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RESEARCH ARTICLE

EXACT TRAVELING WAVE SOLUTIONS FOR THE NON-LINEAR COUPLE DRINFEL'D-SOKOLOV-WILSON (DSW) DYNAMICAL SYSTEM USING EXTENDED JACOBI ELLIPTIC FUNCTION EXPANSION METHOD

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ABSTRACT

The study of water waves is significant for researchers working in many branches of science. The behaviour of waves can be studied by observation or experimental means, but theoretically, mathematical modeling provides solutions to many problems in physics and engineering. Progress in this field is inevitable, with those who work in mathematics, physics, and engineering putting forth interdisciplinary studies.

Jacobi elliptic functions are valuable mathematical tools that can be applied to various aspects of mathematics, physics, and ocean engineering. In this study, traveling wave solutions of the general Drinfel'd-Sokolov-Wilson (DSW) system, introduced as a model of water waves, were obtained by using Jacobi elliptic functions and the wave dynamics were examined. The extended Jacobi elliptic function expansion method is an effective method for generating periodic solutions. It has been observed that the periodic solutions obtained by using Jacobi elliptic function expansions containing different Jacobi elliptic functions may be different and some new periodic solutions can be obtained. 3D simulations were made using Maple™ to see the behaviour of the solutions obtained for different appropriate values of the parameters. 2D simulations are presented for easy observation of wave motion. In addition, we transformed the one of the exact solutions found by the extended Jacobi elliptic function expansion method into the new solution under the symmetry transformation.

Keywords: DSW system, Exact solutions, Extended Jacobi elliptic function expansion method

1. INTRODUCTION

Installation of heavy and complex submarine equipment, submarine pipelines and submarine cables has been studied extensively in recent years due to their importance in ocean engineering. Various mathematical models for submarine installation have been obtained and different dynamic behaviours have been studied in various ways. Submarine installations are in the splash zone, completely submerged or close to the seafloor. If the structure is in the splash zone, shallow water waves, if the structure is completely underwater or close to the seabed, seabed waves are effective in the installation [1].

Autonomous underwater vehicles (AUVs) have an indispensable role in the exploration of the deep-sea, marine surveillance, and underwater rescue operations. Due to the intricate nature of the AUV system and the unpredictable underwater environment, controlling them is a challenging task. The control design faces some challenges like high precision multivariate, strong couplings, nonlinearities, and unknown distortions. To overcome these challenges, various control strategies have been developed to create trajectory-tracking controllers for AUVs, such as back step control, pattern predictive control, fuzzy control, and sliding mode control. However, all of these control schemes

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require calculations based on the underwater and surface wave strength, highlighting the importance of wave theory in this field [2].

When studying equations that describe wave phenomena, it is necessary to analyse travelling wave solutions. These solutions are permanent forms that move at a constant velocity. To obtain travelling wave solutions, the nonlinear evolution equations are usually reduced to associated ordinary differential equations. Solitary wave theory, which is rapidly advancing in several scientific fields, from shallow water waves to plasma physics, places particular interest in different types of travelling wave solutions [3].

The study of water waves is significant for researchers working in many branches of science. The behaviour of waves can be studied by observation or experimental means, but theoretically, mathematical modeling provides solutions to many problems in physics and engineering. Progress in this field is inevitable, with those who work in mathematics, physics, and engineering putting forth interdisciplinary studies.

Partial differential equations have an important place in the theory of waves. Solutions of a nonlinear partial differential equation can be explained by the concept of waves, which helps us to understand many physical phenomena. If the obtained solutions of the equation can be expressed with the time-dependent motion of a wave, the physical event that occurs can be explained more meaningfully. There are many methods in the literature to obtain wave solutions of partial differential equations. Some of these are the F-expansion method [4], the Jacobi elliptic function expansion method [5,6], (G'/G)-expansion method [7], Lie symmetry approach [8], and so on [9-12].

In modeling multiple events, partial differential equation systems consisting of at least two partial differential equations have an important place in the literature. Partial differential equation systems are also systems whose solutions can be obtained by the methods mentioned above.

The DSW system in dispersive water wave was introduced in 1981 by Drinfel'd and Sokolov [13] and Wilson [14] which resulted in the discovery of wave phenomena that have significant applications in fluid dynamics, ocean engineering, and science. This system serves as the fundamental integrable nonlinear system that describes surface gravitational waves propagating over a horizontal seabed. The DSW equation is expressed in the following form:

$$u_t + \alpha v v_x = 0 \tag{1}$$

$$v_t + \beta u v_x + s u_x v + \eta v_{xxx} = 0$$

The discovery of the DSW system and its wave phenomena has opened new avenues of research in various scientific fields and continues to pave the way for advancements in fluid dynamics and ocean engineering. Some researchers with various methods have studied this system: Shen et al. produced lump, soliton, and lump off solutions [15], Bashar et al. used the new auxilary equation (NAE) method [16], Khan et al. used the enhanced (G'/G)-expansion method [17], with a special selection Ren et al. used the consistent Riccati expansion method (CRE) [18], etc.

This article is organized as follows: In Chapter 2, we introduced the Jacobi elliptic functions and extended Jacobi elliptic function expansion method [19]. In Chapter 3, we apply these methods to the DSW system and present many solutions. In addition, we transformed the one of the exact solutions found by the new extended direct algebra method into a new solution under the symmetry transformation. We gave numerical simulations of the solutions obtained for different values of the parameters in 3D in Chapter 4. We also gave 2D plots to see how the wave motion changes as time changes. We have given a 2D drawing to see how the wave motion can be realized as time changes.
In the last chapter, the results obtained in this study are given.

