Sigma J Eng Nat Sci, Vol. 41, No. 2, pp. 226−231, April, 2023

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2023.00034

Research Article

Concerning the generalized Hermite-Hadamard integral inequality

Juan Gabriel GALEANO DELGADO[1], Juan Eduardo NÁPOLES VALDÉS[2], Edgardo PÉREZ REYES[3]

[1]*Departamento de Matemáticas y Estadística, Universidad de Córdoba, Montería, Colombia*
[2]*UNNE, FaCENA, Av. Libertad 5450, Corrientes 3400, and UTN-FRRE, Resistencia, Chaco, Argentina*
[3]*Departamento de Matemáticas y Estadística, Universidad de Córdoba, Montería, Colombia*

ARTICLE INFO

Article history
Received: 22 March 2021
Revised: 25 April 2021
Accepted: 07 May 2021

Keywords:
Hermite-Hadamard Inequality; Generalized Fractional Integral

ABSTRACT

In the work we obtain some Hermite-Hadamard type inequalities for generalized fractional integrals for convex functions by employing a fractional integral operator, establishing, firstly, a basic identity that is used throughout the work. In addition, some classical integral inequalities are special cases of our main findings.

Cite this article as: Galeano Delgado JG, Nápoles Valdés JE, Pérez Reyes E. Concerning the generalized Hermite-Hadamard integral inequality. Sigma J Eng Nat Sci 2023;41(2):226−231.

INTRODUCTION

Fractional calculus is a field of mathematical analysis that studies integrals and derivatives of arbitrary order. The concept of fractional operators appearances almost simultaneously with the development of the classical calculus in a letter written to l'Hˆopital by Leibniz in 1695, where the question of meaning of the fractional derivative has been raised.

Despite Fractional calculus was contemporary with classical calculus, has been gaining attention in the last 40 years and has become one of the most active areas in Mathematics today. For instance, we can find applications in areas like: rheology, viscoelasticity, acoustics, optics, chamical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineerin, etc. [30, 31]. In particular, this has led to the emergence of new comprehensive operators which are natural generalizations of the classical Riemann-Liouville fractional integral. In a previous work (see [11]) the authors define a generalized operator that contains as a particular case several of those reported in the literature.

Some Preliminaries on the Fractional integration

Let $\phi : I \subseteq R \to R$ be a convex function defined on the interval I of real numbers and $a_1, a_2 \in I$ with $a_1 < a_2$. The following inequality

$$\varphi\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \varphi(u)du \leq \frac{\varphi(a_1)+\varphi(a_2)}{2} \qquad (1)$$

holds. This inequality is known in the literature as a Hermite-Hadamard integral inequality for convex functions [12]. Some extensions and generalizations of this

***Corresponding author.**
*E-mail address: jgabrielgaleano@correo.unicordoba.edu.co.

This paper was recommended for publication in revised form by Regional Editor Oscar Castillo

Published by Yıldız Technical University Press, İstanbul, Turkey

inequality, with different fractional and generalized operators and using different convexity operators, can be consulted in [3, 4, 5, 6, 7, 8, 10, 13, 15, 16, 19, 20, 21, 22, 23, 24, 25, 26,33, 34, 35, 36, 37, 38].

Definition 1. The k-generalized fractional Riemann-Liouville integral of order α with α ∈ R, and s/= −1 of an integrable function ϕ(u) on [0,∞), are given as follows (right and left, respectively):

$$ {}^{s}I_{F,a_1+}^{\frac{\alpha}{k}}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u}\frac{F(\tau,s)\varphi(\tau)d\tau}{[\boldsymbol{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \quad (2) $$

$$ {}^{s}J_{F,a_2^-}^{\frac{\alpha}{k}}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{u}^{a_2}\frac{F(\tau,s)\varphi(\tau)d\tau}{[\boldsymbol{F}(\tau,u)]^{1-\frac{\alpha}{k}}}. \quad (3) $$

With $F(\tau,0)=1,\ \boldsymbol{F}(u,\tau)=\int_\tau^u F(\theta,s)d\theta,\ \ \boldsymbol{F}(\tau,u)=\int_u^\tau F(\theta,s)d\theta.$ .With the functions Γ (see [27, 28, 29, 39, 40]) and Γ_k defined by (cf. [9]):}

$$ \Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}\,d\tau, \quad R(z) > 0, \quad (4) $$

$$ \Gamma_k(z) = \int_0^\infty \tau^{z-1}e^{-\tau^k/k}\,d\tau, k > 0. \quad (5) $$

