Numerical solutions of Nonlinear Evolution Equation

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Abstract

Nonlinear evolution of acoustic disturbance such as a sound wave is governed by Burgers equation. This presentation will focus on the evolution of wave motion is governed by Burgers equation, comparing solutions in limited time ranges, such as those are obtained by numerical method known as parabolic method, with exact solutions obtained using the method of matched asymptotic coordinate expansions.
Key words: Burgers Equation

1 Introduction

In this presentation I consider an initial-value problem for BurgersŠ equation, namely,

$$u_t + uu_x - uxx = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u(x,0) = \begin{cases} u_+, & x \geqslant 0, \\ u_-, & x < 0 \end{cases} \tag{2}$$

where $u_? > u_+$. In what follows we label initial-value problem (1), (2) as IVP. In this presentation I develop the large-time structure of the solution of IVP using the method of matched asymptotic coordinate expansions. I begin by examining the asymptotic structure of the solution to IVP as $t \to 0$.

2 ASYMPTOTIC STRUCTURE

Asymptotic solutions of IVP as $t \to 0$

Consideration of initial data (2) indicates that the structure of the asymptotic solution of IVP as $t \to 0$ has three asymptotic regions, namely:

$$RegionI : x = o(1) \qquad u(x,t) = O(1)$$

$$RegionII^+ : x = O(1)(> 0) \qquad u(x,t) = u_+ + o(1)$$

$$RegionII^- : x = O(1)(< 0) \qquad u(x,t) = u_- - o(1).$$

To examine region I, I introduce the scaled coordinate $\eta = xt^\alpha$ where $\alpha > 0$ and $\eta = O(1)$, and after some minor calculation, the expansion in region I is obtained as

$$u(\eta) = A_1 + B_1 erfc\left(\frac{\eta}{2}\right) \tag{3}$$

where A_1 and $B_1$are constants to be determined on matching, and erfc[.] is the complementary function. As $\eta \to \infty$ we move into region II$^+$ and after some calculations:
In region II$^+$, we have that

$$u(x,t) = u_+ + \exp\left(-\frac{x^2}{4t} + \frac{1}{2}lnt + \frac{u_+x}{2} - ln(x) + ln\frac{(u_- - u_+)}{\sqrt{\pi}} + o(1)\right) \quad (4)$$

as $t \to 0$ with $x = O(1)(> 0)$.
As $\eta \to -\infty$ we move into region II^- and after some calculations:
In region II^-, we have that

$$u(x,t) = u_- - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}lnt + \frac{u_-x}{2} - ln(-x) + ln\frac{(u_- - u_+)}{\sqrt{\pi}} + o(1)\right) \quad (5)$$

as $t \to 0$ with $x = O(1)(< 0)$.

The asymptotic structure as $t \to 0$ is now complete with the expansions in regions I, II^+ and II^- providing a uniform approximation to the solution of IVP as $t \to 0$.

Asymptotic solutions of IVP as $|x| \to \infty$

Now, I examine the asymptotic structure of the solution to IVP as $|x| \to \infty$ with $t = O(1)$. I first consider the structure of solution to IVP as $x \to \infty$ with $t = O(1)$. In region III^+, I obtain after some calculations that

$$u(x,t) = u_+ + \exp\left(-\frac{x^2}{4t} + \frac{u_+x}{2} - ln(x) + \left(-\frac{{u_+}^2}{4}t + \frac{1}{2}lnt + ln\frac{(u_- - u_+)}{\sqrt{\pi}}\right) + o(1)\right) \quad (6)$$

as $x \to \infty$, with $t = O(1)$. Expansion (6) remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes non-uniform when $x = O(t)$ as $t \to \infty$.

I now investigate the structure of solution of IVP as $x \to -\infty$, with $t = O(1)$. In region III^-, I obtain after some calculations that

$$u(x,t) = u_- - \exp\left(-\frac{x^2}{4t} + \frac{u_-x}{2} - ln(-x) + \left(-\frac{{u_-}^2}{4}t + \frac{1}{2}lnt + ln\frac{(u_- - u_+)}{\sqrt{\pi}}\right) + o(1)\right) \quad (7)$$

as $x \to -\infty$, with $t = O(1)$. Expansion (7) remains uniform for $t \gg 1$ provided that $(-x) \gg t$, but becomes non-uniform when $(-x) = O(t)$ as $t \to \infty$.

