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RESEARCH ARTICLE

SOLVABILITY OF A FOUR DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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ABSTRACT

In this study, we investigate the following four-dimensional difference equations system

$$\begin{cases} u_n = \frac{\alpha u_{n-3}t_{n-2}+\beta}{\gamma v_{n-1}t_{n-2}u_{n-3}}, \\ v_n = \frac{\alpha v_{n-3}u_{n-2}+\beta}{\gamma w_{n-1}u_{n-2}v_{n-3}}, n \in \mathbb{N}_0, \\ w_n = \frac{\alpha w_{n-3}v_{n-2}+\beta}{\gamma t_{n-1}v_{n-2}w_{n-3}}, \\ t_n = \frac{\alpha t_{n-3}w_{n-2}+\beta}{\gamma u_{n-1}w_{n-2}t_{n-3}}, \end{cases}$$

where the initial values $u_{-d}, v_{-d}, w_{-d}, t_{-d}$, $d \in \{1,2,3\}$ are non-zero real numbers and the parameters α, β are real numbers, γ is non-zero real number. Then, we obtain the solutions of system of third-order difference equations in explicit form. In addition, the solutions according to some special cases of the parameters are examined. Finally, numerical examples are given to demonstrate the theoretical results.

Keywords: Periodicity, System of difference equation, Solution

1. INTRODUCTION

First of all, recall that $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ sembolize natural, non-negative integer, integer, real and complex numbers, respectively. The notation of $\lfloor x \rfloor$ stands for $\mathrm{m} \le \mathrm{x} < \mathrm{m}+1, \mathrm{m} \in \mathbb{Z}$. If $\mathrm{a}, \mathrm{b} \in \mathbb{Z}, a \le b$, the notation $c = \overline{a, \mathrm{b}}$ means to come $\{c \in \mathbb{Z}: a \le c \le b\}$.

Difference equations come into view the study of the evolution of naturally occurring events. The theory of system of difference equations improved until today. Recently, there has been great interest in studying difference equation or difference equations systems [1-3,5-7,9,12-14].

One of the important difference equation is

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1}$$

for $c \neq 0$, $ad \neq bc$ where the initial condition x_0 and the parameters a, b, c, d, are real numbers, which called Riccati difference equation. Further, the general solution of equation (1) can be obtained in the following form

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$$x_n = \frac{x_0(bc-ad)s_{n-1} + (ax_0+b)s_n}{(cx_0-a)s_n + s_{n+1}}, n \in \mathbb{N}, \tag{2}$$

where the sequence $(s_n)_{n\in\mathbb{N}_0}$is satisfying

$$s_{n+1} - (a+d)s_n - (bc-ad)s_{n-1} = 0, n \in \mathbb{N}, \tag{3}$$

where $s_0 = 0, s_1 = 1$ in [11].

In [15], authors acquired the solutions of the following two difference equations systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}},\ n \in \mathbb{N}_0. \tag{4}$$

Then, the solution of the following rational difference equation is obtained

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}}, n \in \mathbb{N}_0, \tag{5}$$

where the initial values x_{-1}, x_0 are non zero real numbers and $\alpha, \beta, \gamma \in \mathbb{R}^+$ [8]. In addition, same authors investigated the two-dimensional case of equation (5) given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}}, n \in \mathbb{N}_0. \tag{6}$$

Finally, the solutions of the following system of difference equations investigated

$$\begin{cases} x_n = \dfrac{a x_{n-3} z_{n-2} + b}{c y_{n-1} z_{n-2} x_{n-3}}, \\ y_n = \dfrac{a y_{n-3} x_{n-2} + b}{c z_{n-1} x_{n-2} y_{n-3}},\ \ n \in \mathbb{N}_0, \\ z_n = \dfrac{a z_{n-3} y_{n-2} + b}{c x_{n-1} y_{n-2} z_{n-3}}, \end{cases} \tag{7}$$

where the initial values $x_{-j}, y_{-j}, z_{-j},\ j \in \{1,2,3\}$, and the parameters, a, b, c, are real numbers [10]. Based on the above mentioned studies, we investigate the following system of difference equations

$$\begin{cases} u_n = \dfrac{\alpha u_{n-3} t_{n-2} + \beta}{\gamma v_{n-1} t_{n-2} u_{n-3}}, \\ v_n = \dfrac{\alpha v_{n-3} u_{n-2} + \beta}{\gamma w_{n-1} u_{n-2} v_{n-3}},\ \ n \in \mathbb{N}_0, \\ w_n = \dfrac{\alpha w_{n-3} v_{n-2} + \beta}{\gamma t_{n-1} v_{n-2} w_{n-3}}, \\ t_n = \dfrac{\alpha t_{n-3} w_{n-2} + \beta}{\gamma u_{n-1} w_{n-2} t_{n-3}}, \end{cases} \tag{8}$$

where the parameters α, β are real numbers, the parameter γ and the initial values $u_{-f}, v_{-f}, w_{-f}, t_{-f}$, $f \in \{1,2,3\}$, are non-zero real numbers. We solve system (8) in explicit form. Then, we examine the solutions according to some special cases of the parameters. Further, numerical examples are given to demonstrate the theoretical results.