2. MATERIAL AND METHODS

In this section, information about Jacobi elliptic functions will be given and the extended Jacobi elliptic function expansion method will be introduced.

2.1. Jacobi Elliptic Function

Legendre, who worked for decades on elliptic integrals, first introduced by John Wallis between 1655 and 1659, introduced the normal forms of elliptic integrals that are still in use. Later, in 1828, Jacobi defined the elliptic functions as inverses of the elliptic integrals. Jacobi elliptic functions are obtained by inverting the elliptic integrals of the first kind. For a given constant m, the function snu is defined with the help of the integral

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}\,.$$

When this integral is reversed, than the Jacobi elliptic function snu is defined as $x = snu$. Similarly, cnu and dnu functions can be defined with the help of following identities:

$$sn^2\xi + cn^2\xi = 1\,,\;\; dn^2\xi + m^2\, sn^2\xi = 1. \tag{2}$$

where $dn(\xi)$ is a third kind Jacobian elliptic function.

With these definitions, $sn0 = 0$, $cn0 = 1$ and $dn0 = 1$ are obtained clearly. Each Jacobi elliptic function depends on a parameter m and this parameter is called the modulus of the Jacobi elliptical function. The aforementioned double periodic functions have the following properties:

$$\frac{d}{d\xi}sn(\xi) = cn(\xi)dn(\xi),\;\; \frac{d}{d\xi}cn(\xi) = -sn(\xi)dn(\xi),\;\; \frac{d}{d\xi}dn(\xi) = -m^2cn(\xi)sn(\xi) \tag{3}$$

Other Jacobi functions which is denoted by Glaisher's symbols and are generated by these three kinds of functions, namely.

$$ns\xi = \frac{1}{sn\xi},\; nc\xi = \frac{1}{cn\xi}\;,\; nd\xi = \frac{1}{dn\xi},\;\; sc\xi = \frac{cn\xi}{sn\xi},\;\; cs\xi = \frac{sn\xi}{cn\xi},\;\; ds\xi = \frac{dn\xi}{sn\xi},\;\; sd\xi = \frac{sn\xi}{dn\xi} \tag{4}$$

that have the relations

$$ns^2\xi - cs^2\xi = 1\,,\;\; ns^2\xi = m^2 + ds^2\xi\;\;,\;\; sc^2\xi + 1 = nc^2\xi\,,\;\; m^2sd^2\xi + 1 = nd^2\xi$$

with the modals m $(0 < m < 1)$.

2.2. Extended Jacobi Elliptic Function Expansion Method

This section presents the extended Jacobi elliptic function expansion method for solving nonlinear evolution equations. Consider a nonlinear evolution equation of the form:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0 \tag{5}$$

By using the transformation $\xi = k(x - wt)$ and $u(x,t) = U(\xi)$, the Eq(5) can be transformed into an ordinary differential equation of the form

$$B(U, U', U'', \dots) = 0 \tag{6}$$

where $= U(\xi), U' = \frac{dU}{d\xi}, U'' = \frac{dU}{d\xi}, \dots$. To find periodic and solitary wave solutions of Eq (5), we assume that $u = u(\xi)$ can be expressed as a finite series of Jacobi elliptic sine and cosine functions.

Using ten different Jacobi elliptic functions, we assume that the solutions of Eq (6) will be in the following forms:

$$u(\xi) = a_0 + \sum_{j=1}^{n} f_i^{\,j-1}(\xi)\left[a_j f_i(\xi) + b_j g_i(\xi)\right], \quad i = 1,2,3,\cdots \tag{7}$$

with

$$\begin{gathered} f_1(\xi) = sn(\xi) \text{ and } g_1(\xi) = cn(\xi) \\ f_2(\xi) = sn(\xi) \text{ and } g_2(\xi) = dn(\xi) \\ f_3(\xi) = ns(\xi) \text{ and } g_3(\xi) = cs(\xi) \\ f_4(\xi) = ns(\xi) \text{ and } g_4(\xi) = ds(\xi) \\ f_5(\xi) = sc(\xi) \text{ and } g_5(\xi) = nc(\xi) \\ f_6(\xi) = sd(\xi) \text{ and } g_6(\xi) = nd(\xi) \end{gathered} \tag{8}$$

where $n, a_j, b_j \quad (j = 0,1,2,3, \dots)$ are constants.

To determine the value of n, we balance the highest power nonlinear term with the highest order derivative. Thus, the highest degree of $\frac{d^pU}{d\xi^p}$ is taken as:

$$O\left(\frac{d^pU}{d\xi^p}\right) = n + p,\ p = 1,2,3, \dots$$

and the nonlinear term as

$$O\left(U^q \frac{d^pU}{d\xi^p}\right) = (q+1)n + p,\ q = 0,1,2,3, \dots$$

Replacing each f_i, g_i in (8) to corresponding f_i, g_i in (7) we get the new ansatz. Then selecting one of these outcomes and substituting it into (6) and equating to zero the coefficients of all powers of elliptic functions, we obtain a system of algebraic equations for a_j, b_j $(j = 0,1,2, \dots)$. By substituting a_j, b_j in (6), the solution of Eq.(6) is obtained. In this solution of Eq.(6), the solution of Eq.(5) is obtained by taking $\xi = k(x - wt)$ [18].