Asymptotic solutions of IVP as $t \to \infty$

The asymptotic expansions (6) and (7), which are defined in region III^+ ($x \to \infty$ with $t = O(1)$) and region III^- ($x \to -\infty$ with $t = O(1)$) remain uniform provided $|x| \gg t$ but become nonuniform when $|x| = O(t)$. After minor calculations I have leading order problem and the solution of this problem is combination of a one-parameter family of linear solutions and the associated envelope solution

$$c_0(y) = \begin{cases} \frac{(y-u_+)^2}{4}, & y > u_+ + 2A \\ , A[y - (u_+ + A)], & u_+ + A < y \leqslant u_+ + 2A \end{cases} \quad (8)$$

for each $A > 0$. When each case is considered separately.

(a)In first case $c_0(y) = \frac{(y-u_+)^2}{4}, \quad y > u_+$:
In region IV^+

$$u(y,t) = u_+ + \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}lnt - H(y) + o(1)\right\} \quad (9)$$

as $t \to \infty$ with $y = O(1)(\in (u_+, \infty))$. In region A, I have that

$$u(\eta,t) = u_+ + \frac{2e^{-\frac{\eta^2}{4}}}{D_2 - \sqrt{\pi}}t^{-\frac{1}{2}} + o(t^{-\frac{1}{2}}) \quad (10)$$

as $t \to \infty$ with $\eta = O(1)$. In region V,

$$u = u_- - o(1) \quad ast \to \infty, \tag{11}$$

and we conclude that this case can be ruled out and that $c_0(y)$ is given by (8_1).

(b) In the second case $c_0(y) = A[y - (u_+ + A)]$:
In region IV$^+$(a) I obtain that

$$u(y,t) = u_+ + \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}lnt - H_R(y) + o(1)\right\} \tag{12}$$

as $t \to \infty$ with $y = O(1)(\in (u_+ + 2A, \infty))$ and where $H_R(y) \sim lny - ln\frac{(u_- - u_+)}{\sqrt{\pi}}$. In region TR$^+$ I have that

$$u(\eta,t) = u_+ + \left(\frac{1}{2}e^{-E_R}erfc\left(\frac{\eta}{2}\right) + o(1)\right)e^{-A^2t - A\eta t} \tag{13}$$

as $t \to \infty$ with $\eta = O(1)$. As $\eta \to -\infty$ move out from region TR$^+$ into region IV$^+$(b). I obtain that

$$u(y,t) = u_+ + \exp\{-A[y-(u_+ + A)]t - E_R\} + t^{-\frac{1}{2}}K_R(y)\exp\left\{-\frac{(y-u_+)^2}{4}t\right\} + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_+)^2}{4}t\right\}\right) \tag{14}$$

as $t \to \infty$ with $y = O(1)(\in (u_+ + A, u_+ + 2A))$ and where $K_R(y) \sim \frac{e^{-E_R}}{\sqrt{\pi}((u_+ + 2A) - y)}$. Now as $y \to (u_+ + A)^+$ move out of region IV$^+$(b) into region TW and in this region we have that

$$U(z) = \frac{u_+ + (2A + u_+)e^{-Az}}{1 + e^{-Az}}, \quad -\infty < z < \infty \tag{15}$$

where the translational invariance has been fixed by requiring $U(0) = 1/2$. Before completing region TW, I summarize regions IV$^\pm$(b).
Region IV$^-$(b)

$$u(y,t) = u_- - \exp\{A[y-c]t - E_L\} + t^{-\frac{1}{2}}K_L(y)\exp\left\{-\frac{(y-u_-)^2}{4}t\right\} + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_-)^2}{4}t\right\}\right) \tag{16}$$

as $t \to \infty$ with $y = O(1)(\in (u_+, c))$ where $c = \frac{u_+ + u_-}{2}$ and $A = \frac{u_- - u_+}{2}$.
Region IV$^+$(b)

$$u(y,t) = u_+ + \exp\{A[y-c]t - E_R\} + t^{-\frac{1}{2}}K_R(y)\exp\left\{-\frac{(y-u_+)^2}{4}t\right\} + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_+)^2}{4}t\right\}\right) \tag{17}$$

