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Whiskered Groupoids and Crossed Modules with Diagrams

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Abstract — In this study, we investigate the relationships between the category of crossed modules of groups and the category of whiskered groupoids. Our first aim is to construct a crossed module structure over groups from a whiskered groupoid with the objects set – a group (regular groupoid) – using the usual functor between the categories of crossed modules and cat groups. Conversely, the second aim is to construct a whiskered groupoid structure with the objects set, which is a group, from a crossed module of groups. While establishing this relationship, we frequently used arrow diagrams representing morphisms to make the axioms more comprehensible. We provide the conditions for the bimorphisms in a whiskered groupoid and give the relations between this structure and internal groupoids in the category of whiskered groupoids with the objects set as a group.

Keywords *Groupoid, crossed module, whiskered categories, bimorphism*

Mathematics Subject Classification (2020) 16E45, 18G45

1. Introduction

The notion of whiskering on a groupoid originally comes from the concept of tensor product in the category of crossed complexes over groupoids defined by Brown and Higgins [1]. If C is a crossed complex of groupoids together with the tensor product over itself, $w : C \otimes C \to C$, then a 1-truncation of C with the biactions of the objects set on the morphisms set gives a whiskered groupoid. In this case, we have the operations $w_{01} : C_0 \times C_1 \to C_1$, $\quad w_{10} : C_1 \times C_0 \to C_1$, and $w_{00} : C_0 \times C_0 \to C_0$ called whiskerings where C_0 is the set of objects and C_1 is the set of morphisms between objects. The operations w_{01} and w_{10} give the left and right actions of C_0 on C_1, respectively. Furthermore, the operation w_{00} gives a monoid structure over C_0. A crossed complex C over groupoids together with the tensor product $\otimes$ over C can be regarded as a crossed differential graded algebra defined by Baues in [2] and further studied by Baues-Tonks in [3]. Thus, we can say that the first component of a crossed differential graded algebra also gives a whiskered groupoid.

The purpose of defining whiskering operations is to explore the conditions under which the composition of morphisms has the commutativity for any given category. For a group G, if each commutator is identity in G, then G is an Abelian group. To define the notion of commutativity for any category $\mathcal{C}$, considering the whiskering operations in $\mathcal{C}$, the left and right multiplications have been introduced by Brown in [4]. In the case $\mathcal{C} := (C_1, C_0)$ is a groupoid together with the whiskering $w_{10} : C_1 \times C_0 \to C_1$

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and $w_{01} : C_0 \times C_1 \to C_1$, the commutator of $a : x \to y$ and $b : u \to v$ in $\mathcal{C}$ can be defined by

$$[a, b] = w_{10}(a, u)^{-1} \circ w_{10}(y, b)^{-1} \circ w_{10}(a, v) \circ w_{01}(x, b)$$

In this equality, the left and right multiplications are given by $l(a, b) = w_{01}(y, b) \circ w_{10}(a, u)$ and $r(a, b) = w_{10}(a, v) \circ w_{01}(x, b)$. Thus, the commutator of the morphisms a, b in $\mathcal{C}$ is $[a, b] = l(a, b)^{-1} r(a, b)$. In the case $l(a, b) = r(a, b)$, the groupoid $\mathcal{C}$ is called a commutative groupoid [4], and then $\mathcal{C}$ is a strict monoidal category.

On the other hand, if C is a groupoid, then the automorphism structure $Aut(C)$ is equivalent to a crossed module introduced by Whitehead in [5]; $\partial : Sc(C) \to Aut(C)$ where $Sc(C)$ is the set of sections of the source map s and the target map is a bijection on C_0. Then, the set $Sc(C)$ has a group structure with the Ehresmannian composition. Using this composition, we give the relationship between crossed modules and whiskered groupoids with the objects set is a group. The crossed module category is equivalent to the category of $\mathcal{G}$-groupoids [6]. The notion of $\mathcal{G}$-groupoid is also defined to be a group-groupoid [7]. Since the set of morphisms is not a group in a whiskered or regular groupoid, this structure is not equivalent to the group-groupoids or cat^1-groups.

As a 2-dimensional analog, we can say that if C is crossed module, then $Aut(C)$ has a braided regular crossed module structure defined by Brown and Gilbert [8], see also [9] for this structure. For the reduced cases of this structure in other contexts, see [10, 11]. Then, this structure can be considered as a whiskered 2-groupoid with the objects set as a group. Brown in [4] has also defined the notion of whiskering for any R-category. Since an R-algebroid can be considered as a small R-category, using the result of [12], it can be studied the R-algebroid version of the results herein.

2. Preliminaries

In this section, we recall the basic definitions of the whiskered categories and crossed modules of groups. For further details, see to [4, 8, 13, 14]. The following sources [15–18] also cover various aspects of this area, including the simplicial objects within categories of some algebraic structures, and could be valuable for the reader's reference.

2.1. Whiskered Groupoids

Suppose that $\mathfrak{C}$ is a (small) category with the set of morphisms (or 1-cells) written by C_1 and the set of objects (or 0-cells) written by C_0. In C_1, particularly, the set of morphisms $a : x \to y$ from x to y is denoted by $C_1(x, y)$, and x and y are called the source and target of the morphism a, respectively. The source and target maps are written $s, t : C_1 \to C_0$. Then, for $a \in C_1(x, y)$, we have $s(a) = x$ and $t(a) = y$.

The category composition in $\mathfrak{C}$ of morphisms $a : x \longrightarrow y$ and $b : y \longrightarrow z$ can be defined by $b \circ a : x \longrightarrow z$. In this case, clearly, $s(b \circ a) = s(a)$ and $t(b \circ a) = t(b)$. We write $C_1(x, x)$ as $C_1(x)$. Brown [4] introduced the notion of 'whiskering' for any category $\mathfrak{C}$ and gave the notions of left and right multiplications for a whiskered category $\mathfrak{C}$ as follows:

Definition 2.1. A whiskering on a category $\mathfrak{C} := (C_1, C_0)$ consists of operations

$$w_{i,j} : C_i \times C_j \longrightarrow C_{i+j}, \ i, j = 0, 1, \ i + j \leqslant 1$$

satisfying the following axioms:

Whisk 1) $w_{0,0}$ gives a monoid structure on C_0;

Whisk 2) $w_{0,1} : C_0 \times C_1 \longrightarrow C_1$ is a left action of the monoid C_0 on the category $\mathfrak{C}$ in the sense that, if $x \in C_0$ and $a : u \longrightarrow v$ in C_1, then

$$w_{0,1}(x, a) : w_{0,0}(x, u) \longrightarrow w_{0,0}(x, v)$$

in $\mathfrak{C}$, so that:

$$w_{0,1}(1, a) = a,\ w_{0,1}(w_{0,0}(x, y), a) = w_{0,1}(x, w_{0,1}(y, a))$$

$$w_{0,1}(x, a \circ b) = w_{0,1}(x, a) \circ w_{0,1}(x, b),\ w_{0,1}(x, 1_y) = 1_{xy}$$

Whisk 3) $w_{1,0} : C_1 \times C_0 \longrightarrow C_1$ is a right action of the monoid C_0 on C_1 with analogous rules.