Definition 1. [4] (Periodicity) A sequence $(z_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there exits $n_0 \geq -k$ such that $z_{n+p} = z_n$ for all $n \geq n_0$.If $n_0 = -k$ then the sequence $(z_n)_{n=-k}^{\infty}$ is said to be periodic with period p.

2. EXPLICIT SOLUTIONS OF THE SYSTEM (8)

Let $(u_n, v_n, w_n, t_n)_{n \geq -3}$ be a solution of system (8). System (8) can be written in the following form

$$\begin{cases} u_n v_{n-1} = \dfrac{\alpha u_{n-3} t_{n-2} + \beta}{\gamma t_{n-2} u_{n-3}} = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma t_{n-2} u_{n-3}}, \\ v_n w_{n-1} = \dfrac{\alpha v_{n-3} u_{n-2} + \beta}{\gamma u_{n-2} v_{n-3}} = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma u_{n-2} v_{n-3}}, \qquad n \in N_0. \\ w_n t_{n-1} = \dfrac{\alpha w_{n-3} v_{n-2} + \beta}{\gamma v_{n-2} w_{n-3}} = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma v_{n-2} w_{n-3}}, \\ t_n u_{n-1} = \dfrac{\alpha t_{n-3} w_{n-2} + \beta}{\gamma w_{n-2} t_{n-3}} = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma w_{n-2} t_{n-3}}, \end{cases} \tag{9}$$

By using the following transformations

$$\begin{cases} u_n v_{n-1} = x_n, \\ v_n w_{n-1} = y_n, \qquad n \geq -2, \\ w_n t_{n-1} = z_n, \\ t_n u_{n-1} = r_n, \end{cases} \tag{10}$$

system (9) is transformed into the following system

$$\begin{cases} x_n = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma r_{n-2}}, \\ y_n = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma x_{n-2}}, \qquad n \in N_0, \\ z_n = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma y_{n-2}}, \\ r_n = \dfrac{\alpha}{\gamma} + \dfrac{\beta}{\gamma z_{n-2}}, \end{cases} \tag{11}$$

which can be written as

$$\begin{cases} x_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)x_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)x_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \\ y_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)y_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)y_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \qquad n \geq 6. \\ z_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)z_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)z_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \\ r_n = \dfrac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)r_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)r_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \end{cases} \tag{12}$$

Then, we consider the following equation

$$\delta_n = \frac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)\delta_{n-8} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)\delta_{n-8} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \qquad n \geq 6, \tag{13}$$

instead of equations in (12).

If we apply the decomposition of indices $n \to 8(m+1) + k$, $k = \overline{-2,5}$ and $m \geq -1$ to (13), then it can be written as follows

$$\delta_{m+1}^{(k)} = \frac{(3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2)\delta_m^{(k)} + \alpha^3\beta + 2\alpha\beta^2\gamma}{(\alpha^3\gamma + 2\alpha\beta\gamma^2)\delta_m^{(k)} + \alpha^2\gamma\beta + \beta^2\gamma^2}, \tag{14}$$

where $\delta_m^{(k)} = \delta_{8m+k}, m \in \mathbb{N}_0, k = \overline{-2,5}$.

Let

$$\begin{cases} A_1 := 3\alpha^2\beta\gamma + \alpha^4 + \beta^2\gamma^2, \\ B_1 := \alpha^3\beta + 2\alpha\beta^2\gamma, \\ C_1 := \alpha^3\gamma + 2\alpha\beta\gamma^2, \\ D_1 := \alpha^2\gamma\beta + \beta^2\gamma^2. \end{cases}$$

From equation (2), the general solution of (14) follows straightforwardly as

$$\delta_m^{(k)} = \frac{-\beta^4\gamma^4\delta_0^{(k)}s_{m-1} + \left(A_1\delta_0^{(k)} + B_1\right)s_m}{\left(C_1\delta_0^{(k)} - A_1\right)s_m + s_{m+1}}, \; m \in \mathbb{N}_0, \tag{15}$$

for $k = \overline{-2,5}$, where the sequence $(s_m)_{m\in\mathbb{N}_0}$ is satisfying

$s_{m+1} - (A_1 + D_1)s_m - (B_1C_1 - A_1D_1)s_{m-1} = 0.$

We use (10) in (12) and from (15), equations in (12) are expressed as

$$\begin{cases} x_{8m+k} = \dfrac{-\beta^4\gamma^4 u_k v_{k-1} s_{m-1} + (A_1 u_k v_{k-1} + B_1) s_m}{(C_1 u_k v_{k-1} - A_1) s_m + s_{m+1}}, \\ y_{8m+k} = \dfrac{-\beta^4\gamma^4 v_k w_{k-1} s_{m-1} + (A_1 v_k w_{k-1} + B_1) s_m}{(C_1 v_k w_{k-1} - A_1) s_m + s_{m+1}}, \qquad m \in N_0, \\ z_{8m+k} = \dfrac{-\beta^4\gamma^4 w_k t_{k-1} s_{m-1} + (A_1 w_k t_{k-1} + B_1) s_m}{(C_1 w_k t_{k-1} - A_1) s_m + s_{m+1}}, \\ r_{8m+k} = \dfrac{-\beta^4\gamma^4 t_k u_{k-1} s_{m-1} + (A_1 t_k u_{k-1} + B_1) s_m}{(C_1 t_k u_{k-1} - A_1) s_m + s_{m+1}}, \end{cases} \tag{16}$$