3. APPLYING THE METHODS TO DSW-SYSTEM

3.1. Extended Jacobi Elliptic Function Expansion Method to DSW System

To apply the extended Jacobi elliptic function expansion method to solve Eq.(1), we substitute the following transformation:

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = k(x - wt)$$

Thus, Eq. (1) can be written in the following form:

$$-kwu' + \alpha k v v' = 0$$

$$-wv' + \beta uv' + su'v + \eta k^2 v''' = 0 \tag{9}$$

From the 1st equation in (9) we have

$$u = \frac{\alpha}{2w} v^2. \tag{10}$$

Using (10) in the 2nd equation in (9) and then integrating, we get undermentioned ODE:

$$-6w^2 v + \alpha(\beta + 2s)v^3 + 6\eta w k^2 v'' = 0 \tag{11}$$

Balancing the highest power nonlinear term and the highest order derivative yields n=1. Since $n = 1$, if (7) is used and $f_1(\xi) = sn(\xi)$ and $g_1(\xi) = cn(\xi)$ are selected from (8), we get

$$v(\xi) = a_0 + a_1 sn(\xi) + b_1 cn(\xi). \tag{12}$$

Using (12) in (11) and collecting all the different powers of $sn^i(\xi)cn^j(\xi), i = 0,1, j = 0..3$, and setting the coefficients of $sn^i(\xi)cn^j(\xi)$ to zero, we obtain a system of nonlinear algebraic equations with respect to a_o, a_1, b_1, k, w which is over-determined system. Solving these equations by the help of Maple™, we get

Set 1:

$$\left\{a_0 = 0,\ a_1 = \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}} mk^2\eta,\quad b_1 = \sqrt{-\frac{6-3m^2}{2\alpha(2s+\beta)}} mk^2\eta,\ w = \frac{\eta k^2(m^2-2)}{2}\right\}. \tag{13}$$

Substituting (13) into (12), we have

$$v_{1,m}(\xi) = \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}} mk^2\eta\, sn(\xi) + \sqrt{-\frac{6-3m^2}{2\alpha(2s+\beta)}} mk^2\eta\, cn(\xi)$$

and using (10) we obtain

$$u_{1,m}(\xi) = -\frac{\alpha}{k^2\eta}\left(\sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}} mk^2\eta\, sn(\xi) + \sqrt{-\frac{-3m^2+6}{2\alpha(2s+\beta)}} mk^2\eta\, cn(\xi)\right)^2.$$

If we calculate the limits of $v_{1,m}(\xi), u_{1,m}(\xi)$ for $m \to 1$, we get $cn(\xi, m) \to sech(\xi)$, $sn(\xi, m) \to tanh(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_1(\xi) = \frac{\sqrt{6}}{2}\sqrt{\frac{1}{\alpha(2s+\beta)}} k^2\eta\, tanh(\xi) + \frac{\sqrt{6}}{2}\sqrt{-\frac{1}{\alpha(2s+\beta)}} k^2\eta sech(\xi)$$

$$u_1(\xi) = -\frac{\alpha}{k^2\eta}\left(\frac{\sqrt{6}}{2}\sqrt{\frac{1}{\alpha(2s+\beta)}} k^2\eta\, tanh(\xi) + \frac{\sqrt{6}}{2}\sqrt{-\frac{1}{\alpha(2s+\beta)}} k^2\eta sech(\xi)\right)^2$$

where $\xi = k(x - wt)$. Therefore the solution of the system (1) is found as

$$\left\{\begin{array}{c} v_1(x,t) = \frac{\sqrt{6}}{2}\sqrt{\frac{1}{\alpha(2s+\beta)}} k^2\eta\, tanh\big(k(x-wt)\big) + \frac{\sqrt{6}}{2}\sqrt{-\frac{1}{\alpha(2s+\beta)}} k^2\eta sech\big(k(x-wt)\big) \\ u_1(x,t) = -\frac{\alpha}{k^2\eta}\left(\frac{\sqrt{6}}{2}\sqrt{\frac{1}{\alpha(2s+\beta)}} k^2\eta\, tanh\big(k(x-wt)\big) + \frac{\sqrt{6}}{2}\sqrt{-\frac{1}{\alpha(2s+\beta)}} k^2\eta sech\big(k(x-wt)\big)\right)^2 \end{array}\right\}. \tag{14}$$

Set 2:

$$\left\{a_0 = 0,\ a_1 = 0,\ \ b_1 = \sqrt{-\frac{12-24m^2}{\alpha(2s+\beta)}}mk^2\eta,\ w = 2\eta k^2 m^2 - \eta k^2\right\} \tag{15}$$

Substituting (15) into (12), we have

$$v_{2,m}(\xi) = \sqrt{-\frac{12-24m^2}{\alpha(2s+\beta)}}mk^2\eta\ cn(\xi)$$

and using (10) we obtain

$$u_{2,m}(\xi) = \frac{(12-24m^2)m^2k^4\eta^2}{2(2\eta k^2m^2-\eta k^2)(2s+\beta)}cn^2(\xi,m)$$