Whisk 4)

$$w_{0,1}(x, w_{1,0}(a, y)) = w_{1,0}(w_{0,1}(x, a), y)$$

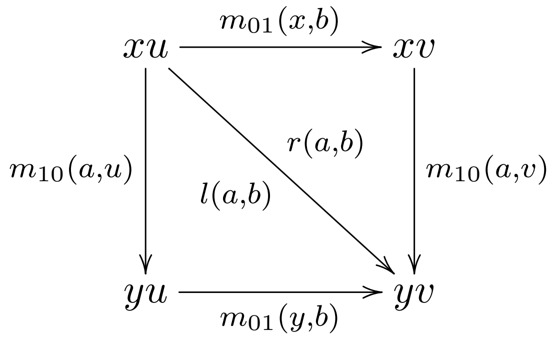
for all $x, y, u, v \in C_0$, $a, b \in C_1$.

Here, a category $\mathfrak{C}$ together with a whiskering is called a whiskered category.

In a whiskered category, for $a : x \to y$, $b : u \to v$, there are two multiplications given by

$$l(a, b) := m_{01}(y, b) \circ m_{10}(a, u) \quad \text{and} \quad r(a, b) := m_{10}(a, v) \circ m_{01}(x, b)$$

These multiplications can be denoted pictorially by



It is well-known that a groupoid is a small category in which every arrow (or morphisms or 1-cells) is an isomorphism. That is, for any morphism a, there is a (necessarily unique) morphism a^{-1} such that $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$ where $e : C_0 \to C_1$ gives the identity morphism at any object. We denote a groupoid as $\mathfrak{C} := (C_1, C_0)$, where C_0 is the set of objects and C_1 is the set of morphisms. For any groupoid $\mathfrak{C}$, if $C_1(x, y)$ is empty whenever x, y are distinct (that is, if $s = t$) then $\mathfrak{C}$ is called totally disconnected groupoid. A groupoid $\mathfrak{C} := (C_1, C_0)$ together with the whiskering operations $w_{i,j} : C_i \times C_j \to C_{i+j}$ for $i + j \leqslant 1$ satisfying the above conditions is called a whiskered groupoid. We denote a whiskered groupoid by $(\mathfrak{C}, w)$. In a whiskered groupoid, if the object set C_0 is a group with the multiplication given by w_{00}, we say that (C_1, C_0) is a regular groupoid as defined by Gilbert in [19]. We use the notation $\mathcal{RG}$ to denote the category of whiskered groupoids whose set of objects is a group with the operation w_{00}, or shortly of regular groupoids.

Example 2.2. Let $C_3 = \{1, x, x^2\} =< x >$ and $C_2 = \{1, y\} =< y >$ be cyclic groups. The action of C_2 on C_3 is given by

$$^{1}1 = 1, ^{1}x = x, ^{1}(x^2) = x^2 \quad \text{and} \quad ^{y}1 = 1, ^{y}x = x^2, ^{y}(x^2) = x$$

Using this action, we can create the semidirect product

$$C_3 \rtimes C_2 = \{(1, 1), (x^2, y), (x, y), (1, y), (x, 1), (x^2, 1)\}$$

with the multiplication of elements given by

$$
\begin{aligned}
(x,y)(x,y) &= (x^y x, y^2) = (x^3, y^2) = (1,1)\\
(x^2,y)(x^2,y) &= (x^2({}^y(x^2)), y^2) = (1,1)\\
(1,y)(1,y) &= (1,1)\\
(x^2,1)(x^2,1) &= (x^2({}^1(x^2)),1) = (x,1)\\
(x,1)(x,1) &= (x^1(x),1) = (x^2,1)
\end{aligned}
$$

and

$$
\begin{aligned}
&(x,y)(x^2,1) = (x^y(x^2), y) = (x^2, y)\\
&(x^2,1)(x,y) = (1,y)\\
&(x,y)(x,1) = (x^y(x), y) = (xx^2, y) = (1,y)\\
&(x,1)(x,y) = (x^2, y)
\end{aligned}
$$

It can be observed that this is a non-Abelian group and isomorphic to S_3. In this case, we can consider $C_3 \rtimes C_2$ as the set of morphisms G_1 and C_2 as the set of objects G_0. The elements of $C_3 \rtimes C_2$ can be regarded as morphisms.

$$(x^2,y),(1,y),(x,y) : y \to y \text{ and } (1,1),(x,1),(x^2,1) : 1 \to 1$$

The compositions of these morphisms are defined by

$$
\begin{aligned}
(x^2,y)\circ(1,y) &= (x^2,y)\\
(x^2,y)\circ(x,y) &= (1,y)\\
(1,y)\circ(x,y) &= (x,y)\\
(x,1)\circ(x^2,1) &= (1,1)
\end{aligned}
$$

The idendity map $e : G_0 \to G_1$ is defined on elements by $e(1) = (1,1)$ and $e(y) = (1,y)$. The whiskering operation $w_{01} : G_0 \times G_1 \to G_1$ is defined on elements by

$$w_{01}(y,(x,y)) = ({}^y x, y^2) = (x^2,1), \quad w_{01}(y,(x^2,y)) = ({}^y(x^2), y^2) = (x,1)$$
$$w_{01}(y,(1,y)) = ({}^y 1, y^2) = (1,1)$$

and

$$w_{01}(y,(1,1)) = ({}^y 1, y) = (1,y), \quad w_{01}(y,(x,1)) = ({}^y x, y) = (x^2,y)$$

and

$$w_{01}(y,(x^2,1)) = ({}^y(x^2), y) = (x,y)$$

and the whiskering operation $w_{10} : G_1 \times G_0 \to G_1$ is defined on elements by

$$w_{10}((x,y),y) = (x,y^2) = (x,1), \quad w_{10}((x^2,y),y) = (x^2,y^2) = (x^2,1)$$
$$w_{10}((1,y),y) = (1,y^2) = (1,1)$$

and

$$w_{10}((1,1),y) = (1,y), \quad w_{10}((x,1),y) = (x,y), \quad w_{10}((x^2,1),y) = (x^2,y)$$

It can be easily seen that these maps satisfy the whiskering axioms. Therefore, we obtain a regular groupoid (G_1, G_0, w_{ij}) which is also isomorphic to (S_3, H, w_{ij}) where $C_2 \cong H = \{I, (12)\}$ is the subgroup of S_3.