as $t \to \infty$ with $y = O(1)(\in (c, u_-))$.
In summary, I have in region TW, that

$$u(z,t) = \frac{u_+ + u_- e^{-Az}}{1 + e^{-Az}} + O(t^{-3/2}e^{-\frac{A^2}{4}t}) \tag{18}$$

as $t \to \infty$ with $z = o(1)$ where $A = \frac{u_- - u_+}{2}$, $z = x - s(t)$ and

$$s(t) = ct + O\left(t^{-3/2}e^{-\frac{A^2}{4}t}\right) \tag{19}$$

as $t \to \infty$. We recall that

$$c = \frac{u_+ + u_-}{2}\begin{cases} > 0 & \text{when} \quad u_- > u_+ > -u_- \quad \text{with} \quad u_- > 0 \\ < 0 & \text{when} \quad u_+ < u_- < -u_+ \quad \text{with} \quad u_+ < 0 \\ = 0 & u_+ = -u_- \quad \text{with} \quad u_- > 0 \end{cases}$$

3 Numerical Solutions

In this section representative numerical solutions of IVP are presented which confirm the analysis presented in this chapter. I solved IVP using the numerical method outlined in [7]. In order to obtain numerical solutions of IVP I use a parabolic method with $N = 100$ where N is the number of grid points time step $\triangle t = 0.001$ and the length $\triangle x = 0.005$.

In this Section we consider two sets of problem parameters which illustrate the situation when the wave speed c is positive and when it is negative. The two cases, we will consider are:

1. $u_- = 1, \quad u_+ = 0$
2. $u_+ = -1, \quad u_- = 0$

We now consider these two cases in turn.

$\mathbf{u_- = 1, \quad u_+ = 0}$

In Figure 1 we plot the numerical solution of IVP against x at times $t = 10 \;\; t = 15 \;\; t = 20$ and $\;t = 25$ clearly, the solution converges to the PTW rapidly as $t \to \infty$.

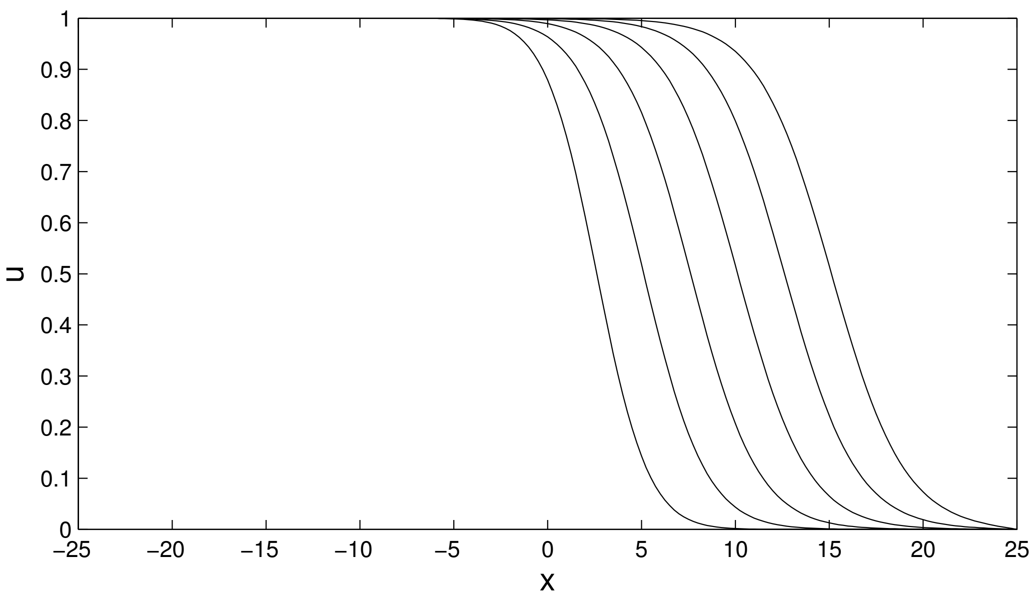

Figure 1: Numerical solution of IVP at times $t = 10 \;\; t = 15 \;\; t = 20$ and $\;t = 25$.