If we calculate the limits of $v_{2,m}(\xi), u_{2,m}(\xi)$ for $m \to 1$, we get $cn(\xi,m) \to sech(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_2(\xi) = \frac{2\sqrt{3}k^2\eta}{\sqrt{\alpha(2s+\beta)}}sech(\xi)$$

$$u_2(\xi) = \frac{6\eta k^2}{2s+\beta}sech^2(\xi)$$

since $\xi = k(x-wt)$. Therefore, the solution of the system (1) is found as

$$\left\{v_2(x,t) = \frac{2\sqrt{3}k^2\eta}{\sqrt{\alpha(2s+\beta)}}sech\big(k(x-wt)\big),\ u_2(x,t) = \frac{6\eta k^2}{2s+\beta}sech^2(k(x-wt))\right\}. \tag{16}$$

Set 3:

$$\left\{a_0 = 0\ ,\ \ a_1 = \sqrt{\frac{12m^2+12}{\alpha(2s+\beta)}}mk^2\eta\ ,\ \ b_1 = 0,\ w = -\eta k^2(m^2+1),\ k = k\ \ \right\} \tag{17}$$

Substituting (17) into (12), we have

$$v_{3,m}(\xi) = \sqrt{\frac{12m^2+12}{\alpha(2s+\beta)}}mk^2\eta\ sn(\xi,m)$$

and using (10) we have

$$u_{3,m}(\xi) = -\frac{6m^2k^2\eta}{2s+\beta}sn^2(\xi,m).$$

If we calculate the limits of $v_{3,m}(\xi), u_{3,m}(\xi)$ for $m \to 1$, we get $sn(\xi,m) \to tanh(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_3(\xi) = 2\sqrt{6}k^2\eta\sqrt{\frac{1}{\alpha(2s+\beta)}}tanh(\xi)$$

$$u_3(\xi) = -\frac{6\eta k^2 tanh^2(\xi)}{2s+\beta},$$

since $\xi = k(x-wt)$. Then the solution of the system (1)

$$\left\{v_3(x,t) = 2\sqrt{6}k^2\eta\sqrt{\frac{1}{\alpha(2s+\beta)}}tanh\big(k(x-wt)\big)\,,\; u_3(x,t) = -\frac{6\eta k^2 tanh^2(k(x-wt))}{2s+\beta}\right\}. \tag{18}$$

Now since $n = 1$, if (7) is used and $f_3(\xi) = ns(\xi)$ and $g_3(\xi) = cs(\xi)$ are selected from (8), we get

$$v(\xi) = a_0 + a_1 ns(\xi) + b_1 cs(\xi). \tag{19}$$

Using (18) in (11) and collecting all the different powers of $ns^i(\xi)cs^j(\xi), i = 0,1,\dots 6\,, j = 0,1$, and setting the coefficients of $ns^i(\xi)cs^j(\xi)$ to zero, we obtain a system of nonlinear algebraic equations with respect to $k,\ w, a_o, a_1, b_1$ which is over-determined system. Solving these equations by the help of the Maple™, we get

Set 1:

$$\left\{a_0 = 0, \quad a_1 = \sqrt{\frac{12m^2+12}{\alpha(2s+\beta)}}k^2\eta, \quad b_1 = 0, \quad w = -\eta k^2(1+m^2)\right\} \tag{20}$$

Substituting (20) into (19), we have

$$v_{4,m}(\xi) = \sqrt{\frac{12m^2+12}{\alpha(2s+\beta)}}k^2\eta\, ns(\xi)$$

and using (10) we obtain

$$u_{4,m}(\xi) = -\frac{12m^2+12}{4(2s+\beta)}k^2\eta\, ns^2(\xi).$$

If we calculate the limits of $v_{4,m}(\xi), u_{4,m}(\xi)$ for $m \to 1$, we get $ns(\xi, m) \to coth(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_4(\xi) = 2\sqrt{6}\sqrt{\frac{1}{\alpha(2s+\beta)}}k^2\eta\, coth(\xi)$$

$$u_4(\xi) = -\frac{6k^2\eta}{2s+\beta}\, coth^2(\xi)$$

where $\xi = k(x-wt)$. Then the solution of the system (1) is found as

$$\left\{v_4(x,t) = 2\sqrt{6}\sqrt{\frac{1}{\alpha(2s+\beta)}}k^2\eta\, coth\big(k(x-wt)\big)\;\;,\; u_4(x,t) = -\frac{6\eta k^2 coth^2(k(x-wt))}{2s+\beta}\right\}. \tag{21}$$

Now since $n = 1$, if (7) is used and $f_4(\xi) = ns(\xi)$ and $g_4(\xi) = ds(\xi)$ are selected from (8), we get

$$v(\xi) = a_0 + a_1 ns(\xi) + b_1 ds(\xi) \tag{22}$$

Using (22) in (11) and collecting all the different powers of $sn^i(\xi)dn^j(\xi), i = 0,1, j = 0..3$, and setting the coefficients of $sn^i(\xi)dn^j(\xi)$ to zero, we obtain a system of nonlinear algebraic equations with respect to a_o, a_1, b_1, k, w which is over-determined. Solving these equations by the help of the Maple™, we get

Set 1:

$$\left\{a_0 = 0, \quad a_1 = \sqrt{\frac{12(1+m^2)}{\alpha(2s+\beta)}}k^2\eta\,, \quad b_1 = 0, k = k, \;\; w = -\eta k^2(m^2+1)\right\}. \tag{23}$$