Example 2.3. With the same groups as previous example, define the action of C_2 on C_3 by ${}^y x = x$. Then, we have

$$^1 1 = 1, ^1 x = x, ^1 (x^2) = x^2 \text{ and } ^y 1 = 1, ^y x = x, ^y (x^2) = x^2$$

Using this action, in the semidirect product $C_3 \rtimes C_2$, we have, for $g = (x, y) \in C_3 \rtimes C_2$

$$\begin{aligned}
g^2 &= (x, y)(x, y) = (x^2, 1) \\
g^3 &= (x^2, 1)(x, y) = (1, y) \\
g^4 &= (1, y)(x, y) = (x, 1) \\
g^5 &= (x, 1)(x, y) = (x^2, y) \\
g^6 &= (x^2, y)(x, y) = (x^2(^1x), y^2) = (1, 1)
\end{aligned}$$

and then $C_3 \rtimes C_2 =< (x, y) >\cong C_6 = \{1, g, g^2, g^3, g^4, g^5\}$ is a cyclic group. The elements of $C_3 \rtimes C_2$ can be regarded as morphisms

$$g = (x, y), g^3 = (1, y), g^5 = (x^2, y) : y \to y \text{ and } g^6 = (1, 1), g^2 = (x^2, 1), g^4 = (x, 1) : 1 \to 1$$

The compositions of these morphisms are defined by

$$\begin{aligned}
g^5 \circ g^3 &= (x^2, y) \circ (1, y) = (x^2, y) = g^5 \\
g^5 \circ g &= (x^2, y) \circ (x, y) = (1, y) = g^3 \\
g^3 \circ g &= (1, y) \circ (x, y) = (x, y) = g \\
g^4 \circ g^2 &= (x, 1) \circ (x^2, 1) = (1, 1) = g^6 = g^2 \circ g^4
\end{aligned}$$

The idendity map $e : G_0 \to G_1$ is defined on elements by $e(1) = g^6 = 1$ and $e(y) = g^3$. The whiskering operation $w_{01} : G_0 \times G_1 \to G_1$ is defined on elements by

$$w_{01}(y, g)) = (^y x, y^2) = (x, 1) = g^4$$

$$w_{01}(y, g^5) = (^y(x^2), y^2) = (x^2, 1) = g^2$$

$$w_{01}(y, g^3)) = (^y 1, y^2) = (1, 1) = g^6$$

and

$$w_{01}(y, g^6)) = (^y 1, y) = (1, y) = g^3$$

$$w_{01}(y, g^4) = (^y x, y) = (x, y) = g$$

$$w_{01}(y, g^2) = (^y(x^2), y) = (x^2, y) = g^5$$

and the whiskering operation $w_{10} : G_1 \times G_0 \to G_1$ is defined on elements by

$$w_{10}(g, y) = (x, y^2) = (x, 1) = g^4$$

$$w_{10}(g^5, y) = (x^2, y^2) = (x^2, 1) = g^2$$

$$w_{10}(g^3, y) = (1, y^2) = (1, 1) = g^6$$

and

$$w_{10}(g^6, y) = (1, y) = g^3$$

$$w_{10}(g^4, y) = (x, y) = g$$

$$w_{10}(g^2, y) = (x^2, y) = g^5$$

It can be easily seen that these maps satisfy the whiskering axioms. Therefore, we obtain a regular groupoid (C_6, C_2, w_{ij}).

Example 2.4. Consider the Klein 4-group $K = \{1, a, b, c\}$ with $a^2 = b^2 = c^2 = 1$ and the subgroup $N = \{1, b\}$. Let $G_0 = K$. Using the action of K on N given on elements by

$$^c1 = 1, \quad ^b1 = 1, \quad ^a1 = 1 \quad \text{and} \quad ^ab = aba = ac = b, \quad ^bb = b, \quad ^cb = cbc = ca = b$$

We can create a semidirect product group

$$G_1 = N \rtimes K = \{(1,1), (1,a), (1,b), (1,c), (b,1), (b,a), (b,b), (b,c)\}$$

which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The elements of this group can be regarded as morphisms:

$$(1,a) : a \to a, \quad (1,b) : b \to b, \quad (1,c) : c \to c, \quad (b,1) : 1 \to b$$

and

$$(b,a) : a \to ba = c, \quad (b,b) : b \to 1, \quad (b,c) : c \to a, \quad (1,1) : 1 \to 1$$

The compositions are defined on morphisms by, for example,

$$(b,c) \circ (b,a) = (b^2, a) = (1,a) : a \to a \quad \text{and} \quad (b,b) \circ (b,1) = (1,1)$$

Then, we can define the whiskering operations. The operations w_{01} and w_{10} are defined for $a \in K$ by

$$w_{01}(a,(1,a))) = (1,1) = w_{10}((1,a),a), \quad w_{01}(a,(1,b)) = (1,ab) = (1,c) = w_{10}((1,b),a)$$
$$w_{01}(a,(1,c))) = (1,ac) = (1,b) = w_{10}((1,c),a)$$

and

$$w_{01}(a,(b,1)) = (b,a) = w_{10}((b,1),a), \quad w_{01}(a,(b,b)) = (b,c) = w_{10}((b,b),a)$$
$$w_{01}(a,(b,c)) = (b,b) = w_{10}((b,c),a)$$

The whiskering operations can be defined similarly for elements $b, c \in K$. Thus, we have a regular groupoid $(N \rtimes K, K, w_{ij})$ which is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, w_{ij})$.

3. Crossed Modules and Regular Groupoids

In this section, we provide the close relationship between the category of crossed modules of groups and the category of regular groupoids. Crossed modules were introduced by Whitehead in [5]. This structure is an algebraic model for homotopy connected 2-types of topological spaces. Recall that a crossed module is a group homomorphism $\partial : M \to P$ together with an action of P on M, written ^pm, for $p \in P$ and $m \in M$, satisfying the conditions $\partial(^pm) = p\partial(m)p^{-1}$ and $^{\partial m}m' = mm'm^{-1}$, for all $m, m' \in M, p \in P$. We denote the category of crossed modules of groups by $\mathcal{XM}$. For further work about some categorical and algebraic properties of crossed modules in various settings and their examples, see to [20–24].

Example 3.1. Some algebraic examples of crossed modules are as follows:

i. The automorphism map $\phi : G \to Aut(G)$ defined by $\phi(g) = I_g$, for $g \in G$ is a crossed module, where I_g is the inner automorphism of G.

ii. If M is a P-module, there is a well-defined P-action on M. This, together with the zero homomorphism $0 : M \to P$, yields a crossed module.

iii. Let N be normal subgroup of G. Then, G acts on N by conjugation. This action and the inclusion map $i : N \to G$ form a crossed module.

