On Wintgen Ideal Submanifolds Satisfying Some Pseudo-symmetry Type Curvature Conditions

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

Let $M$ be a Wintgen ideal submanifold of dimension $n$ in a real space form $\mathbb{R}^{n+m}(k)$ of dimension $(n+m)$ and of constant curvature $k$, $n \geq 4$, $m \geq 1$. We determine necessary and sufficient conditions for $M$ to be a submanifold satisfying pseudo-symmetry type curvature conditions of the form: the derivation-commutator $R \cdot C - C \cdot R$ (resp., the tensors $R \cdot C$ and $C \cdot R$) formed by the Riemann-Christoffel curvature tensor $R$ and the Weyl conformal curvature tensor $C$ of $M$, and some Tachibana tensors formed by the metric tensor $g$, the Ricci tensor $\text{Ricc}$ and the tensors $R$ and $C$ of $M$ are linearly dependent.

Keywords: Tachibana tensor, pseudo-symmetry type curvature condition, generalized Einstein metric condition, submanifold, Wintgen ideal submanifold, DDVV conjecture.

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1. Preliminaries

1.1. Pseudo-symmetry type curvature conditions

Let $M = M^n$ be a connected $n$-dimensional, $n \geq 4$, Riemannian manifold of class $C^\infty$. Further, let $\nabla$ be its Levi-Civita connection and $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$. We define on $M$ the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\mathfrak{X}(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$

respectively, where $X, Y, Z \in \mathfrak{X}(M)$ and $A$ is a symmetric $(0,2)$-tensor on $M$. The Ricci tensor $\text{Ricc}$, the Ricci operator $\mathcal{S}$, the scalar curvature $\kappa$ and the endomorphism $\mathcal{C}(X,Y)$ of $(M,g)$ are defined by

$$\text{Ricc}(X,Y) = \text{tr}(Z \rightarrow \mathcal{R}(Z,X)Y),$$

$$g(SU,V) = \text{Ricc}(U,V) = \sum_{i=1}^{n} g(\mathcal{R}(E_i,U)V, E_i), \quad \kappa = \text{tr}\mathcal{S},$$

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge_g SY + SX \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z,$$

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respectively, where \( X, Y, Z, U, V \in \mathfrak{X}(M) \) and \( \{E_k\}_{k \in \{1, \ldots, n\}} \) is a local orthonormal tangent frame field. The Riemann-Christoffel tensor \( R \) and the Weyl conformal curvature tensor \( C \) of \( M \) are defined by

\[
R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W),
\]

\[
C(X, Y, Z, W) = g(\mathcal{C}(X, Y)Z, W),
\]

respectively, where \( X, Y, Z, W \in \mathfrak{X}(M) \). Let \( A \) and \( B \) be the \((0,2)\)-tensors; their Kulkarni-Nomizu product \( A \wedge B \) is defined by

\[
(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) - A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X),
\]

where \( X, Y, X_1, X_2 \in \mathfrak{X}(M) \). Now the tensor \( C \) can be expressed in the form

\[
C = R - \frac{1}{n-2} g \wedge \text{Ricc} + \frac{\kappa}{2(n-1)(n-2)} g \wedge g.
\]

Let \( A \) be a symmetric \((0,2)\)-tensor and \( T \) a \((0,k)\)-tensor, \( k \geq 1 \); we define the \((0,k+2)\)-tensors \( R \cdot T \) and \( Q(A, T) \) by

\[
(R \cdot T)(X_1, X_2, \ldots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= - T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, X_2, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),
\]

\[
Q(A \cdot T)(X_1, X_2, \ldots, X_k; X, Y) = ((A \wedge A) \cdot T)(X_1, \ldots, X_k)
\]

\[
= - T((A \wedge A)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, X_2, \ldots, X_{k-1}, (A \wedge A)X_k),
\]

respectively, where \( X, Y, X_1, X_2, \ldots, X_n \in \mathfrak{X}(M) \). If we set in the above formulas \( T = R, T = \text{Ricc}, T = C, A = g \) or \( A = \text{Ricc} \), then we obtain the tensors: \( R \cdot R, R \cdot \text{Ricc}, R \cdot C, Q(g, R), Q(g, \text{Ricc}), Q(g, C), Q(g, g \wedge \text{Ricc}), Q(\text{Ricc}, R) \) and \( Q(\text{Ricc}, C) \). Similarly we define the tensors: \( C \cdot R, C \cdot C \) and \( C \cdot \text{Ricc} \). The derivation-commutator \( R \cdot C - C \cdot R \) reads (see, e.g., \([20, 28]\))