Substituting (23) into (22), we have

$$v_{5,m}(\xi) = \sqrt{\frac{12(1+m^2)}{\alpha(2s+\beta)}}k^2\eta \;\, ns(\xi)$$

and using (10) we obtain

$$u_{5,m}(\xi) = -3\,\frac{k^2\eta(m^2+1)}{2s+\beta}\; ns^2(\xi).$$

If we calculate the limits of $v_{5,m}(\xi), u_{51,m}(\xi)$ for $m \to 1$, we get $ns(\xi, m) \to 1/\tanh(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_5(\xi) = 2\sqrt{\frac{6}{\alpha(2s+\beta)}}k^2\eta \;\, \frac{1}{tanh(\xi)}$$

$$u_5(\xi) = -\frac{6k^2\eta}{(2s+\beta)}\,\frac{1}{tanh^2(\xi)}$$

where $\xi = k(x - wt)$. Then the solution of the system (1) is found as

$$\left\{v_5(x,t) = 2\sqrt{\frac{6}{\alpha(2s+\beta)}}k^2\eta \;\, \frac{1}{tanh\big(k(x-wt)\big)},\; u_5(x,t) = -\frac{6k^2\eta}{(2s+\beta)}\,\frac{1}{tanh^2\big(k(x-wt)\big)}\right\}. \tag{24}$$

Set 2:

$$\left\{a_0 = 0,\; a_1 = 0,\; b_1 = \sqrt{\frac{12(1-2m^2)}{\alpha(2s+\beta)}}k^2\eta,\;\, k = k,\;\; w = -\eta k^2(m^2+1)\right\} \tag{25}$$

Substituting (25) into (22), we have

$$v_{6,m}(\xi) = \sqrt{\frac{12(1-2m^2)}{\alpha(2s+\beta)}}k^2\eta \;\, ds(\xi)$$

and using (10) we obtain

$$u_{6,m}(\xi) = -\frac{k^2\eta(2m^2-1)}{2s+\beta}\; ds^2(\xi).$$

If we calculate the limits of $v_{6,m}(\xi), u_{6,m}(\xi)$ for $m \to 1$, we get $ds(\xi, m) \to sec\,\mathrm{h}(\xi)\,/\tanh(\xi)$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_6(\xi) = 2\sqrt{\frac{-3}{\alpha(2s+\beta)}}k^2\eta \;\, \frac{\mathrm{sech}\,(\xi)}{tanh(\xi)}$$

$$u_6(\xi) = -\frac{6k^2\eta}{(2s+\beta)}\,\frac{sech^2(\xi)}{tanh^2(\xi)}$$

where $\xi = k(x - wt)$. Then the solution of the system (1) is found as

$$\left\{v_6(x,t) = 2\sqrt{\frac{-3}{\alpha(2s+\beta)}}k^2\eta \;\, \frac{\mathrm{sech}\,(\xi)}{tanh(\xi)},\; u_6(x,t) = -\frac{6k^2\eta}{(2s+\beta)}\,\frac{sech^2(\xi)}{tanh^2(\xi)}\right\}. \tag{26}$$

Set 3:

$$\left\{a_0 = 0,\ a_1 = \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta,\quad b_1 = \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta,\ w = \frac{\eta k^2(m^2-2)}{2}\right\}. \tag{27}$$

Substituting (27) into (21), we have

$$v_{7,m}(\xi) = \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta\, ns(\xi) + \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta\, ds(\xi)$$

and using (10) we obtain

$$u_{7,m}(\xi) = -\frac{\alpha}{k^2\eta}\left(\sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta\, ns(\xi) + \sqrt{-\frac{3m^2-6}{2\alpha(2s+\beta)}}k^2\eta\, ds(\xi)\right)^2.$$

If we calculate the limits of $v_{7,m}(\xi), u_{7,m}(\xi)$ for $m \to 1$, we get $ns(\xi, m) \to \frac{1}{\tanh(\xi)}, ds(\xi, m) \to \frac{sech(\xi)}{\tanh(\xi)}$ and therefore the equations above degenerates into a solution of the DSW equation that is both periodic and exact. It can be written as

$$v_7(\xi) = \sqrt{\frac{6}{\alpha(2s+\beta)}}k^2\eta\ \frac{(1+sech(\xi))}{2tanh(\xi)}$$

$$u_7(\xi) = -\frac{3k^2\eta(1+sech(\xi))^2}{2(2s+\beta)tanh^2(\xi)}$$

where $\xi = k(x - wt)$. Then the solution of the system (1) is found as

$$\left\{v_7(x,t) = \sqrt{\frac{6}{\alpha(2s+\beta)}}k^2\eta\ \frac{(1+sech(\xi))}{2tanh(\xi)},\ u_7(x,t) = -\frac{3k^2\eta(1+sech(\xi))^2}{2(2s+\beta)tanh^2(\xi)}\right\}. \tag{28}$$

3.2. New Exact Solution by Lie Transformation Groups

Lie symmetry analysis is one of the most general and effective methods for obtaining exact solutions of nonlinear partial differential equations. In the last few decades, Lie's method has been applied to a number of physical and engineering models. Solutions of partial differential equations can be transformed into another solution under the act of any symmetry group. Solutions that do not change under a symmetry transformation are called invariant solutions.

In this section, we will transform the exact solution (26) of the system (1) obtained by the extended Jacobi elliptic function method into a new solution under the symmetry transformation.

Essential aim is to yield one new exact solution by the transformation groups of which makes the Eq. (1) invariant.