3.1. From Crossed Modules to Regular Groupoids

Let $\partial : M \to N$ be a crossed module. We obtain a whiskered groupoid $\mathcal{C} := (C_1, C_0)$ together with the operations w_{10} and w_{01}. Let $C_0 = N$. By using the action of N on M, we can consider the semidirect product group $M \rtimes N$ with the group operation given by $(m,n)(m',n') = (m({}^{n}m'), nn')$, for $m, m' \in M$ and $n, n' \in N$. Then, by taking $C_0 = N$ and $C_1 = M \rtimes N$, we can create a whiskered groupoid as follows: The source and target maps from C_1 to C_0 are given by $s(m,n) = n$ and $t(m,n) = \partial(m)n$ for all $(m,n) \in C_1$. The groupoid composition is given by $(m',n') \circ (m,n) = (m'm, n)$ if $n' = \partial(m)n$. Finally, the whiskering operations w_{01} and w_{10} are given by respectively $w_{01}(p,(m,n)) = ({}^{p}m, pn)$ and $w_{10}((m,n),p) = (m, np)$, for all $m \in M, n, p \in N$. For these operations, we have

$$s(w_{01}(p,(m,n))) = s({}^{p}m, pn) = pn = ps(m,n)$$

and

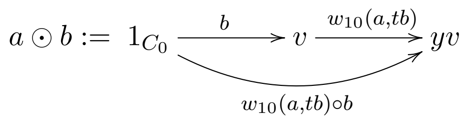
$$\begin{aligned} t(w_{01}(p,(m,n))) &= t({}^{p}m, pn) \\ &= \partial({}^{p}m)pn \\ &= p\partial(m)p^{-1}pn \quad \text{(Since } \partial \text{ cros. mod)} \\ &= p\partial(m)n = pt(m,n) \end{aligned}$$

Similarly, we obtain easily that $s(w_{10}((m,n),p)) = s(m,np) = np = s(m,n)p$ and $t(w_{10}((m,n),p)) = t(m,np) = \partial(m)np = t(m,n)p$ for all $(m,n) \in C_1$ and $n, p \in C_0$.

Consequently, we obtain a whiskered groupoid. In this structure, the operation w_{00} can be taken as the group operation of $C_0 = N$. Thus, we can define a functor from the category of crossed modules of groups to the category of regular groupoids. We denote it by $S : \mathcal{XM} \to \mathcal{RG}$.

3.2. From Regular Groupoids to Crossed Modules

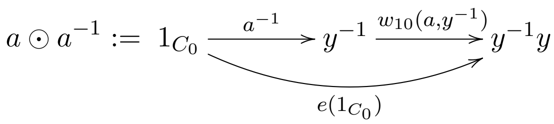
Let $\mathcal{C} := (C_1, C_0, w_{i,j})$ be a whiskered groupoid with the set of objects C_0 is a group according to the multiplication given by the operation w_{00}. In this case, from [4,8] we can say, using the Ehresmannian composition, that the set $K = \{a \in C_1 : s(a) = 1_{C_0}\}$ is a group with the group operation given by $a \odot b = w_{10}(a, t(b)) \circ b$, for any $a : 1_{C_0} \to y$ and $b : 1_{C_0} \to v$ in K, $y, v \in C_0$, and the target map t from K to C_0 is a homomorphism of groups. We can show this multiplication pictorially by



For $1_{C_0} \in C_0$, we have $e(1_{C_0}) : 1_{C_0} \to 1_{C_0}$ is the identity element of K. Indeed for any $a : 1_{C_0} \to y \in K$, we obtain

$$a \odot e(1_{C_0}) = w_{10}(a, 1_{C_0}) \circ 1_{C_0} = a = e(1_{C_0}) \odot a$$

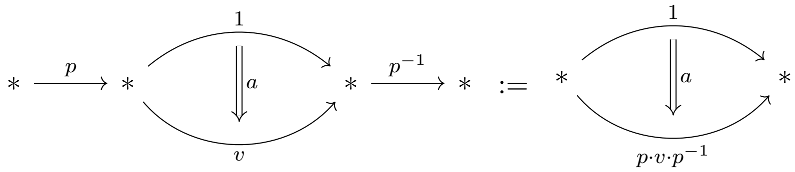
The inverse of $a : 1_{C_0} \to y$ is $a^{-1} : 1_{C_0} \to y^{-1}$ where y^{-1} is the inverse of y in the group C_0. Thus, we have $a \odot a^{-1} = w_{10}(a, y^{-1}) \circ a^{-1} = e(1_{C_0})$. This can be represented by the diagram:



We show that the target map t is a homomorphism of groups from K to C_0. For $a : 1_{C_0} \to y$ and $b : 1_{C_0} \to v$ in K, and $y, v \in C_0$, we obtain $t(a \odot b) = t(w_{10}(a, tb)) \circ b = yv = t(a)t(b)$.

The group action of $p \in C_0$ on $a : 1_{C_0} \to y \in K$ is given by ${}^{p}a = w_{01}(p, w_{10}(a, p^{-1})) = w_{10}(w_{01}(p, a), p^{-1})$.

The group C_0 is acting on itself by conjugation. This action can be represented pictorially by



Thus, we obtain that the homomorphism t is C_0-equivariant relative to the action of C_0 on K given above. Indeed, we have

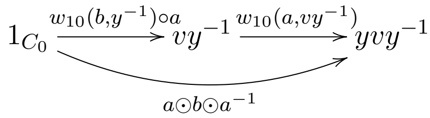
$$t({}^{p}a) = pvp^{-1} = pt(a)p^{-1}$$

for $p \in C_0$ and $a \in K$, and so t is a pre-crossed module of groups.

Furthermore, for any $a : 1_{C_0} \to y, b : 1_{C_0} \to v \in K$, we have

$$a \odot b \odot a^{-1} = w_{10}(a, vy^{-1}) \circ w_{10}(b, y^{-1}) \circ a^{-1} = w_{01}(t(a), w_{10}(b, (ta)^{-1}))$$

This can be represented pictorially by

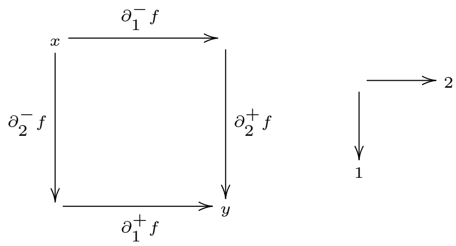


Therefore, we obtain ${}^{t(a)}b = a \odot b \odot a^{-1}$ and this is second crossed module axiom. So, we can say that t is a crossed module of groups. Thus, we have a crossed module $t : K \to C_0$ from the regular groupoid $(\mathcal{C}, w) := (C_1, C_0, w_{i,j})$. We can define a functor from the category of regular groupoids to the category of crossed modules as $F : \mathcal{RG} \to \mathcal{XM}$.

Remark 3.2. We see that there are functors between the categories $\mathcal{RG}$ and $\mathcal{XM}$. However, these functors do not give an equivalence between these categories. Consider a regular groupoid (C_1, C_0, w_{ij}). In this structure, we know that C_0 is a group with the multiplication given by w_{00}, and the set of morphisms C_1 is not a group. If we apply the functor $F : \mathcal{RG} \to \mathcal{XM}$ to this regular groupoid, we obtain $F((C_1, C_0, w_{ij})) := K \to C_0$ and we see that this is a crossed module of groups. If we apply the functor $S : \mathcal{XM} \to \mathcal{RG}$ to this crossed module, we have $S(K \to C_0) := (K \rtimes C_0, C_0, w_{ij})$. Since in the regular groupoid (C_1, C_0, w_{ij}), the set of morphisms C_1 is not a group, there is no isomorphism between $K \rtimes C_0$ and C_1. That is, $K \rtimes C_0 \ncong C_1$; therefore, we can say that these categories are not equivalent.