The asymptotic wave speed, $\dot{s}(t)$, is given by

$$\dot{s}(t) = \frac{1}{2} + O(\chi(t)) \tag{20}$$

as $t \to \infty$. Clearly, In Figure 2 the numerically calculated wave speed rapidly approaches the expected value of $\frac{1}{2}$ as $t \to \infty$.

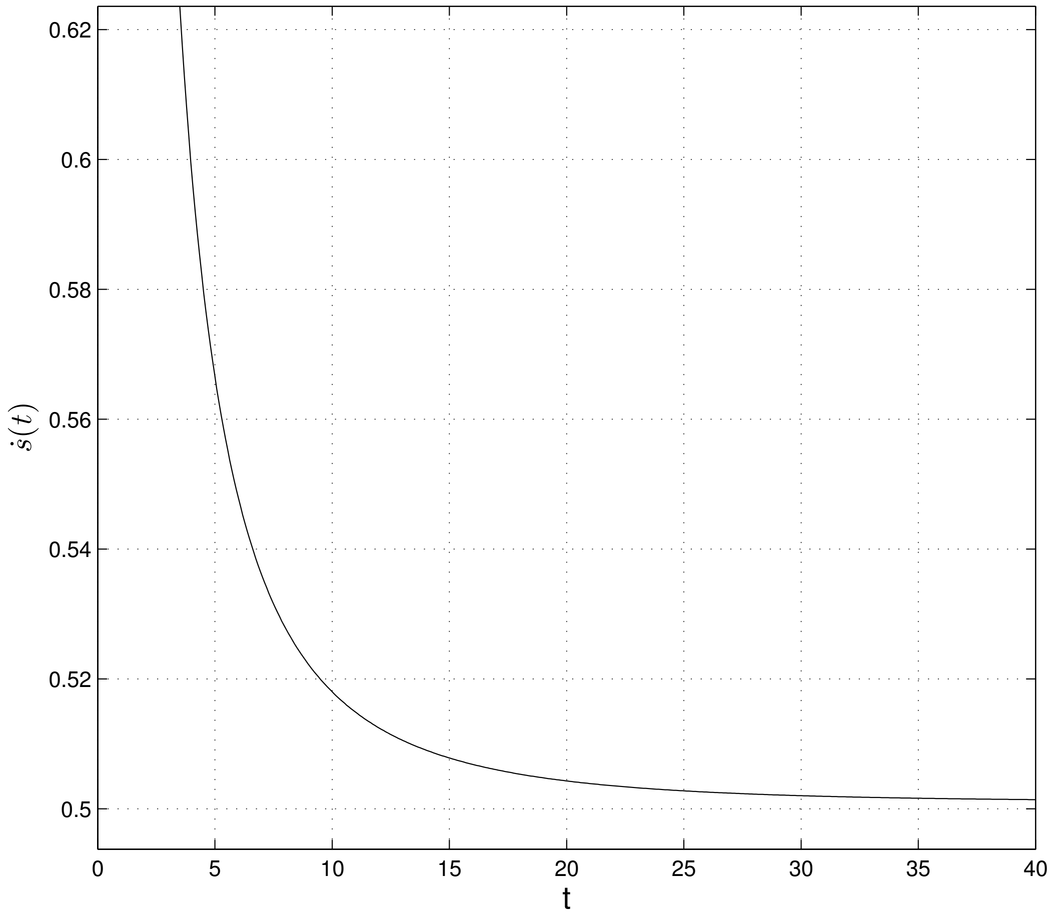

Figure 2: Numerical solution of $\dot{s}(t)$ versus t.

In Figure 3 I observe that the numerically calculated curve rapidly approaches $\phi_0 = 0$ as $t \to \infty$.