\[
(n-2)(R \cdot C - C \cdot R) = Q \left( \text{Ricc} - \frac{\kappa}{n-1} g, R \right) - g \wedge (R \cdot \text{Ricc}) + P,
\]

where the \((0,6)\)-tensor \( P \) is defined by

\[
P(X_1, X_2, X_3, X_4; X, Y) = g(X, X_1)R(S(Y), X_2, X_3, X_4) - g(Y, X_1)R(S(X), X_2, X_3, X_4)
\]

\[
+ g(X, X_2)R(X_1, S(Y), X_3, X_4) - g(Y, X_2)R(X_1, S(X), X_3, X_4)
\]

\[
+ g(X, X_3)R(X_1, X_2, S(Y), X_4) - g(Y, X_3)R(X_1, X_2, S(Y), X_4)
\]

\[
+ g(X, X_4)R(X_1, X_2, X_3, S(Y)) - g(Y, X_4)R(X_1, X_2, X_3, S(X))
\]

\(X, Y, X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)\).

A Riemannian manifold \((M, g)\) is said to be semi-symmetric (see \([47]\)) if \( R \cdot R = 0 \) on \( M \). An extension form for semi-symmetric manifolds are the pseudo-symmetric manifolds. A Riemannian manifold \((M, g)\) is said to be pseudo-symmetric (see \([16, 20]\)) if at every point of \( M \) the tensors \( R \cdot R \) and \( Q(g, R) \) are linearly dependent, which means that there exists a function \( L_R \) on the set \( U_R = \{ x \in M \mid R - (\kappa/(2n(n-1)))g \wedge g \neq 0 \text{ at } x \} \) such that

\[
R \cdot R = L_R Q(g, R)
\]

on \( U_R \). Every semi-symmetric manifold is pseudo-symmetric. The converse is not true (see, e.g., \([14, \text{Corollary 3.2 (i)}]\)). Pseudo-symmetric manifolds also are called pseudo-symmetric (in the sense of Deszcz) or Deszcz symmetric spaces (see, e.g., \([10, 13, 48, 50, 51, 52]\)).

A Riemannian manifold \((M, g)\) is said to be Weyl-semi-symmetric (see \([16, 20]\)) if

\[
R \cdot C = 0
\]

on \( M \). An extension form for Weyl-semi-symmetric manifolds are the Weyl pseudo-symmetric manifolds. A Riemannian manifold \((M, g)\) is said to be Weyl-pseudo-symmetric (see \([16, 20]\)) if at every point of \( M \) the
tensors \( R \cdot C \) and \( Q(g, C) \) are linearly dependent, which means that there exists a function \( L_C \) on the set \( U_C = \{ x \in M \mid C \neq 0 \text{ at } x \} \) such that
\[
R \cdot C = L_C Q(g, C)
\]
on \( U_C \). It is obvious that every pseudo-symmetric manifold is Weyl-pseudo-symmetric. The converse is not true.

A Riemannian manifold \((M, g)\) is said to be a manifold with pseudo-symmetric Weyl tensor (see [16, 17, 20]) if at every point of \( M \) the tensors \( C \cdot C \) and \( Q(g, C) \) are linearly dependent, which means that there exists a function \( L \) on the set \( U \) such that
\[
C \cdot C = LQ(g, C)
\]
on \( U \). Every Chen ideal submanifold in a space of constant curvature is a manifold with pseudo-symmetric Weyl tensor [22, 27, 28, 32].

A Riemannian manifold \((M, g)\) is said to be Ricci-pseudo-symmetric (see [16, 20, 30]) if at every point of \( M \) the tensors \( R \cdot \text{Ricc} \) and \( Q(g, \text{Ricc}) \) are linearly dependent, which means that there exists a function \( L_{\text{Ricc}} \) on the set \( U_{\text{Ricc}} = \{ x \in M \mid \text{Ricc} - (\kappa/n)g \neq 0 \text{ at } x \} \) such that
\[
R \cdot \text{Ricc} = L_{\text{Ricc}} Q(g, \text{Ricc})
\]
on \( U_{\text{Ricc}} \). Every Cartan hypersurface \( M \) in the sphere \( S^{n+1}, n = 6, 12 \) or 24, is a non-pseudo-symmetric, Ricci-pseudo-symmetric with non-pseudo-symmetric Weyl tensor hypersurfaces [33] (see also [31]). We also mention that Ricci-pseudo-symmetric manifolds also are called Ricci pseudo-symmetric in the sense of Deszcz, or simply Deszcz Ricci-symmetric (see, e.g., [10]).