Equation (1) accepts a three dimensional Lie algebra having the generators given below:

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \mathrm{X}_2 = \frac{\partial}{\partial x}, \mathrm{X}_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.$$

Thus, one parameter $G_i, (i = 1,2,3)$ groups produced by $X_i, (i = 1,2,3)$ can be obtained in the following form:

$$G_1: (t, x, u, v) \to (t + \varepsilon, x, u, v),$$

$$G_2: (t, x, u, v) \to (t, x + \varepsilon, u, v),$$

$$G_3: (t, x, u, v) \to (te^{3\varepsilon}, xe^{\varepsilon}, ue^{-2\varepsilon}, ve^{-2\varepsilon}).$$

Let us consider the nontrivial generator X_3, then the transformation $\{\bar{u} = f_1(\bar{x}, \bar{t}), \bar{v} = f_2(\bar{x}, \bar{t})\}, \{u(z,t) = f(xe^{\varepsilon}, te^{3\varepsilon})e^{2\varepsilon}, v(z,t) = f(xe^{\varepsilon}, te^{3\varepsilon})e^{2\varepsilon}\}$ is obtained as corresponding transformation where

$$\begin{array}{ll} \bar{x} = xe^{\varepsilon} & \bar{x} = xe^{\varepsilon} \\ \bar{t} = te^{3\varepsilon} & \bar{t} = te^{3\varepsilon} \\ \bar{v} = ve^{-2\varepsilon} & \bar{v} = ve^{-2\varepsilon} \end{array}$$

We know from the theory of Lie groups that using these variables the solution $\{u(x,t), v(x,t)\}$ of the Eq. (1) transforms into another solution of the Eq. (1). Under this symmetry transformation, using solution (26) of the DSW system obtained by the extended Jacobi elliptic function expansion method we reach to the following solution

$$u_{new} = -\frac{6\eta k^2 \operatorname{sech}\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2}{(2s+\beta)\tanh\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2}$$

$$v_{new} = -\frac{6\eta k^2 \operatorname{sech}\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2 e^{2\varepsilon}}{(2s+\beta)\tanh\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2}.$$

Then the solution of the system (1) is found as

$$\left\{ v_{new}(x,t) = -\frac{6\eta k^2 \operatorname{sech}\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2 e^{2\varepsilon}}{(2s+\beta)\tanh\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2},\ u_{new}(x,t) = -\frac{6\eta k^2 \operatorname{sech}\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2}{(2s+\beta)\tanh\left(k(-wte^{3\varepsilon}+xe^{\varepsilon})\right)^2} \right\}. \quad (29)$$

New solutions can be obtained in the same way by using other solutions.

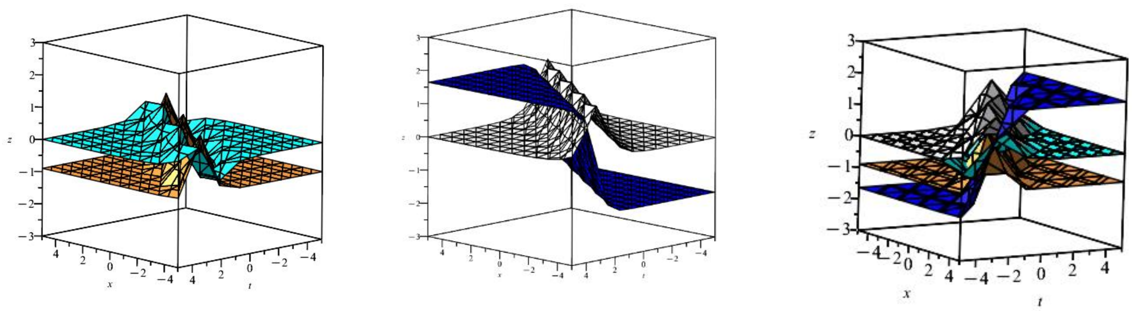
4. GRAPHICAL REPRESENTATIONS OF THE RESULT

Graphs are a powerful tool commonly used to visually represent and communicate data and information. The purpose of graphics is to make data more meaningful and accessible. They are also used to reduce complexity, facilitate focus, and provide quick understanding. In this section, we visualized the exact solutions we obtained using the extended Jacobi elliptic method with specific parameter values in 2D and 3D graphs. These graphs were intended to assist readers in better understanding and interpreting the solutions.

Figure 1 given below is given for to show the behaviour of (14) by using the parameters $\{\alpha = -1, \beta = 3, s = 1, \eta = 3, k = 1, w = -\frac{3}{2}\}$ (in complex plane)
Figure 2 given below is given for to show the behaviour of (14) by using the parameters $\{\alpha = -1, \beta = 3, s = 1, \eta = 3, k = 1, w = -\frac{3}{2}\}$ (in real plane)
Figure 3 given below is given for to show the behaviour of (21) by using the parameters $\{\alpha = 1, \beta = 1,\ s = 1, \eta = -2, k = \frac{1}{2}, w = 1\}$
Figure 4 given below is given for to show the behaviour of (24) by using the parameters $\{\alpha = 1, \beta = 1,\ s = 1, \eta = -2, k = 1, w = 4\}$
Figure 5 given below is given for to show the behaviour of (28) by using the parameters $\{\alpha = 1, \beta = 1,\ s = 1, \eta = 1, k = 1, w = -\frac{1}{2}\}$
Figure 6 given below is given for to show the behaviour of (29) by using the parameters $\{\alpha = 1, \beta = 1,\ s = 1, \eta = 1, k = 1, \varepsilon = \frac{1}{3}, w = -\frac{1}{2}\}$