for $k = \overline{-2,5}$. From (10), we get

$$\begin{cases} u_n = \dfrac{x_n}{v_{n-1}} = \dfrac{x_n w_{n-2}}{y_{n-1}} = \dfrac{x_n z_{n-2}}{y_{n-1} t_{n-3}} = \dfrac{x_n z_{n-2} u_{n-4}}{y_{n-1} r_{n-3}} = \dfrac{x_n z_{n-2} x_{n-4}}{y_{n-1} r_{n-3} v_{n-5}} \\ \quad = \dfrac{x_n z_{n-2} x_{n-4} w_{n-6}}{y_{n-1} r_{n-3} y_{n-5}} = \dfrac{x_n z_{n-2} x_{n-4} z_{n-6}}{y_{n-1} r_{n-3} y_{n-5} t_{n-7}} = \dfrac{x_n z_{n-2} x_{n-4} z_{n-6}}{y_{n-1} r_{n-3} y_{n-5} r_{n-7}} u_{n-8}, \\ v_n = \dfrac{y_n}{w_{n-1}} = \dfrac{y_n t_{n-2}}{z_{n-1}} = \dfrac{y_n r_{n-2}}{z_{n-1} u_{n-3}} = \dfrac{y_n r_{n-2} v_{n-4}}{z_{n-1} x_{n-3}} = \dfrac{y_n r_{n-2} y_{n-4}}{z_{n-1} x_{n-3} w_{n-5}} \\ \quad = \dfrac{y_n r_{n-2} y_{n-4} t_{n-6}}{z_{n-1} x_{n-3} z_{n-5}} = \dfrac{y_n r_{n-2} y_{n-4} r_{n-6}}{z_{n-1} x_{n-3} z_{n-5} u_{n-7}} = \dfrac{y_n r_{n-2} y_{n-4} r_{n-6}}{z_{n-1} x_{n-3} z_{n-5} x_{n-7}} v_{n-8}, \\ w_n = \dfrac{z_n}{t_{n-1}} = \dfrac{z_n u_{n-2}}{r_{n-1}} = \dfrac{z_n x_{n-2}}{r_{n-1} v_{n-3}} = \dfrac{z_n x_{n-2} w_{n-4}}{r_{n-1} y_{n-3}} = \dfrac{z_n x_{n-2} z_{n-4}}{r_{n-1} y_{n-3} t_{n-5}} \\ \quad = \dfrac{z_n x_{n-2} z_{n-4} u_{n-6}}{r_{n-1} y_{n-3} r_{n-5}} = \dfrac{z_n x_{n-2} z_{n-4} x_{n-6}}{r_{n-1} y_{n-3} r_{n-5} v_{n-7}} = \dfrac{z_n x_{n-2} z_{n-4} x_{n-6}}{r_{n-1} y_{n-3} r_{n-5} y_{n-7}} w_{n-8}, \\ t_n = \dfrac{r_n}{u_{n-1}} = \dfrac{r_n v_{n-2}}{x_{n-1}} = \dfrac{r_n y_{n-2}}{x_{n-1} w_{n-3}} = \dfrac{r_n y_{n-2} t_{n-4}}{x_{n-1} z_{n-3}} = \dfrac{r_n y_{n-2} r_{n-4}}{x_{n-1} z_{n-3} u_{n-5}} \\ \quad = \dfrac{r_n y_{n-2} r_{n-4} v_{n-6}}{x_{n-1} z_{n-3} x_{n-5}} = \dfrac{r_n y_{n-2} r_{n-4} y_{n-6}}{x_{n-1} z_{n-3} x_{n-5} w_{n-7}} = \dfrac{r_n y_{n-2} r_{n-4} y_{n-6}}{x_{n-1} z_{n-3} x_{n-5} z_{n-7}} t_{n-8}, \end{cases} \tag{17}$$

for $n \geq 5$. From (17), we have

$$\begin{cases} u_{8m+h} = \dfrac{x_{8m+h} z_{8m+h-2} x_{8m+h-4} z_{8m+h-6}}{y_{8m+h-1} r_{8m+h-3} y_{8m+h-5} r_{8m+h-7}} u_{8(m-1)+h}, \\ v_{8m+h} = \dfrac{y_{8m+h} r_{8m+h-2} y_{8m+h-4} r_{8m+h-6}}{z_{8m+h-1} x_{8m+h-3} z_{8m+h-5} x_{8m+h-7}} v_{8(m-1)+h}, \qquad m \in N_0, \\ w_{8m+h} = \dfrac{z_{8m+h} x_{8m+h-2} z_{8m+h-4} x_{8m+h-6}}{r_{8m+h-1} y_{8m+h-3} r_{8m+h-5} y_{8m+h-7}} w_{8(m-1)+h}, \\ t_{8m+h} = \dfrac{r_{8m+h} y_{8m+h-2} r_{8m+h-4} y_{8m+h-6}}{x_{8m+h-1} z_{8m+h-3} x_{8m+h-5} z_{8m+h-7}} t_{8(m-1)+h}, \end{cases} \tag{18}$$

for $h = \overline{5,12}$.