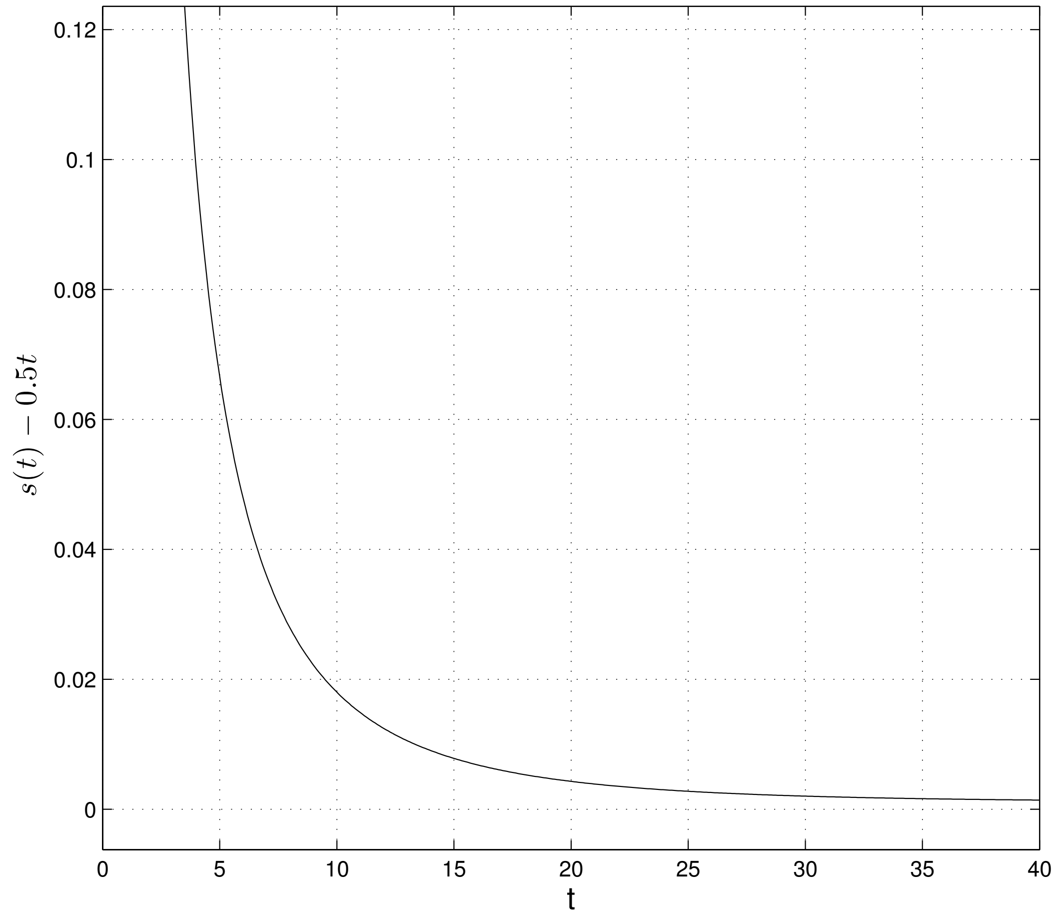

Figure 3: Numerical solution of $s(t) - 0.5t$ versus t.

Finally, in Figure 4 I observe that the numerically calculated curve rapidly approaches the line of gradient $-\frac{1}{16}$ as $t \to \infty$. However, numerical error grows rapidly for $t > 20$.

