm{B}}A_3'X') \subseteq \mathrm{C}(V_{\mathrm{A}}A_3'X') \Rightarrow \{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\},$$

where A_3 is as given in (11).

Remark 2: Theorem 2 can be expressed as follows since $V_{\mathrm{A}}A_2X' = XA_2'V_{\mathrm{A}}$ from (9)

$$\mathrm{C}(V_{\mathrm{B}}A_2X') \subseteq \mathrm{C}(XA_2'V_{\mathrm{A}}) \Rightarrow \{\mathrm{BLUE}(X\beta)_{\mathrm{A}}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\}.$$

Now, we consider the cases to investigate what happens if we replace the model A with the model $\mathrm{A}_0 = \{y, X\beta, I_n\}$. When the equivalence of the BLUEs of $X\beta$ under the models A_0 and B is considered, we obtain the results given in the following corollary.

Corollary 2: Consider the models $\mathrm{A}_0 = \{y, X\beta, I_n\}$ and $\mathrm{B} = \{y, X\beta, V_{\mathrm{B}}\}$. Then the following statements are equivalent:

$$\{\mathrm{BLUE}(X\beta)_{\mathrm{A}_0}\} \subseteq \{\mathrm{BLUE}(X\beta)_{\mathrm{B}}\} \Leftrightarrow \mathrm{C}(A_2'V_{\mathrm{B}}Q) \subseteq \mathrm{C}(A_2'Q),$$

where A_2 is as given in (11).

Corollary 3: Consider the models $\mathrm{A}_0 = \{y, X\beta, I_n\}$ and $\mathrm{B} = \{y, X\beta, V_\mathrm{B}\}$. Then, $\mathrm{C}(V_\mathrm{B} A_2 X') \subseteq \mathrm{C}(P_X)$ implies $\{\mathrm{BLUE}(X\beta)_{\mathrm{A}_0}\} \subseteq \{\mathrm{BLUE}(X\beta)_\mathrm{B}\}$, where A_2 is as given in (11).

Finally, we illustrate these results with the following simple example with $n = 2$ and $p = 1$. Let us consider the model matrix and the covariance matrices for the models A and B as follows:

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ V_\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } V_\mathrm{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then we obtain the equal BLUEs for $X\beta$ and equal column space as given in Corollary 1.

4. CONCLUDING REMARKS

In this paper, some results are obtained related to the equality of the BLUEs under two full models, having different covariance matrices. Differing from the existing results in the literature, the considerations and main results in this study are based on the matrix C given in (6) which is a generalized inverse of the special bordered matrix $\begin{pmatrix} V_\mathrm{M} & X \\ X' & 0 \end{pmatrix}$ given in (5). We note that considering the problems by means of this approach have some advantages from the computational point of view since BLUEs and some results on them are obtained without any further calculations except for a few matrix multiplication when a generalized inverse of the matrix given in the Pandora's Box equation computed by a suitable procedure as also noted by Rao (1971). In other words, estimation and inference from a linear model reduces to the numerical problem of finding a generalized inverse of the symmetric matrix given in the Pandora's Box equation.

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