Multiplying the equalities which are obtained from (18), from 0 to m, it follows that

$$\begin{cases} u_{8m+k+7} = u_{k-1} \displaystyle\prod_{l=0}^{m} \left(\frac{x_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor} z_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{y_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor} r_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}} \times \frac{x_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor} z_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{y_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor} r_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}} \right), \\ v_{8m+k+7} = v_{k-1} \displaystyle\prod_{l=0}^{m} \left(\frac{y_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor} r_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{z_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor} x_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}} \times \frac{y_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor} r_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{z_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor} x_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}} \right), \\ w_{8m+k+7} = w_{k-1} \displaystyle\prod_{l=0}^{m} \left(\frac{z_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor} x_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{r_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor} y_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}} \times \frac{z_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor} x_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{r_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor} y_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}} \right), \\ t_{8m+k+7} = t_{k-1} \displaystyle\prod_{l=0}^{m} \left(\frac{r_{8\left(l+\left\lfloor\frac{k+9}{8}\right\rfloor\right)+k+7-8\left\lfloor\frac{k+9}{8}\right\rfloor} y_{8\left(l+\left\lfloor\frac{k+7}{8}\right\rfloor\right)+k+5-8\left\lfloor\frac{k+7}{8}\right\rfloor}}{x_{8\left(l+\left\lfloor\frac{k+8}{8}\right\rfloor\right)+k+6-8\left\lfloor\frac{k+8}{8}\right\rfloor} z_{8\left(l+\left\lfloor\frac{k+6}{8}\right\rfloor\right)+k+4-8\left\lfloor\frac{k+6}{8}\right\rfloor}} \times \frac{r_{8\left(l+\left\lfloor\frac{k+5}{8}\right\rfloor\right)+k+3-8\left\lfloor\frac{k+5}{8}\right\rfloor} y_{8\left(l+\left\lfloor\frac{k+3}{8}\right\rfloor\right)+k+1-8\left\lfloor\frac{k+3}{8}\right\rfloor}}{x_{8\left(l+\left\lfloor\frac{k+4}{8}\right\rfloor\right)+k+2-8\left\lfloor\frac{k+4}{8}\right\rfloor} z_{8\left(l+\left\lfloor\frac{k+2}{8}\right\rfloor\right)+k-8\left\lfloor\frac{k+2}{8}\right\rfloor}} \right), \end{cases} \tag{19}$$

for $h = k + 7,\ k = \overline{-2,5}$.

By substituting the formulas in (16) into (19), we obtain

$$\begin{cases} u_{8m+k+7} = u_{k-1}\prod_{l=0}^{m}\left(\dfrac{\dfrac{-\beta^4\gamma^4 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 u_{k+7-8\lfloor\frac{k+9}{8}\rfloor} v_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 v_{k+6-8\lfloor\frac{k+8}{8}\rfloor} w_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\ \times \dfrac{\dfrac{-\beta^4\gamma^4 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 w_{k+5-8\lfloor\frac{k+7}{8}\rfloor} t_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 t_{k+4-8\lfloor\frac{k+6}{8}\rfloor} u_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\ \times \dfrac{\dfrac{-\beta^4\gamma^4 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 u_{k+3-8\lfloor\frac{k+5}{8}\rfloor} v_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 v_{k+2-8\lfloor\frac{k+4}{8}\rfloor} w_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\ \left.\times \dfrac{\dfrac{-\beta^4\gamma^4 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 w_{k+1-8\lfloor\frac{k+3}{8}\rfloor} t_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 t_{k-8\lfloor\frac{k+2}{8}\rfloor} u_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right), \end{cases} \tag{20}$$

$$\begin{cases} v_{8m+k+7} = v_{k-1}\prod_{l=0}^{m}\left(\dfrac{\dfrac{-\beta^4\gamma^4 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 v_{k+7-8\lfloor\frac{k+9}{8}\rfloor} w_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 w_{k+6-8\lfloor\frac{k+8}{8}\rfloor} t_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}}\right. \\ \times \dfrac{\dfrac{-\beta^4\gamma^4 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 t_{k+5-8\lfloor\frac{k+7}{8}\rfloor} u_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 u_{k+4-8\lfloor\frac{k+6}{8}\rfloor} v_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\ \times \dfrac{\dfrac{-\beta^4\gamma^4 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 v_{k+3-8\lfloor\frac{k+5}{8}\rfloor} w_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 w_{k+2-8\lfloor\frac{k+4}{8}\rfloor} t_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\ \left.\times \dfrac{\dfrac{-\beta^4\gamma^4 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 t_{k+1-8\lfloor\frac{k+3}{8}\rfloor} u_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 u_{k-8\lfloor\frac{k+2}{8}\rfloor} v_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}}\right), \end{cases} \tag{21}$$