It is clear that if k→1 we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$. As well, we define the k-beta function as follows $B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}\,d\tau$, notice that $B_k(u,v) = \frac{1}{k}B\left(\frac{u}{k},\frac{v}{k}\right)$ and $B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

Remark 2. We want to point out that for adequate kernel F choices, we can obtain as particular cases, several fractional integral operators. So we have:

i) The classic Riemann integral is obtained with $F(t,\alpha) = t^{\alpha-1}$, α = 1 and β = k (with notation changed).

ii) If $F(t,\alpha) = t^{\alpha-1}$ and β = k we obtain the fractional Riemann-Liouville integral.

iii) Considering $F(t,\alpha) = t^s$ with s = 1, we can write the right sided operator as follows $\left({}^{\beta}_{k}J^{\alpha}_{F,a+}f\right)(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\frac{\beta}{k}}}$ and similarly the left sided integral. The k-Riemann-Liouville fractional integral of Mubeen and Habibullah (see [18]).

iv) Katugampola fractional integral of [14] is obtained, taking $F(t,\alpha) = t^{-\alpha}$ (the notation is changed).

v) If we put $F = t^{-s}$ with s = 1, then we get the right sided Hadamard fractional integral of [12].

vi) An integral operator with non-singular kernel can also be obtained from our Definition 1. Thus, considering $F(t,\alpha) = exp\left[-\frac{1-\alpha}{\alpha}t\right]$, if α = 1 we have that F = 1. In this case $F(\boldsymbol{F}_+(x,t),\beta) = exp\left[\frac{1-\beta}{\beta}(x-t)\right]$, a slight modification of the operator defined by Kirane and Toberek in [2].

The main purpose of this paper, using the generalized fractional integral operator of the Riemann- Liouville type, from Definition 1, is to establish several integral inequalities of Hermite-Hadamard type, which contain as particular cases, several of those reported in the literature.

Main Results

Let ϕ : I° →R be a given function, where $a_1,a_2 \in$ Io with $0 < a_1 < a_2 < \infty$. We assume that $\phi \in L^\infty[a_1,a_2]$ such that ${}^{s}I_{F,a_1+}^{\frac{\alpha}{k}}\varphi(u)$ and ${}^{s}J_{F,a_2^-}^{\frac{\alpha}{k}}\varphi(u)$ are well defined. We define $\tilde{\varphi}(u) := \varphi(a_1+a_2-u),\ u\in[a_1,a_2]$ and $G(u) := \varphi(u)+\tilde{\varphi}(u),\ u\in[a_1,a_2]$. Notice that by using the change of variables $w = \frac{\tau-a_1}{u-a_1}$, we have that (2) becomes in

$$ {}^{s}J_{F,a_1^+}^{\frac{\alpha}{k}}\varphi(u) = \frac{(u-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wu+a_1(1-w),s)\varphi(wu+a_1(1-w))dw}{[\boldsymbol{F}(u,wu+a_1(1-w))]^{1-\frac{\alpha}{k}}}, \quad (6) $$

where $u > a_1$.

Theorem 3. For α,k > 0 and s≠-1. If φ is a convex function on $[a_1,a_2]$, then we get

$$ \varphi\left(\frac{a_1+a_2}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\ {}^{s}J_{F,a_1^+}^{\frac{\alpha}{k}}G(a_2) + \ {}^{s}J_{F,a_2^-}^{\frac{\alpha}{k}}G(a_1)\right] \le \frac{\varphi(a_1)+\varphi(a_2)}{2}, \quad (7) $$

Proof. For w ∈ [0,1], let $\eta_1 = a_1w+(1-w)a_2$ and $\eta_2 = (1-w)a_1+a_2w$. As φ is convex, we obtain

$$ \varphi\left(\frac{a_1+a_2}{2}\right) = \varphi\left(\frac{\eta_1+\eta_2}{2}\right) \le \frac{\varphi(\eta_1)}{2} + \frac{\varphi(\eta_2)}{2}. $$

That is,

$$ \varphi\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{2}\varphi(a_1w+(1-w)a_2) + \frac{1}{2}\varphi((1-w)a_1+a_2w). \quad (8) $$