4. Bimorphisms within Whiskered (Regular) Groupoids

In this section, using the axioms of the crossed module, we give the bimorphism conditions in the regular groupoid obtained from a crossed module. We know from [4] that for the ordered set $I = \{-, +\}$ with $- < +$, a square or a 2-cube, in any category C is a functor $f : I^2 \to C$ and this is written as a diagram



where $Sf = x$, $tf = y$. The squares in C form a double category $\square C$ with compositions $\circ_1, \circ_2$ as given in [4].

Definition 4.1. [4] Let C be a category. A bimorphism $m : (C, C) \to \square C$ assigns to each pair of morphisms $a, b \in C$ a square $m(a, b) \in \square C$ such that if ad, bc are defined in C then

$$m(ad, c) = m(a, c) \circ_1 m(d, c)$$

$$m(a, bc) = m(a, b) \circ_2 m(a, c)$$

Remark 4.2. If we assume that C and D are crossed complexes of groupoids as provided in [1], the tensor product $C \otimes D$ of crossed complexes C, D constructed by Brown and Higgins in [1], is given by the universal bimorphism $(C, D) \to C \otimes D$.

Proposition 4.3. [4] If C is a whiskered category then a bimorphism

$$* : (C, C) \to \square C$$

is defined for $a : x \to y, b : u \to v$ by

$$a * b = \begin{pmatrix} & w_{01}(x, b) & \\ w_{10}(a, u) & & w_{10}(a, v) \\ & w_{01}(y, b) & \end{pmatrix}$$

According to the results obtained above, we can give the following proposition.

Proposition 4.4. For the regular groupoid

$$(C_1, C_0) := \ C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 = N \ , \circ, w_{ij})$$

which is obtained from the crossed module $\partial : M \to N$, the multiplication $a * b$ given by

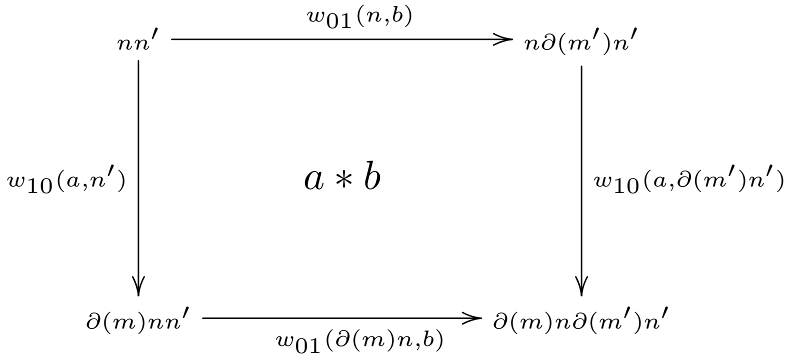
$$m(a, b) = a * b = \begin{pmatrix} & w_{01}(n, (m', n')) & \\ w_{10}((m, n), n') & & w_{10}((m, n), \partial(m')n') \\ & w_{01}(\partial(m)n, (m', n')) & \end{pmatrix}$$

is a bimorphism for $a = (m, n), b = (m', n') \in M \rtimes N$.

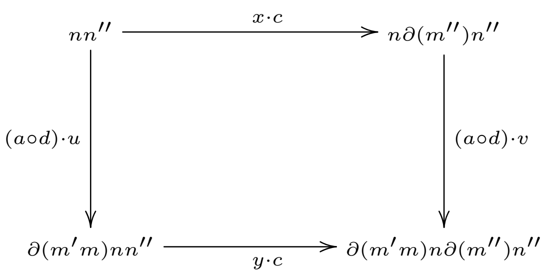
Proof. We must show that

$$m(a \circ d, c) = m(a, c) \circ_1 m(d, c), \quad \text{and } m(a, b \circ c) = m(a, b) \circ_2 m(a, c)$$

For $a = (m, n) : n \to \partial(m)n$ and $b = (m', n') : n' \to \partial(m')n'$, we have already obtained the following diagram:



and then we have $l(a, b) = r(a, b)$. To prove the above equality suppose that $a = (m, n) : n \to \partial(m)n$, $\quad d = (m', \partial(m, n)) : \partial(m)n \to \partial(m')\partial(m)n$. In this case, we have $a \circ d = (m'm, n) : n \to \partial(m')\partial(m)n$. For $c = (m'', n'') : n'' \to \partial(m'')n''$, we can draw the multiplication $m(a \circ d, c)$ by the following picture



where

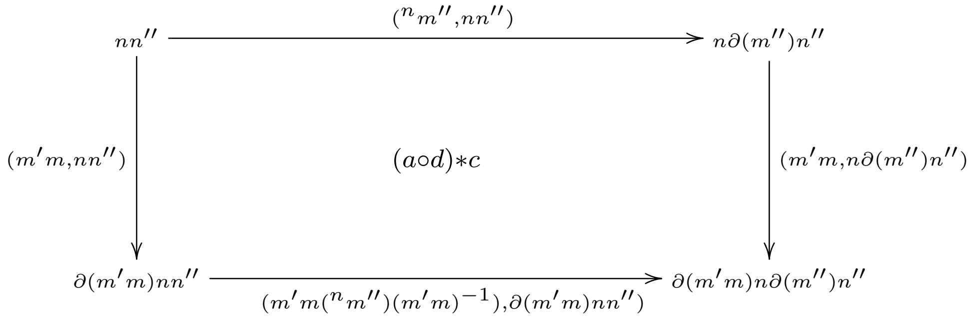
$$(a \circ d) \cdot u = (m'm, n) \cdot n'' = (m'm, nn'')$$

$$(a \circ d) \cdot v = (m'm, n) \cdot \partial(m'')n'' = (m'm, n\partial(m'')n'')$$

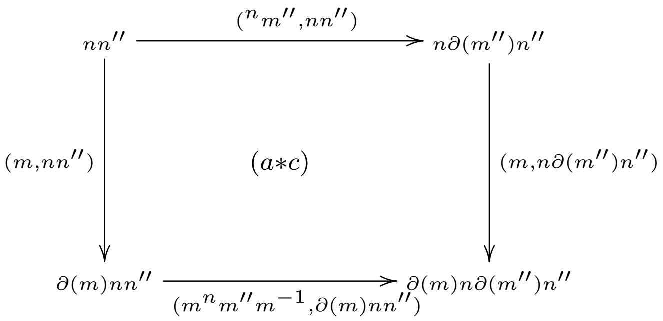
$$x \cdot c = n \cdot (m'', n'') = ({}^{n}(m''), nn'')$$

$$\begin{aligned} y \cdot c = \partial(m'm)n \cdot (m'', n'') &= ({}^{\partial(m'm)}({}^{n}m''), \partial(m'm)nn'') \\ &= (m'm({}^{n}m'')(m'm)^{-1}), \partial(m'm)nn'') \end{aligned}$$

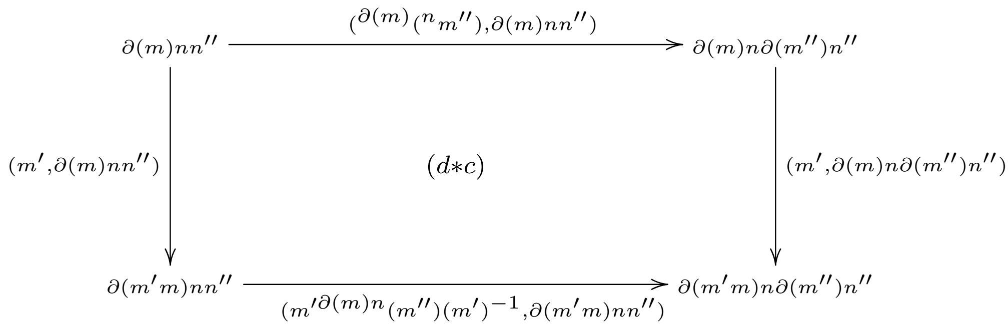
Thus,



On the other hand, we investigate the notion of $m(a, c) \circ_1 m(d, c)$. In the following diagram, we show the multiplication $m(a, c)$.