$$
\left\{
\begin{aligned}
w_{8m+k+7} = w_{k-1} \prod_{l=0}^{m} \Bigg(& \frac{\dfrac{-\beta^4\gamma^4 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 w_{k+7-8\lfloor\frac{k+9}{8}\rfloor} t_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 t_{k+6-8\lfloor\frac{k+8}{8}\rfloor} u_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 u_{k+5-8\lfloor\frac{k+7}{8}\rfloor} v_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 v_{k+4-8\lfloor\frac{k+6}{8}\rfloor} w_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 w_{k+3-8\lfloor\frac{k+5}{8}\rfloor} t_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 t_{k+2-8\lfloor\frac{k+4}{8}\rfloor} u_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 u_{k+1-8\lfloor\frac{k+3}{8}\rfloor} v_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 v_{k-8\lfloor\frac{k+2}{8}\rfloor} w_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}} \Bigg),
\end{aligned}
\right. \tag{22}
$$

$$
\left\{
\begin{aligned}
t_{8m+k+7} = t_{k-1} \prod_{l=0}^{m} \Bigg(& \frac{\dfrac{-\beta^4\gamma^4 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} s_{l+\lfloor\frac{k+9}{8}\rfloor-1} + \left(A_1 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor}}{\left(C_1 t_{k+7-8\lfloor\frac{k+9}{8}\rfloor} u_{k+7-8\lfloor\frac{k+9}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+9}{8}\rfloor} + s_{l+\lfloor\frac{k+9}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} s_{l+\lfloor\frac{k+8}{8}\rfloor-1} + \left(A_1 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor}}{\left(C_1 u_{k+6-8\lfloor\frac{k+8}{8}\rfloor} v_{k+6-8\lfloor\frac{k+8}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+8}{8}\rfloor} + s_{l+\lfloor\frac{k+8}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} s_{l+\lfloor\frac{k+7}{8}\rfloor-1} + \left(A_1 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor}}{\left(C_1 v_{k+5-8\lfloor\frac{k+7}{8}\rfloor} w_{k+5-8\lfloor\frac{k+7}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+7}{8}\rfloor} + s_{l+\lfloor\frac{k+7}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} s_{l+\lfloor\frac{k+6}{8}\rfloor-1} + \left(A_1 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor}}{\left(C_1 w_{k+4-8\lfloor\frac{k+6}{8}\rfloor} t_{k+4-8\lfloor\frac{k+6}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+6}{8}\rfloor} + s_{l+\lfloor\frac{k+6}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} s_{l+\lfloor\frac{k+5}{8}\rfloor-1} + \left(A_1 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor}}{\left(C_1 t_{k+3-8\lfloor\frac{k+5}{8}\rfloor} u_{k+3-8\lfloor\frac{k+5}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+5}{8}\rfloor} + s_{l+\lfloor\frac{k+5}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} s_{l+\lfloor\frac{k+4}{8}\rfloor-1} + \left(A_1 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor}}{\left(C_1 u_{k+2-8\lfloor\frac{k+4}{8}\rfloor} v_{k+2-8\lfloor\frac{k+4}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+4}{8}\rfloor} + s_{l+\lfloor\frac{k+4}{8}\rfloor+1}}} \\
& \times \frac{\dfrac{-\beta^4\gamma^4 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} s_{l+\lfloor\frac{k+3}{8}\rfloor-1} + \left(A_1 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor}}{\left(C_1 v_{k+1-8\lfloor\frac{k+3}{8}\rfloor} w_{k+1-8\lfloor\frac{k+3}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+3}{8}\rfloor} + s_{l+\lfloor\frac{k+3}{8}\rfloor+1}}}{\dfrac{-\beta^4\gamma^4 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} s_{l+\lfloor\frac{k+2}{8}\rfloor-1} + \left(A_1 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} + B_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor}}{\left(C_1 w_{k-8\lfloor\frac{k+2}{8}\rfloor} t_{k-8\lfloor\frac{k+2}{8}\rfloor-1} - A_1\right) s_{l+\lfloor\frac{k+2}{8}\rfloor} + s_{l+\lfloor\frac{k+2}{8}\rfloor+1}}} \Bigg),
\end{aligned}
\right. \tag{23}
$$

for $m \in N_0, k = \overline{-2,5}$.

3. PARTICULAR CASES OF SYSTEM (8)

In this section, we will consider the solutions according to some special cases of the parameters.