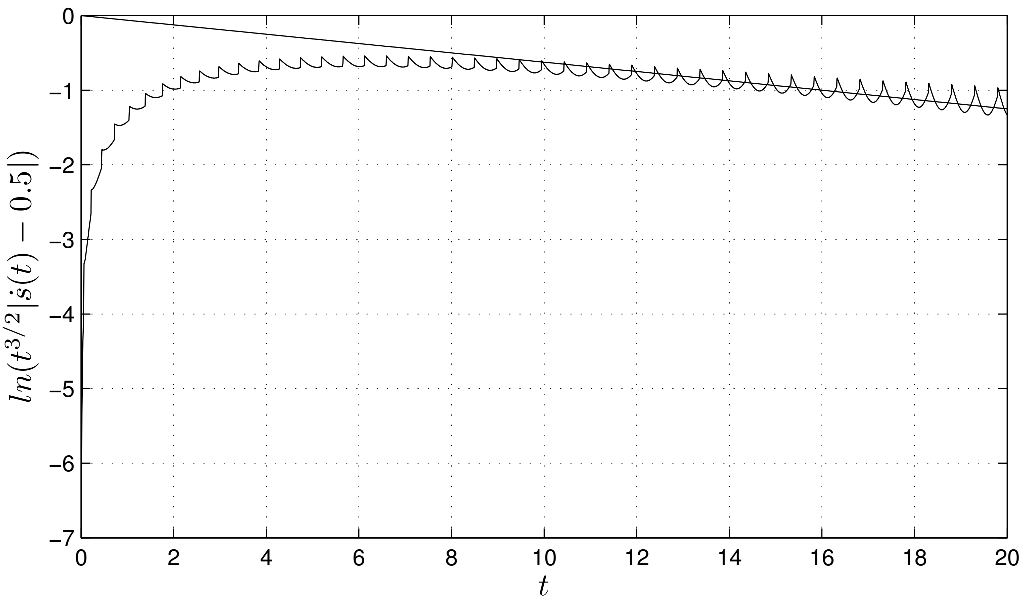

Figure 4: Numerical solution of $\ln\left(t^{3/2}|\dot{s}(t) - \frac{1}{2}|\right)$ versus t.

$\mathbf{u_- = 0, \quad u_+ = -1}$

In Figure 5 I plot the numerical solution of IVP against x at times $t = 10$ $t = 15$ $t = 20$ and $t = 25$ clearly, the solution converges to the PTW rapidly as $t \to \infty$.

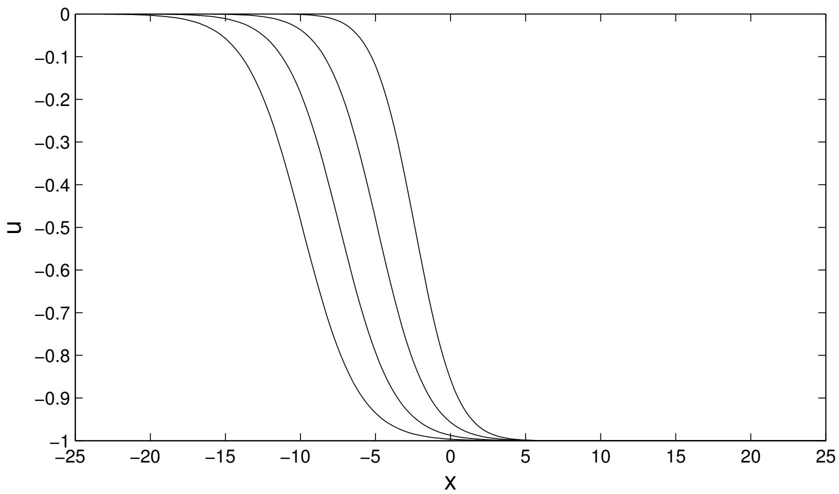

Figure 5: Numerical solution of IVP at times $t = 10$ $t = 15$ $t = 20$ and $t = 25$.

Clearly, In Figure 6 the numerical calculate wave speed rapidly approaches the expected value of $-\frac{1}{2}$ as $t \to \infty$.

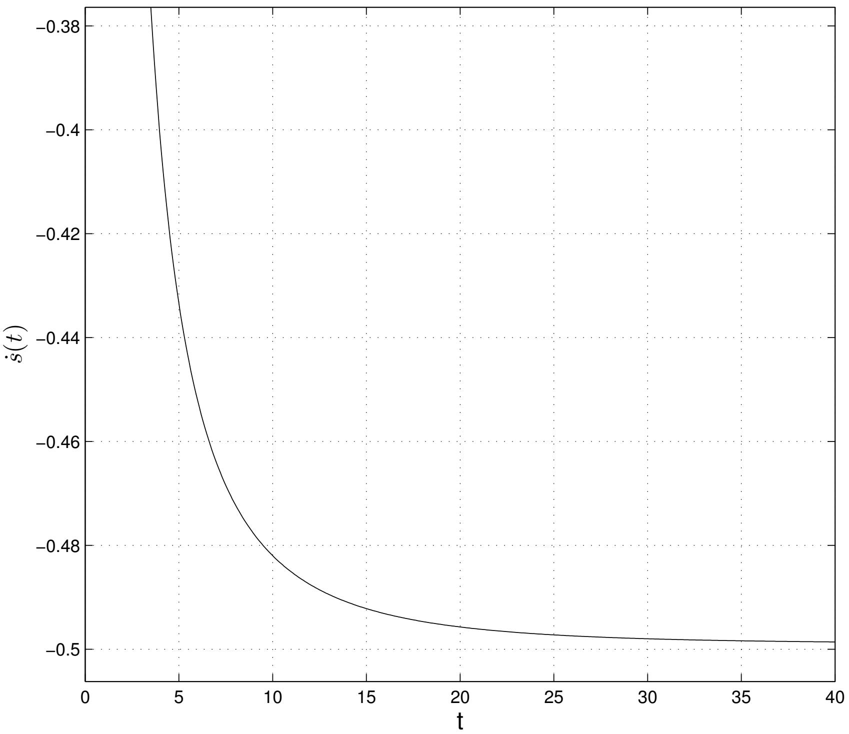

Figure 6: Numerical solution of $\dot{s}(t)$ versus t.