Similarly, we have



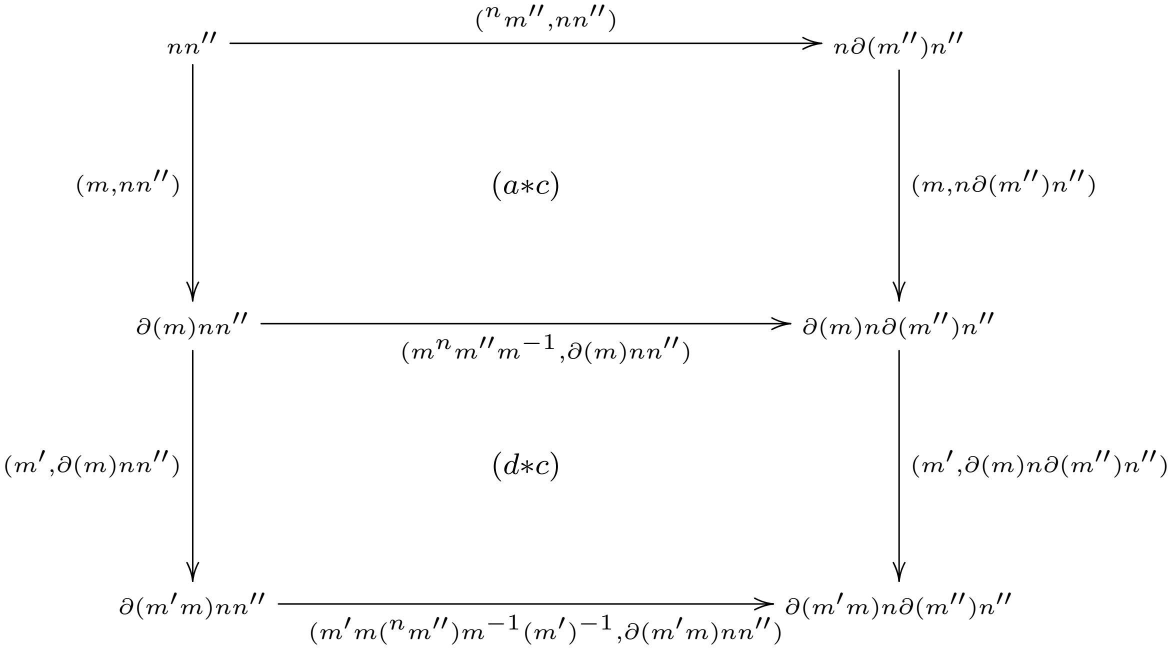
Since ∂ is a crossed module, in $d * c$, we have

$$s(d * c) = ({}^{\partial(m)}({}^{n}m''), \partial(m)nn'') = t(a * c) = (m{}^{n}m''m^{-1}, \partial(m)nn'')$$

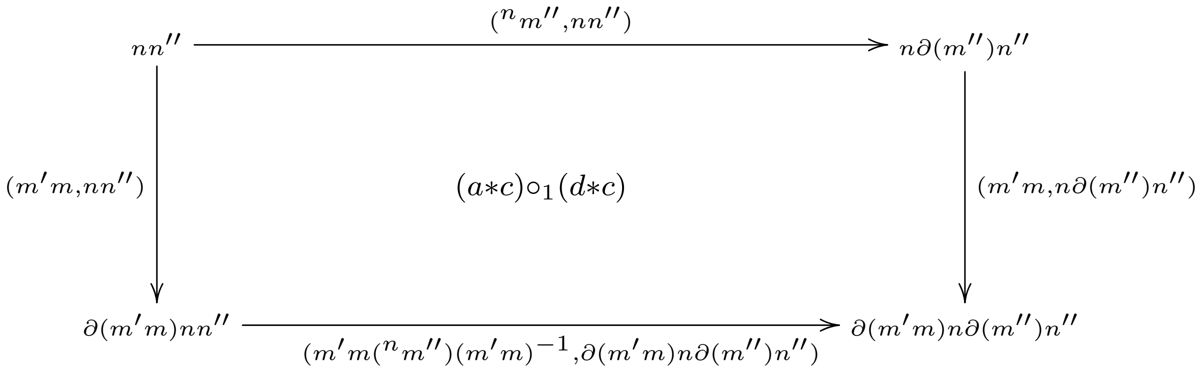
and

$$t(d * c) = (m'^{\partial(m)n}(m'')(m')^{-1}, \partial(m')\partial(m)nn'') = (m'm({}^{n}m'')m^{-1}(m')^{-1}, \partial(m')\partial(m)nn'')$$

Then, we can compose them as



Therefore, from these diagrams, the composition $(a * c) \circ_1 (d * c)$ is given pictorially by



Thus,

$$(a \circ d) * c = (a * c) \circ_1 (d * c)$$

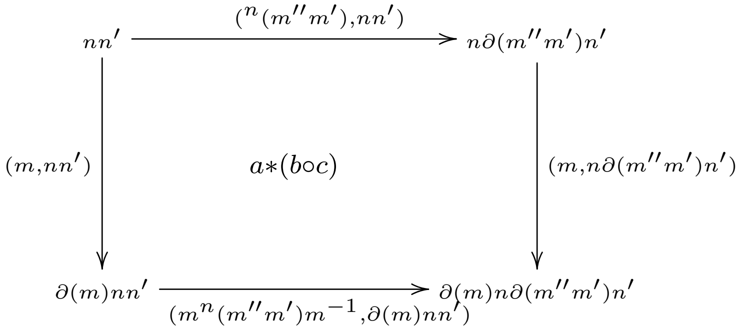
We must show that

$$a * (b \circ c) = (a * b) \circ_2 (a * c)$$

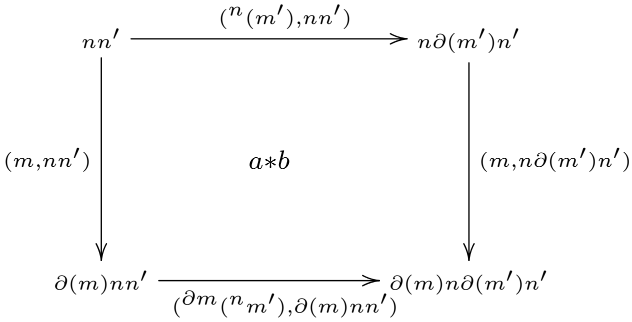
For $b = (m', n') : n' \to \partial(m')n'$ and $c = (m'', n'') : n'' \to \partial(m'')n''$ with $n'' = \partial(m')n'$, we have

$$b \circ c = (m', n') \circ (m'', \partial(m')n') = (m''m', n')$$

We can draw the following diagram for the multiplication $a * (b \circ c)$:



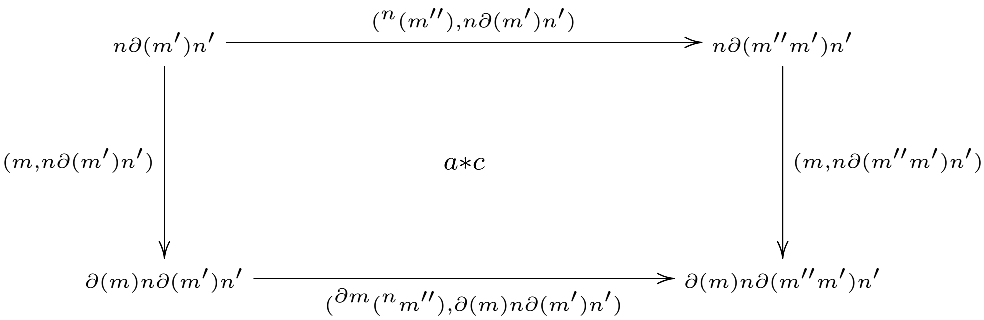
On the other hand, we have



where since ∂ is a crossed module, we obtain

$$t(a*b) = ({}^{\partial m}({}^{n}m'), \partial(m)nn') = (m({}^{n}m')m^{-1}, \partial(m)nn')$$

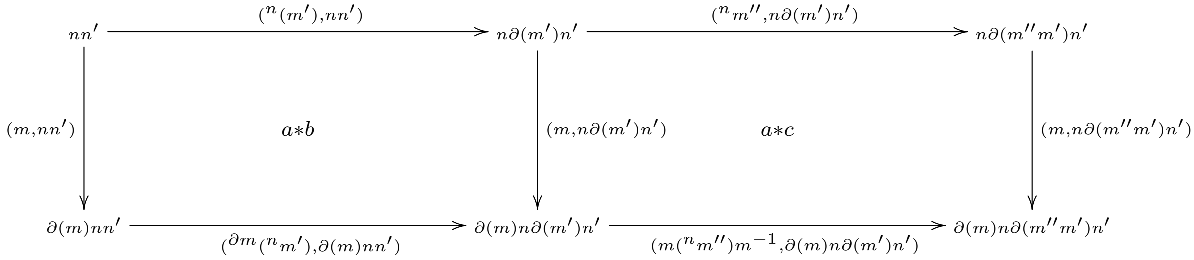
Similarly,



where

$$s(a*c) = (m, n\partial(m')n') = t(a*b)$$

For the horizontal composition $\circ_2$, we obtain the following diagram:



From this diagram, we have

$$({}^{n}m', nn') \circ_2 ({}^{n}m'', n\partial(m')n') = ({}^{n}(m''m'), nn')$$

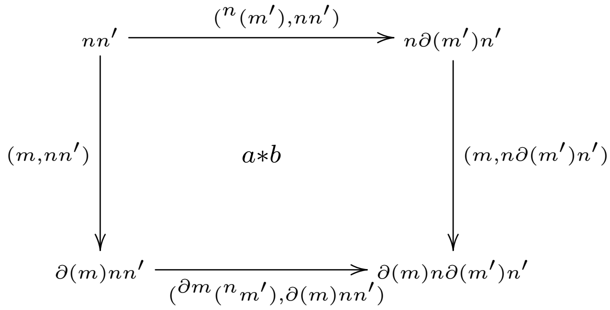
and

$$(m^n m' m^{-1}, \partial(m)nn') \circ_2 (m^n m'' m^{-1}, \partial(m)n\partial(m')n') = (m^n m'' m^{-1} m^n m' m^{-1}, \partial(m)nn')$$
$$= (m^n (m'' m') m^{-1}, \partial(m)nn')$$

Thus, we obtain $(a * b) \circ_2 (a * c) = a * (b \circ c)$. Therefore, in the regular groupoid

$$(C_1, C_0) := (\; C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 = N \;, \circ, w_{ij})$$

associated to the crossed module $\partial : M \to N$, the multiplication given by the following diagram



is a bimorphism for $a = (m, n)$ and $b = (m', n')$. □

Proposition 4.5. In the regular groupoid

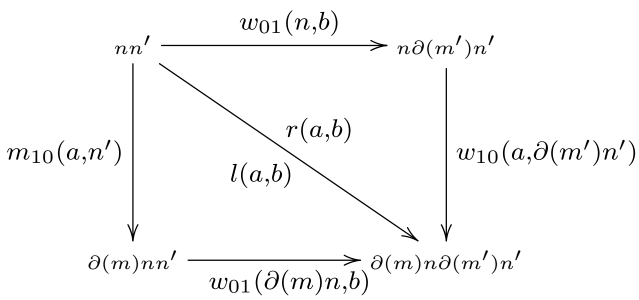
$$(C_1, C_0) := (\; C_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 = N \;, \circ, w_{ij})$$

associated to the crossed module $\partial : M \to N$, we have $l(a, b) = r(a, b)$ so this category is a strict monoidal category.

Proof. For $a = (m, n)$ and $b = (m', n') \in M \rtimes N$,

$$\begin{aligned} l(a,b) =& m_{0,1}(\partial(m)n, b) \circ m_{1,0}(a, n') \\ =& ({}^{\partial(m)n}m', \partial(m)nn') \circ (m, nn') \\ =& ({}^{\partial(m)n}m'm, nn') \\ =& (m({}^{n}m')m^{-1}m, nn') \\ =& (m, n\partial(m')n') \circ ({}^{n}m', nn') \\ =& m_{1,0}(a, \partial(m')n') \circ m_{0,1}(n, b) \\ =& r(a,b) \end{aligned}$$

Therefore, for $a, b \in C_1$, $ab = l(a, b) = r(a, b)$ and thus the regular groupoid (C_1, C_0) is a strict monoidal category. We can illustrate this equality in the following diagram:



□

Proposition 4.6. In the regular groupoid

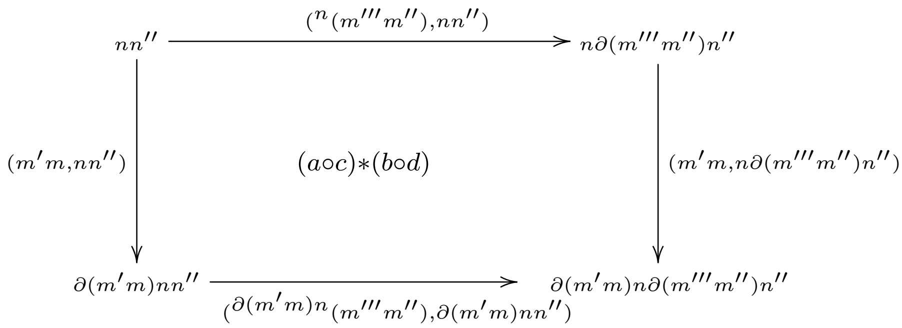
$$(C_1, C_0) := (\ C_1 = M \rtimes N \overset{s,t}{\underset{e}{\rightleftarrows}} C_0 = N\ , \circ, m_{ij})$$

associated to the crossed module $\partial : M \to N$, the interchange law is hold,

$$(a \circ c) * (b \circ d) = (a * b) \circ (c * d)$$

Thus, (C_1, C_0) is an internal category in the category of regular groupoids.