3.1. Case 1 $\alpha = \gamma = 1, \beta = 0$.

In this case, system (8) can be written to the following form

$$\begin{cases} u_n = \dfrac{u_{n-3}t_{n-2}}{v_{n-1}t_{n-2}u_{n-3}} = \dfrac{1}{v_{n-1}} = w_{n-2} = \dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{v_{n-3}u_{n-2}}{w_{n-1}u_{n-2}v_{n-3}} = \dfrac{1}{w_{n-1}} = t_{n-2} = \dfrac{1}{u_{n-3}} = v_{n-4}, \\ w_n = \dfrac{w_{n-3}v_{n-2}}{t_{n-1}v_{n-2}w_{n-3}} = \dfrac{1}{t_{n-1}} = u_{n-2} = \dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{t_{n-3}w_{n-2}}{u_{n-1}w_{n-2}t_{n-3}} = \dfrac{1}{u_{n-1}} = v_{n-2} = \dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{24}$$

for $n \geq 3$.

We obtain the solutions of equations in (24) as in the following form

$$u_{4m+j} = u_{j-4}, \quad v_{4m+j} = v_{j-4}, \quad w_{4m+j} = w_{j-4}, \quad t_{4m+j} = t_{j-4}, \quad m \in N_0, \quad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = \gamma = 1, \beta = 0$.

Example 3.1. Consider the system (8) with the parameters $\alpha = \gamma = 1, \beta = 0$ and the initial conditions $u_{-3} = 0.1$, $u_{-2} = 2.4$, $u_{-1} = 3$, $v_{-3} = 4.06$, $v_{-2} = 0.05$, $v_{-1} = 0.6$, $w_{-3} = 70.54$, $w_{-2} = 0.86$, $w_{-1} = 9.05$, $t_{-3} = 0.3$, $t_{-2} = 1.7$, $t_{-1} = 3.2$, the solutions are given as in Figure 1.

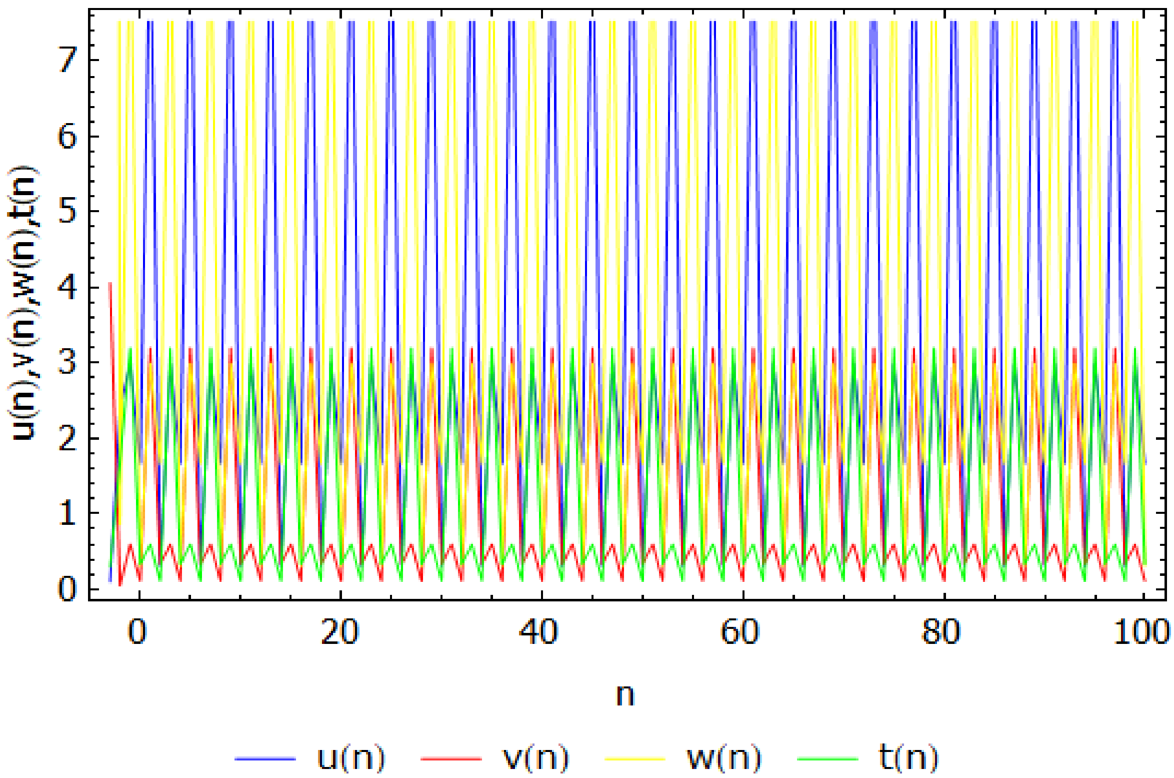

Figure 1. Plots of u_n, v_n, w_n, t_n in case $\alpha = \gamma = 1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

3.2. Case 2 $\alpha = \gamma = -1, \beta = 0.$

In this case, system (8) becomes

$$\begin{cases} u_n = \dfrac{-u_{n-3}t_{n-2}}{-v_{n-1}t_{n-2}u_{n-3}} = \dfrac{1}{v_{n-1}} = w_{n-2} = \dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{-v_{n-3}u_{n-2}}{-w_{n-1}u_{n-2}v_{n-3}} = \dfrac{1}{w_{n-1}} = t_{n-2} = \dfrac{1}{u_{n-3}} = v_{n-4}, \\ w_n = \dfrac{-w_{n-3}v_{n-2}}{-t_{n-1}v_{n-2}w_{n-3}} = \dfrac{1}{t_{n-1}} = u_{n-2} = \dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{-t_{n-3}w_{n-2}}{-u_{n-1}w_{n-2}t_{n-3}} = \dfrac{1}{u_{n-1}} = v_{n-2} = \dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{25}$$

for $n \geq 3$.