Now, multiplying both sides of (8) by

$$ \frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} $$

and integrating over (0,1) with respect to w, we have

$$ \frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\varphi\left(\frac{a_1+a_2}{2}\right)\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\boldsymbol{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} \le $$

$$ \le \frac{1}{2}\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\boldsymbol{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} + $$

$$ +\frac{1}{2}\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\boldsymbol{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}}. $$

Additionally, we note that

$$\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} = \frac{k[F(a_2,a_1)]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)}.$$

Using the identity $\tilde{\varphi}((1-w)a_1+a_2w) = \varphi(a_1w+(1-w)a_2)$, from (6) we get

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} = {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\tilde{\varphi}(a_2)$$

and

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} = {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2).$$

Therefore, we obtain

$$\frac{[F(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\varphi\left(\frac{a_1+a_2}{2}\right) \le \frac{{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)}{2}. \quad (9)$$

Analogously, multiplying both sides of (8) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_1+a_2(1-w),s)}{[\mathbf{F}(wa_1+a_2(1-w),a_1)]^{1-\frac{\alpha}{k}}},$$

integrating over (0,1) with respect to w, and from (6), we also have

$$\frac{[F(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\varphi\left(\frac{a_1+a_2}{2}\right) \le \frac{{}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)}{2}. \quad (10)$$

Thus, from inequalities (9) and (10), we obtain

$$\varphi\left(\frac{a_1+a_2}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right],$$

which is the left-hand side of (7). Besides, since φ is convex, for w ∈ [0,1] we get

$$\varphi(a_1w+(1-w)a_2) + \varphi((1-w)a_1+a_2w) \le \varphi(a_1)+\varphi(a_2). \quad (11)$$

Multiplying both sides of (11) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[F(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}}$$

and integrating over (0,1) with respect to w, we get

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} +$$

$$+\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}} \le$$

$$\frac{(a_2-a_1)[\varphi(a_1)+\varphi(a_2)]}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbf{F}(a_2,wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}}.$$

That is,

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) \le \frac{[F(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}[\varphi(a_1)+\varphi(a_2)]. \quad (12)$$

Similarly, if we multiple both sides of (11) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_1+a_2(1-w),s)}{[\boldsymbol{F}(wa_1+a_2(1-w),a_1)]^{1-\frac{\alpha}{k}}},$$

integrating over (0,1) with respect to w, and from (6), we obtain

$${}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1) \le \frac{[F(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}[\varphi(a_1)+\varphi(a_2)]. \quad (13)$$

Finally, adding (12) and (13), we get

$$\frac{\Gamma_k(\alpha+k)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right] \le \frac{\varphi(a_1)+\varphi(a_2)}{2}.$$

Therefore, the proof is complete.

Remark 4. If in the Theorem 3 we put $F(\tau,s)=\tau^s$, we have $[\boldsymbol{F}_+(u,\tau)]^{1-\frac{\alpha}{k}} = \left[\frac{u^{s+1}-\tau^{s+1}}{s+1}\right]^{1-\frac{\alpha}{k}}$, so the above Theorem becomes Theorem 2.1 of [1].

The next result will be crucial going forward.

Lemma 5. For α,k > 0 and s≠-1. If φ is a differentiable function on Io such that $\varphi' \in L[a_1,a_2]$, then we have

$$\frac{\varphi(a_1)+\varphi(a_2)}{2} - \frac{\Gamma_k(\alpha+k)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) + {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]$$
$$= \frac{(a_2-a_1)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1 \chi_{\alpha,s}(\tau)\varphi'(\tau a_1+(1-\tau)a_2)d\tau, \quad (14)$$

where

$$\chi_{\alpha,s}(\tau) = [F(\tau a_1+(1-\tau)a_2,a_1)]^{\frac{\alpha}{k}} - [F(\tau a_2+(1-\tau)a_1,a_1)]^{\frac{\alpha}{k}}$$
$$+[F(a_2,\tau a_2+(1-\tau)a_1)]^{\frac{\alpha}{k}} - [F(a_2,\tau a_1+(1-\tau)a_2)]^{\frac{\alpha}{k}}.$$