Proof. For $a = (m, n)$, $c = (m', \partial(m)n)$, $b = (m'', n'')$, and $d = (m''', \partial(m'')n'')$, we have $a \circ c = (m'm, n)$ and $b \circ d = (m'''m'', n'')$ and then



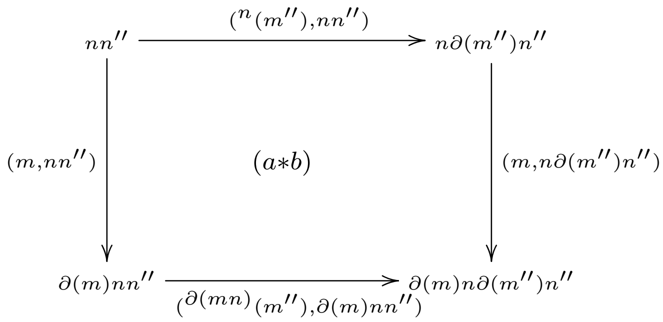
where

$$({}^{\partial(m'm)}({}^{n}(m'''m''), \partial(m')\partial(m)nn'') = (m'm^{n}(m'''m'')(m'm)^{-1}, \partial(m'm)nn'')$$

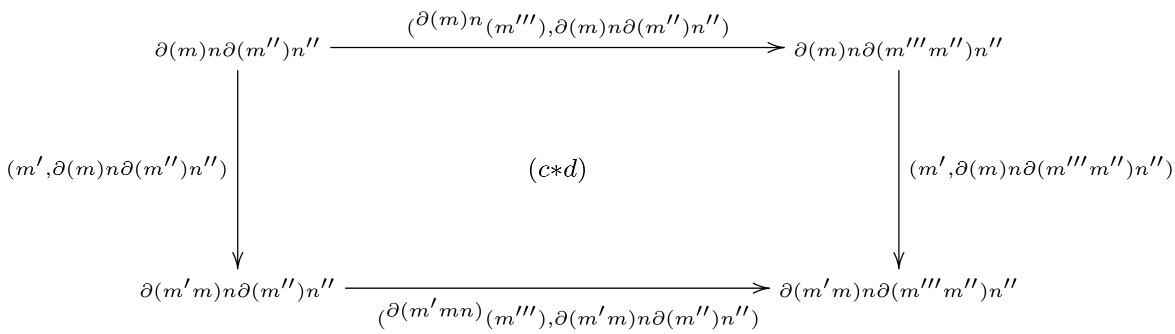
and thus,

$$t(m'm^{n}(m'''m'')(m'm)^{-1}, \partial(m'm)nn'') = \partial(m')\partial(m)n\partial(m''')\partial(m'')n''$$

On the other hand, for $a = (m, n)$ and $b = (m'', n'')$,



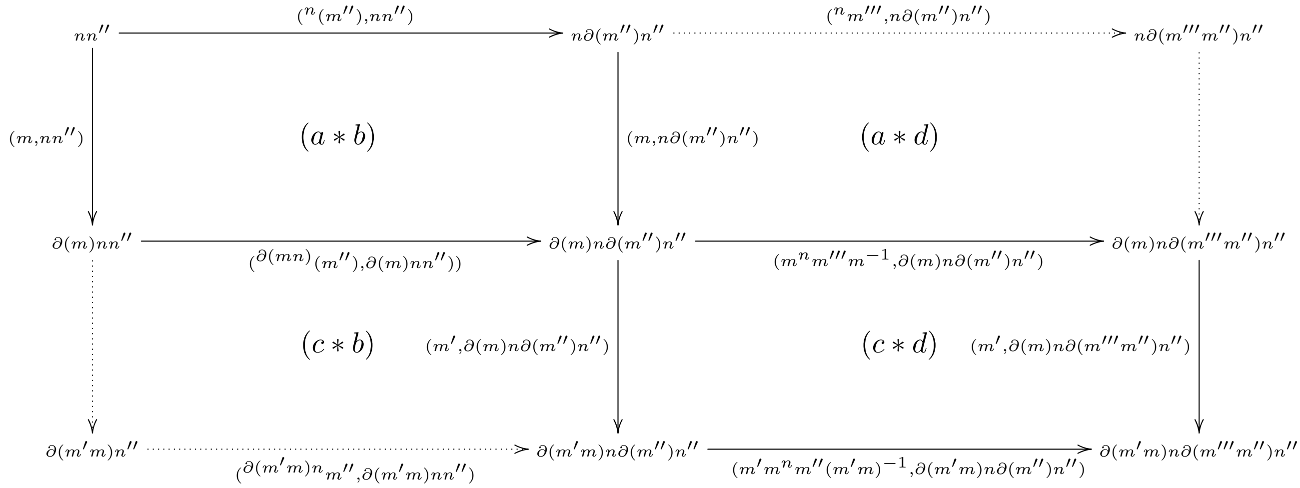
and for $c = (m', \partial(m)n)$ and $d = (m''', \partial(m'')n'')$,



and thus,

$$(a \circ c) * (b \circ d) = (a * b) \circ (c * d)$$

We can illustrate this result in the following diagram,



In this diagram, we can see that

$$((a*b)\circ_2(c*b))\circ_1((a*d)\circ_2(c*d))=((a*b)\circ_1(a*d))\circ_2((c*b)\circ_1(c*d))$$

□

5. Conclusion

In this study, we give the close relationship between crossed modules and whiskered groupoids with the objects set in a group. Thus, we see that a whiskered groupoid can be regarded as a crossed module of groups. If C is a crossed module, then the automorphism structure $Aut(C)$ defined by Brown and Gilbert in [8] has a braided regular crossed module structure. Then, this structure can be considered as a whiskered 2-groupoid with the objects set in a group. For a future study, as a two-dimensional analog of the result herein, the notion of a whiskered 2-groupoid can be defined using the properties of the braiding map on a crossed module. An R-algebroid can be considered as a small R-category, using the results of [12] and [23]; it can also be studied as the R-algebroid version of the results.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis, supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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