We obtain the solutions of equations in (25) as in the following form

$$u_{4m+j} = u_{j-4}, \quad v_{4m+j} = v_{j-4}, \quad w_{4m+j} = w_{j-4}, \quad t_{4m+j} = t_{j-4}, \quad m \in N_0, \quad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = \gamma = -1, \beta = 0$.

Example 3.2. Consider the system (8) with the initial values $u_{-3} = 0.05$, $u_{-2} = 6$, $u_{-1} = 0.63$, $v_{-3} = 7$, $v_{-2} = 0.6$, $v_{-1} = 0.68$, $w_{-3} = 7.5$, $w_{-2} = 0.6$, $w_{-1} = 7.5$, $t_{-3} = 5.3$, $t_{-2} = 8.6$, $t_{-1} = 2.1$, and the parameters $\alpha = \gamma = -1, \beta = 0$ the solutions are represented as in Figure 2.

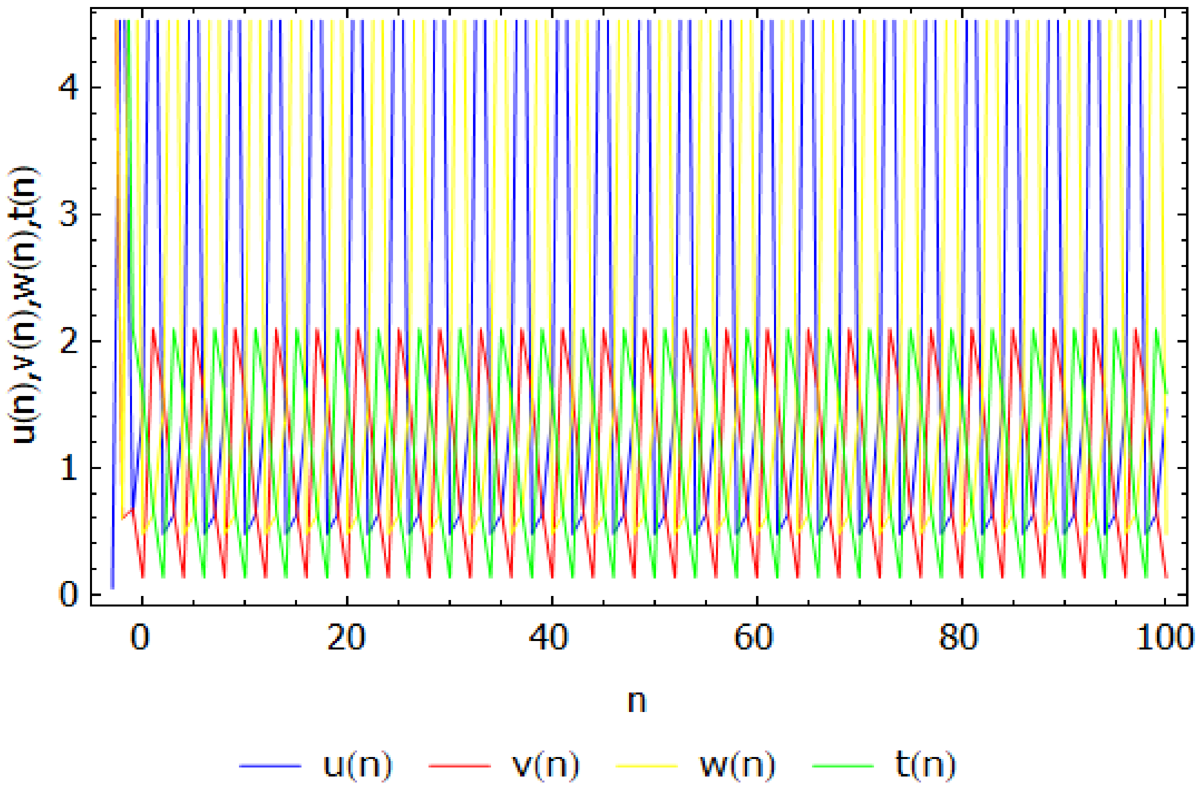

Figure 2. Plots of u_n, v_n, w_n, t_n in case $\alpha = \gamma = -1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