Proof. Integrating by parts, we get

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2) = \frac{[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}G(a_1)}{\Gamma_k(\alpha+k)} +$$

$$+\frac{(a_2-a_1)}{\Gamma_k(\alpha+k)}\int_0^1\left[\boldsymbol{F}\left(a_2,a_2w+a_1(1-w)\right)\right]^{\frac{\alpha}{k}}G'(a_1(1-w)+a_2w)dw. \quad (15)$$

Similarly, we obtain

$${}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)=\frac{[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}G(a_2)}{\Gamma_k(\alpha+k)}-$$

$$-\frac{(a_2-a_1)}{\Gamma_k(\alpha+k)}\int_0^1[\boldsymbol{F}(a_2w+a_1(1-w),a_1)]^{\frac{\alpha}{k}}G'(a_1(1-w)+a_2w)dw. \quad (16)$$

Now, using the fact that $G(u)=\varphi(u)+\tilde{\varphi}(u)$, and computing (15) and (16), we have

$$\frac{4[\mathbf{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{a_2-a_1}\left(\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[F(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+{}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right)=$$

$$=\int_0^1\left[[\boldsymbol{F}(a_2w+a_1(1-w),a_1)]^{\frac{\alpha}{k}}-\left[\boldsymbol{F}\left(a_2,a_2w+a_1(1-w)\right)\right]^{\frac{\alpha}{k}}\right]G'(a_2w+(1-w)a_1)dw. \quad (17)$$

Notice that we have

$$G'\left(a_2w+a_1(1-w)\right)=\varphi'\left(a_2w+a_1(1-w)\right)-\varphi'\left(a_1w+a_2(1-w)\right), w\in[0,1].$$

Thus, we can obtain

$$\begin{aligned}&\int_0^1[\mathbf{F}(a_2w+a_1(1-w),a_1)]^{\frac{\alpha}{k}}G'\left(a_2w+a_1(1-w)\right)dw\\&=\int_0^1[\boldsymbol{F}(a_1\tau+a_2(1-\tau),a_1)]^{\frac{\alpha}{k}}\varphi'\left(a_1\tau+a_2(1-\tau)\right)d\tau \quad (18)\\&-\int_0^1[\boldsymbol{F}(a_2\tau+a_1(1-\tau),a_1)]^{\frac{\alpha}{k}}\varphi'\left(a_1\tau+a_2(1-\tau)\right)d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\left[\boldsymbol{F}\left(a_2,a_2w+a_1(1-w)\right)\right]^{\frac{\alpha}{k}}G'\left(a_2w+a_1(1-w)\right)dw\\&=\int_0^1\left[\boldsymbol{F}\left(a_2,a_1\tau+a_2(1-\tau)\right)\right]^{\frac{\alpha}{k}}\varphi'\left(a_1\tau+a_2(1-\tau)\right)d\tau \quad (19)\\&-\int_0^1[\boldsymbol{F}(a_2,a_2\tau+a_1(1-\tau),a_1)]^{\frac{\alpha}{k}}\varphi'\left(a_1\tau+a_2(1-\tau)\right)d\tau.\end{aligned}$$

Thus, from (17), (18) and (19) we obtain (14).

Remark 6. Under the Remark 4 conditions, this result becomes Lemma 2.1 of [1].

Now, for $\alpha,k>0$, $s\neq-1$ and $u,v\in[a_1,a_2]$, we introduce the following operator:

$$\psi(s,u,v):=\int_{a_1}^{\frac{a_1+a_2}{2}}|u-w||\boldsymbol{F}(v,w)|^{\frac{\alpha}{k}}dw-\int_{\frac{a_1+a_2}{2}}^{a_2}|u-w||\boldsymbol{F}(v,w)|^{\frac{\alpha}{k}}dw.$$

Using Lemma 5, we can get the following result.