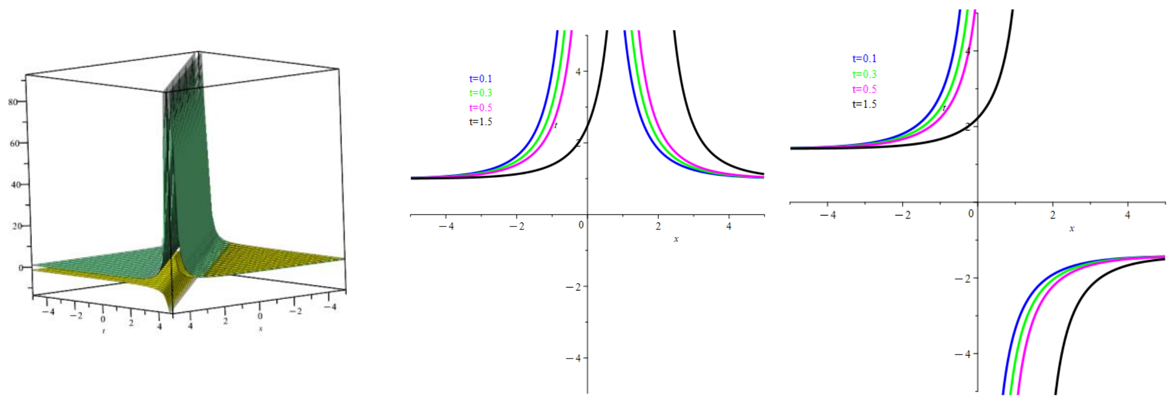
$u_1(x,t)$ $\qquad$ $v_1(x,t)$

Figure 1. Profile of solution (14)



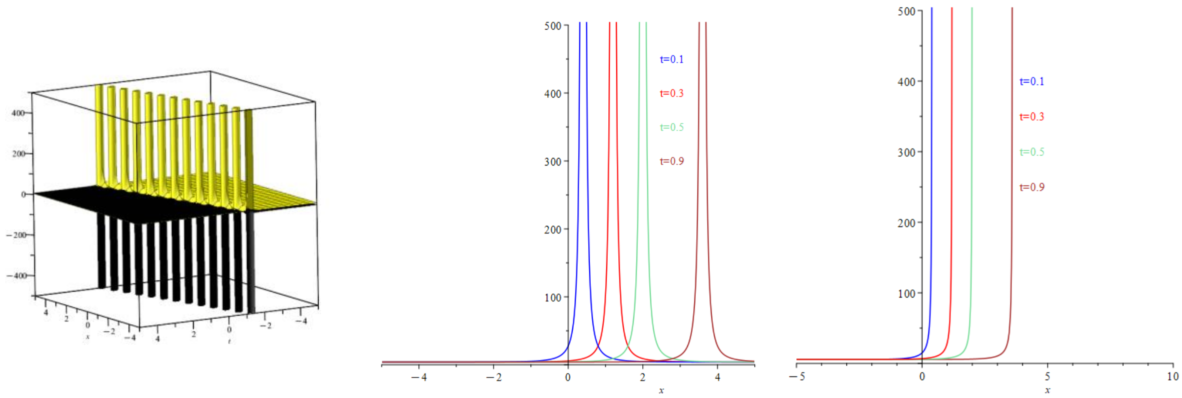
$\{Re(u_1(x,t)), Im(u_1(x,t)\}$ $\qquad$ $\{Re(v_1(x,t)), Im(v(x,t))\}$ $\qquad$ $\{Re(u_1(x,t)), Im(u_1(x,t)), Re(v_1(x,t)), Im(v(x,t))\}$

Figure 2. Profile of solution (14)



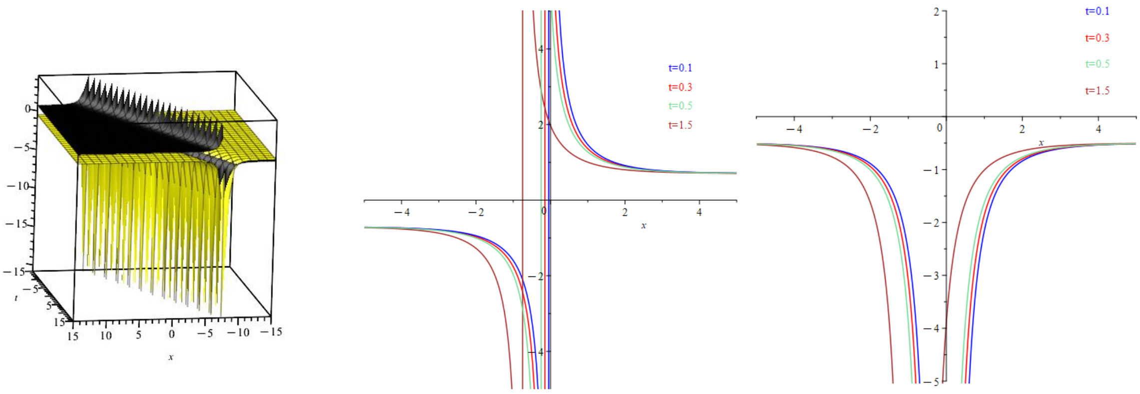
$\{u_4(x,t), v_4(x,t)\}$ $\qquad$ $u_4(x,t_i)$ $\qquad$ $v_4(x,t_i)$