3.3. Case 3 $\alpha = 1, \gamma = -1, \beta = 0$.

In this case, system (8) becomes

$$\begin{cases} u_n = \dfrac{u_{n-3}t_{n-2}}{-v_{n-1}t_{n-2}u_{n-3}} = -\dfrac{1}{v_{n-1}} = w_{n-2} = -\dfrac{1}{t_{n-3}} = u_{n-4}, \\ v_n = \dfrac{v_{n-3}u_{n-2}}{-w_{n-1}u_{n-2}v_{n-3}} = -\dfrac{1}{w_{n-1}} = t_{n-2} = -\dfrac{1}{u_{n-3}} = v_{n-4}, \\ w_n = \dfrac{w_{n-3}v_{n-2}}{-t_{n-1}v_{n-2}w_{n-3}} = -\dfrac{1}{t_{n-1}} = u_{n-2} = -\dfrac{1}{v_{n-3}} = w_{n-4}, \\ t_n = \dfrac{t_{n-3}w_{n-2}}{-u_{n-1}w_{n-2}t_{n-3}} = -\dfrac{1}{u_{n-1}} = v_{n-2} = -\dfrac{1}{w_{n-3}} = t_{n-4}, \end{cases} \tag{26}$$

for $n \geq 3$.

We obtain the solutions of equations in (26) as in the following form

$$u_{4m+j} = u_{j-4}, \quad v_{4m+j} = v_{j-4}, \quad w_{4m+j} = w_{j-4}, \quad t_{4m+j} = t_{j-4}, \quad m \in N_0, \quad j = \overline{3,6}.$$

Now, we give numerical example that represent the solutions of system (8) when $\alpha = 1, \gamma = -1, \beta = 0$.

Example 3.3. Consider the system (8) with the initial values $u_{-3} = 4$, $u_{-2} = 6.2$, $u_{-1} = 0.2$, $v_{-3} = 0.09$, $v_{-2} = 1.7$, $v_{-1} = 0.6$, $w_{-3} = -7.5$, $w_{-2} = -0.63$, $w_{-1} = 2.6$, $t_{-3} = -3.2$, $t_{-2} - 6.8$, $t_{-1} = 2.1$ and the parameters $\alpha = 1, \gamma = -1, \beta = 0$ the solutions are represented as in Figure 3.

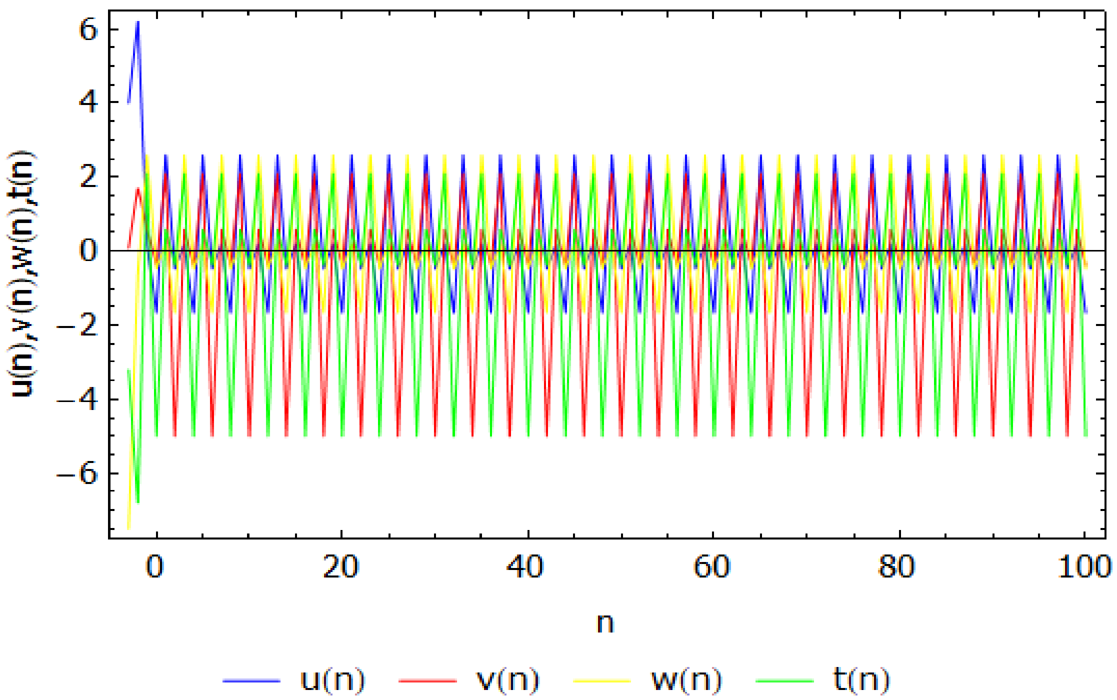

Figure 3. Plots of u_n, v_n, w_n, t_n in case $\alpha = 1, \gamma = -1, \beta = 0$

Therefore, the solutions of system (8) are eventually periodic with period 4.

4. CONCLUSION

In this paper, four-dimesional system of difference equations is solved in explicit form by using convenient transformation. In addition, the periodic solutions of aforementioned system of difference equations are obtained according to some special cases of the parameters. Finally, to support obtained results, we give numerical examples.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

The authors contributed equally to this work.

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