Theorem 7. For $\alpha,k>0$ and $s\neq-1$. If φ is a differentiable function on Io such that $\varphi'\in L[a_1,a_2]$ and $|\varphi'|$ is convex on $[a_1,a_2]$, then

$$\begin{aligned}&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+{}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right|\\&\leq\frac{\Theta(s,\alpha,a_1,a_2)}{4[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}(a_2-a_1)}(|\varphi'(a_1)|+|\varphi'(a_2)|),\end{aligned} \quad (20)$$

where

$$\Theta(s,\alpha,a_1,a_2)=\psi(s,a_2,a_2)+\psi(s,a_1,a_2)-\psi(s,a_2,a_1)-\psi(s,a_1,a_1).$$

Proof. Using Lemma 5 and the convexity of $|\varphi'|$, we have

$$\begin{aligned}&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left[{}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+{}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right|\\&\leq\frac{(a_2-a_1)}{4[\mathbf{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\int_0^1|\chi_{\alpha,s}(\tau)||\varphi'(\tau a_1+(1-\tau)a_2)|d\tau\end{aligned} \quad (21)$$

$$\leq\frac{(a_2-a_1)}{4[\boldsymbol{F}(a_2,a_1)]^{\frac{\alpha}{k}}}\left(|\varphi'(a_1)|\int_0^1\tau|\chi_{\alpha,s}(\tau)|d\tau+|\varphi'(a_2)|\int_0^1(1-\tau)\,|\chi\_\{\alpha,s\}(\tau)|\,d\tau\right).$$

Observe that

$$\int_0^1\tau|\chi_{\alpha,s}(\tau)|d\tau=\frac{1}{(a_2-a_1)^2}\int_{a_1}^{a_2}|\rho(w)|(a_2-w)dw,$$

where

$$\rho(w)=[\boldsymbol{F}(w,a_1)]^{\frac{\alpha}{k}}-[\boldsymbol{F}(a_2+a_1-w,a_1)]^{\frac{\alpha}{k}}+[\boldsymbol{F}(a_2,a_2+a_1-w)]^{\frac{\alpha}{k}}-[\boldsymbol{F}(a_2,w)]^{\frac{\alpha}{k}}.$$

Note that ρ is non-decreasing function on $[a_1,a_2]$. Moreover, we get $\rho(a_1)<0$ and $\rho\left(\frac{a_1+a_2}{2}\right)=0$. Thus, we obtain

$$\begin{cases}\rho(w)\leq 0 \ \ if\ a_1\leq w\leq\dfrac{a_1+a_2}{2},\\ \rho(w)>0\ \ if\ \dfrac{a_1+a_2}{2}<w\leq a_2.\end{cases}$$

Thus, we get $(a_2-a_1)^2\int_0^1\tau|\chi_{\alpha,s}(\tau)|d\tau=\lambda_1+\lambda_2+\lambda_3+\lambda_4$, where

$$\lambda_1=\int_{a_1}^{\frac{a_1+a_2}{2}}(a_2-w)[\boldsymbol{F}(a_2,w)]^{\frac{\alpha}{k}}dw-\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2-w)[\boldsymbol{F}(a_2,w)]^{\frac{\alpha}{k}}dw,$$

$$\lambda_2=-\int_{a_1}^{\frac{a_1+a_2}{2}}(a_2-w)[\boldsymbol{F}(w,a_1)]^{\frac{\alpha}{k}}dw+\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2-w)[\boldsymbol{F}(w,a_1)]^{\frac{\alpha}{k}}dw,$$

$$\lambda_3=\int_{a_1}^{\frac{a_1+a_2}{2}}(a_2-w)[\boldsymbol{F}(a_2+a_1-w,a_1)]^{\frac{\alpha}{k}}dw-\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2-w)[\boldsymbol{F}(a_2+a_1-w,a_1)]^{\frac{\alpha}{k}}dw,$$

$$\lambda_4=-\int_{a_1}^{\frac{a_1+a_2}{2}}(a_2-w)[\boldsymbol{F}(a_2,a_2+a_1-w)]^{\frac{\alpha}{k}}dw+\int_{\frac{a_1+a_2}{2}}^{a_2}(a_2-w)[\boldsymbol{F}(a_2,a_2+a_1-w)]^{\frac{\alpha}{k}}dw.$$

We note that $\lambda_1=\psi(s,a_2,a_2)$ and $\lambda_2=-\psi(s,a_2,a_1)$, and using the change of variable $r=a_2+a_1-w$, we obtain that $\lambda_3=-\psi(s,a_1,a_1)$ and $\lambda_4=-\psi(s,a_1,a_2)$. Therefore, we get

$$\int_0^1\tau|\chi_{\alpha,s}(\tau)|d\tau=\frac{\psi(s,a_2,a_2)+\psi(s,a_1,a_2)-\psi(s,a_2,a_1)-\psi(s,a_1,a_1)}{(a_2-a_1)^2}. \quad (22)$$

Similarly, we obtain

$$\int_0^1 (1-\tau)\left|\chi_{\alpha,s}(\tau)\right|d\tau = \frac{\psi(s,a_2,a_2)+\psi(s,a_1,a_2)-\psi(s,a_2,a_1)-\psi(s,a_1,a_1)}{(a_2-a_1)^2}. \quad (23)$$

Therefore, the inequality (20) follows from (21), (22) and (23).