In Figure 7 I plot $s(t) + \frac{1}{2}t$ versus t. I observe that the numerically calculated curve rapidly approaches $\phi_0 = 0$ as $t \to \infty$.

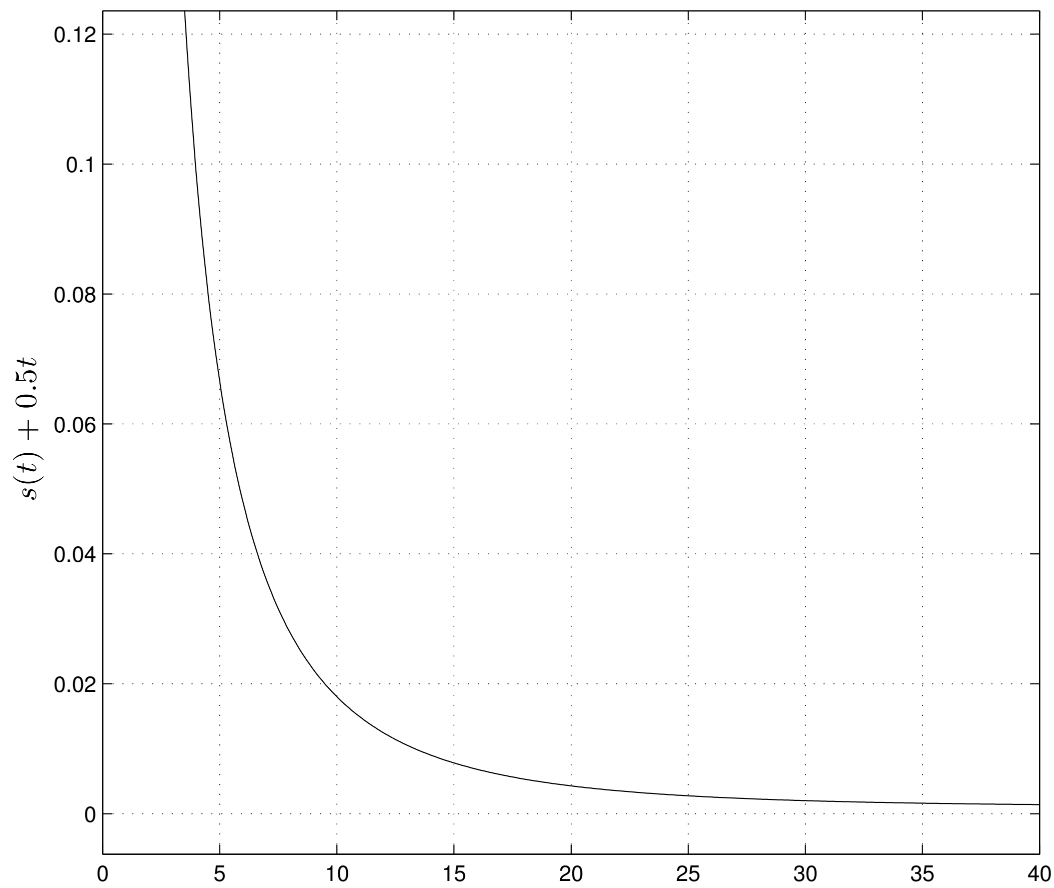

Figure 7: Numerical solution of $s(t) + 0.5t$ versus t.

4 Conclusions

it is worth noting that the structure of solution of IVP as $t \to \infty$ depends critically on interaction between the selected envelope-touching solution of equation in region IV^{+} and the selected envelope solution of equivalent equation in region IV^{-}. These results are in argument with the numerical simulations of Section Numerical Solutions. Figure 1 and 5 are solid forms of propagation in fluids [Figure 8].

Figure 8: Propagation in fluids

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