Figure 3. Profile of solution (21)



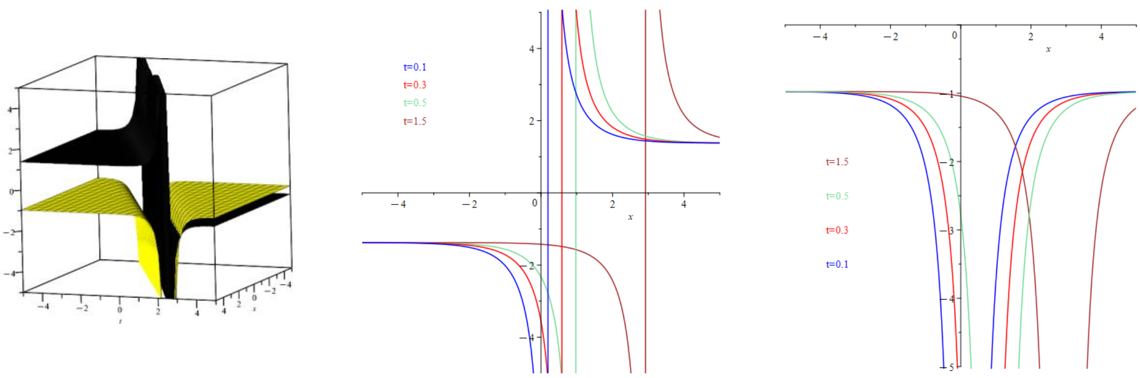
$\{u_5(x,t), v_5(x,t)\}$ $\quad u_5(x,t_i)$ $\quad v_5(x,t_i)$

Figure 4. Profile of solution (24)



$\{u_7(x,t), v_7(x,t)\}$ $\quad u_7(x,t_i)$ $\quad v_7(x,t_i)$

Figure 5. Profile of solution (28)



$\{u_{new}(x,t), v_{new}(x,t)\}$ $\quad u_{new}(x,t_i)$ $\quad v_{new}(x,t_i)$

Figure 6. Profile of solution (29)

5. CONCLUSIONS

In this study, the extended Jacobi elliptic function expansion method was applied to the DSW system and the exact solutions of this system were obtained. Research has demonstrated that the periodic wave solutions derived from the Jacobi elliptic function expansion may vary depending on the choice of the Jacobi elliptic function used. Consequently, this approach can lead to the discovery of a multitude of novel periodic solutions as well as shock wave or solitary wave solutions. The physical characterization of some solutions obtained in our study is depicted in two- and three-dimensional graphics. We produced new solutions with the help of Lie symmetry groups previously given in the literature. The new solution is graphed in Figure 6.

When we reviewed the literature, we noticed that some of the solutions obtained in [17] structurally resemble the solutions we obtained in (14), (16), (18), and (21). However, the enhanced (G'/G)-expansion method used in [17] includes parameters associated with hyperbolic function variables due to the auxiliary equation used in the application of the method. On the other hand, the extended Jacobi elliptic function expansion method we used does not require an auxiliary equation. Therefore, although the solutions we obtained structurally resemble each other, it is observed that different solutions emerge when looking at the hyperbolic function variables they contain. Additionaly it is seen that the solutions (16) and (18) obtained in this study coincide with the solutions obtained using the Jacobi elliptic function expansion method of the same system [20]. According to our research, other solutions obtained are not available in the literature.

Figure 1 shows the multi soliton solution of (14) at special parameter values in the complex space. In Figure 2, a kink shape soliton and a bell shape soliton together represent the solution. Figure 3 shows a bright soliton and singular kink shape soliton. Figure 4 shows the periodic wave solution. The multi soliton solution is also seen in Figure 5. In Figure 6, we see that a singular kink wave and a kink wave together represent the solution.

From the open literature, we notice that Lie symmetry analysis of system (1) is carried out with special coefficient selection. Zhang and Zhao made a special case Lie symmetry analysis of the system by choosing $\alpha=2,\beta=3k,s=3b,\eta=-a$ [21]. They systematically constructed the Lie symmetries together with some symmetry reductions and group invariant solutions corresponding to this reduction. In their study, we saw that the symmetry generators obtained in the special case of the DSW-system and the generators we obtained for the general DSW-system are the same. Therefore, with the new solution generation method we use, an even richer solution set of the system can be obtained.

The obtained solutions were checked one by one by substituting them in the equations of the system with Maple. Numerical simulations of the solutions obtained from the method discussed were performed for specific parameter values.

We think that the new wave solutions obtained by applying extended Jacobi elliptic expansion method from the DSW system, resulting from the interaction between water waves and long waves, will have a significant impact on the field of ocean engineering. Furthermore, this research has the potential to provide novel perspectives on the behaviour of various scientific phenomena.
The solutions obtained can be used as an auxiliary function in the modeling of autonomous underwater vehicles, in the installation of heavy and complex submarine equipment, in the placement of submarine pipelines and submarine cables. We also believe that it will be useful for those working in the fields of physics and engineering in interpreting ocean waves, the physics of underwater sound and how to make sense of sounds underwater.

CONFLICT OF INTEREST

The researcher declared that she had no conflicts of concern relating to the publication of this research.

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