Remark 8. As before, this result contains as a particular case Theorem 2.2 of [1].

CONCLUSIONS

We want to point out, by way of conclusion, that we have obtained some results, including a Hermite-Hadamard Inequality, which generalize several previous results for k-integral operators and that, if we choose other functions F(τ, s), we will obtain additional inequalities not reported in the literature. Thus, putting F(τ, s) = τs and k = 1, we obtain integral inequalities for the Katugampola fractional integral, not known by the authors.

Regarding the above, we can add the following. If in the Theorem 3 we consider the integral operator of the Definition 1 with F ≡ 1 and k = 1, that is, we have the Riemann-Liouville fractional integral , this result is a slight variant of Theorem 2 from [32].

The above is still valid for Lemma 1 of [17], if we put F(τ, s) = ψ′(τ) and k = 1.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Agarwal P, Jleli M, Tomar M. Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. J Inequal Appl 2017;1:1–10. [CrossRef]

[2] Ahmad B, Alsaedi A, Kirane M, Toberek BT. Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. J Comput Appl Math 2019;353:120–129. [CrossRef]

[3] Ali MA, Nápoles JE, Kashuri A, Zhang Z. Fractional non conformable Hermite-Hadamard inequalities for generalized φ-convex functions. Fasc Math 2020;64:5–16.

[4] Bayraktar B, Butt SI, Shaokat S, Nápoles Valdés JE. New Hadamard-type inequalities via (s,m1,m2)-convex functions. Vestn Udmurt Univ Mat Mekhanika Komp'yuternye Nauki 2021;31:597–612. [CrossRef]

[5] Bayraktar B, Nápoles VJE, Rabossi F. On generalizations of integral inequalities. Probl Anal Issues Anal 2022;11:3–23. [CrossRef]

[6] Bermudo S, K'orus P, Nápoles JE. On q-Hermite-Hadamard inequalities for general convex functions. Acta Math Hung 2020;162:364–374. [CrossRef]

[7] Bohner M, Kashuri A, Mohammed PO, Nápoles VJE. Hermite-Hadamard-type inequalities for integrals arising in conformable fractional calculus. Hacet J Math Stat 2022;51:775–786.

[8] Bracamonte M, Gimenez J, Vivas M. Hermite-Hadamard-Fej'er type inequalities for (s,m)-Convex functions with modulus c, in second sense. Appl Math Inf Sci 2016;10:2045–2053. [CrossRef]

[9] Díaz R, Pariguan E. On hypergeometric functions and Pochhammer k-symbol. Divulg Mat 2007;15:179–192.

[10] Galeano Delgado JD, Lloreda J, Nápoles VJE, Pérez Reyes E. Certain integral inequalities of hermite-hadamard type for h-convex functions. J Math Control Sci Appl 2021;7:129–140.

[11] Galeano J. D., Nápoles, J. E., and Pérez, E. (n/d), On a general formulation of the fractional operator Riemann-Liouville and related inequalities, submitted.

[12] Hadamard J. Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann. J Math Pures App 1893;9:171–216.

[13] Kashuri A, Nápoles V, Juan E, Ali MA, Din GMU. New Integral inequalities using quasi-convex functions via generalized ıntegral operators and their applications. Appl Math E-Notes 2022;22:221–231.

[14] Katugampola UN. New approach generalized fractional integral. Appl Math Comput 2011;218:860–865. [CrossRef]

[15] Kórus P, Nápoles Valdés JE. On some integral inequalities for (h, m)-convex functions in a generalized framework. Carpathian J Math 2022. Preprint

[16] Mehmood S, Nápoles Valdés JE, Fatima N, Aslam W. Some integral inequalities via fractional derivatives. Adv Stud Euro Tbil Math J 2022;15:31–44. [CrossRef]

[17] Mohammed PO. Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. Math Meth Appl Sci 2019;44:2314–2324. [CrossRef]

[18] Mubeen S, Habibullah GM. K-fractional integrals and applications. Int J Contemp Math Sci 2012;7:89–94.

[19] Nápoles JE. Hermite-Hadamard inequality in generalized context, VI Colloquium on Applied Mathematics and II International Meeting of Applied Mathematics, Unimilitar, Bogotá, Colombia, November 11–13, 2020.

[20] Nápoles JE. New generalized fractional integral inequalities of Hermite-Hadamard type for harmonically convex functions, XVI International Meeting of Mathematics, Barranquilla, Colombia, November 17–20, 2020. [CrossRef]

[21] Nápoles Valdés JE. On the Hermite-Hadamard type inequalities involving generalized integrals. Contrib Math 2022;5:45–51. [CrossRef]

[22] Nápoles Valdés JE, Bayraktar B. On the generalized inequalities of the Hermite-Hadamard type. Filomat 2021;35:4917–4924. [CrossRef]

[23] Nápoles Valdés JE, Bayraktar B, Butt SI. New integral inequalities of Hermite-Hadamard type in a generalized context. Punjab Univ J Math 2021;53:765–777. [CrossRef]

[24] Nápoles Valdés JE, Rabossi F, Ahmad H. Inequalities of the Hermite-Hadamard type, for functıons (h,m)-convex modified of the second typE. Commun Combin Cryptogr Computer Sci 2021;1:33–43.

[25] Nápoles Valdes JE, Rabossi F, Samaniego AD. Convex functions: Ariadne's thread or Charlotte's Spiderweb?, Adv Math Models Appl 2020;5:176–191.

[26] Nápoles Valdes JE, Rodríguez JM, Sigarreta JM. New Hermite-Hadamard type inequalities involving non-conformable integral operators. Symmetry 2019;11:1108. [CrossRef]

[27] Qi F, Guo BN. Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function. Rev Real Acad Cienc Exactas F'ıs Nat Ser A Mat 2017;111:425–434. [CrossRef]

[28] Qi F, Habib S, Mubeen S, Naeem MN. (2019), Generalized k-fractional conformable integrals and related inequalities. AIMS Mathematics 2019;4:343–358. [CrossRef]

[29] Rainville ED. Special Functions. New York: Macmillan Company; 1960.

[30] Ross, B. Fractional Calculus and its Applications. Berlin: Springer; 1975. [CrossRef]

[31] Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. 1st ed. Amsterdam: Taylor and Francis; 1993.

[32] Sarikaya MZ, Set E, Yaldiz H, Basak N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math Comput Model 2013;57:2403–2407. [CrossRef]

[33] Vivas M. Jensen type inequalities for (s,m)-convex functions in second sense. Appl Math Inf Sci 2016;10:1689–1696. [CrossRef]

[34] Vivas M, Hernández J, Merentes N. New Hermite-Hadamard and Jensen type inequalities for h-Convex functions on fractal sets. Rev Colomb de Mat 2016;50:145–164. [CrossRef]

[35] Vivas-Cortez M, Kermausuor S, Nápoles Valdés JE. Hermite-Hadamard type inequalities for coordinated quasi-convex functions via generalized fractional integrals. In: Debnath P, Srivastava HM, Kumam P, Hazarika B, editors. Fixed point theory and fractional calculus.1st ed. Singapore: Springer; 2022. pp. 275–296. [CrossRef]

[36] Vivas-Cortez M, Kórus P, Nápoles Valdés JE. Some generalized Hermite-Hadamard-Fejér inequality for convex functions. Adv Differ Equ 2021;1:1–11. [CrossRef]

[37] Vivas M, Medina J. Hermite-Hadamard type inequalities for harmonically convex functions on n-coordinates. Appl Math Inf Sci Lett 2018;6:1–6. [CrossRef]

[38] Vivas-Cortez M, Nápoles Valdés JE, Guerrero JA. Some Hermite-Hadamard weighted integral inequalities for (h,m)-convex modified functions. Appl Math Inf Sci 2022;16:25–33. [CrossRef]

[39] Yang ZH, Tian JF. Monotonicity and inequalities for the gamma function. J Inequal Appl 2017;1:1–15. [CrossRef]

[40] Yang ZH, Tian JF. Monotonicity and sharp inequalities related to gamma function. J Math Inequal 2018;12:1–22. [CrossRef]