A Riemannian manifold \((M, g)\) is said to be Ricci-Weyl-pseudo-symmetric (see [16, 20]) if at every point of \( M \) the tensors \( R \cdot C \) and \( Q(\text{Ricc}, C) \) are linearly dependent, which means that there exists a function \( L' \) on the set \( U = \{ x \in M \mid Q(\text{Ricc}, C) \neq 0 \text{ at } x \} \) such that
\[
R \cdot C = L'Q(\text{Ricc}, C)
\]
on \( U \). It is obvious that every semi-symmetric manifold \((R \cdot R = 0)\) satisfies (1.5) with \( L' = 0 \).

We refer to [18] (see also [19, 21]) for a recent survey on manifolds satisfying (1.1)–(1.5) and other conditions of this kind. Such conditions are called pseudo-symmetry type curvature conditions. It seems that (1.1) is the most important condition of that family of curvature conditions (see [17, 23, 24, 25, 28, 38, 39, 48, 49, 50, 51, 52]).

We also refer to [19] (see also [20]) for a survey of results on semi-Riemannian manifolds \((M, g)\) and in particular, hypersurfaces in spaces of constant curvature or Chen ideal submanifolds in Euclidean spaces, satisfying pseudo-symmetry type curvature conditions of the form: the derivation-commutator \( R \cdot C - C \cdot R \), formed by the tensors \( R \) and \( C \), and a finite sum of the Tachibana tensors of the form \( Q(A, T) \) are linearly dependent, where \( A \) is a symmetric \((0, 2)\)-tensor and \( T \) a generalized curvature tensor. Conditions of this kind belong to the family of curvature conditions called generalized Einstein metric conditions [19, 20, 26, 30].

### 1.2. Submanifolds in space forms

Throughout all sections of the present paper, let \( M = M^n \) be a connected Riemannian manifold of class \( C^\infty \) of dimension \( n \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \) of dimension \( (n+m) \) and of constant curvature \( \bar{k}, n \geq 4, m \geq 1 \).

On \( \mathbb{R}^{n+m}(\bar{k}) \), we denote by \( \bar{g} \) and \( \bar{\nabla} \) respectively the Riemannian metric and the corresponding Levi-Civita connection. On the submanifold \( M \), the induced Riemannian metric and the corresponding Levi-Civita connection on \( M \) will be denoted by \( g, \nabla \). We will write as \( X, Y, \ldots \) the tangent vector fields on \( M \), and as \( \xi, \eta, \ldots \) the normal vector fields on \( M \).

The well-known formulae of Gauss and Weingarten are given by
\[
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]
\[
\bar{\nabla}_X \xi = \nabla_X \xi - A_\xi Y,
\]
respectively, whereby \( \nabla^\perp \) is the normal connection induced in the normal bundle of \( M \) in \( \mathbb{R}^{n+m}(\bar{k}) \), \( h \) is the second fundamental form of the submanifold \( M \) and \( A_\xi \) is the shape operator or the Weingarten map on \( M \) with respect to the normal vector field \( \xi \). We have
\[
g(A_\xi(X), Y) = g(\xi, h(X, Y)).
\]
or still
\[ h(X, Y) = \sum_{\alpha=1}^{m} g(A_\alpha(X), Y) \xi_\alpha, \]
whereby \( \{\xi_\alpha\}_{\alpha \in \{1, \ldots, m\}} \) is any local orthonormal normal framefield on \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \) and \( A_\alpha = A_{\xi_\alpha} \).

The mean curvature vector field of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \) is defined by
\[ \vec{H} = \frac{1}{n} \sum_{i=1}^{n} h(E_i, E_i) = \frac{1}{n} \sum_{\alpha=1}^{m} (\operatorname{tr} A_\alpha) \xi_\alpha, \]
whereby \( \{E_s\}_{s \in \{1, \ldots, n\}} \) is any local orthonormal tangent framefield on \( M \), and \( \{\xi_\alpha\}_{\alpha \in \{1, \ldots, m\}} \) is any local orthonormal normal framefield on \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \) and its length \( H = \|\vec{H}\| \) is the mean curvature of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \).

The submanifold \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \) is totally geodesic when \( h = 0 \). \( M \) is totally umbilical when \( h = g \vec{H} \). It is minimal when \( \vec{H} = 0 \), or equivalently, when its squared mean curvature function \( H^2 = g(\vec{H}, \vec{H}) \) vanishes identically. \( M \) is pseudo-umbilical when the mean curvature vector field \( \vec{H} \) determines an umbilical normal direction on \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \), i.e., when \( A_{\vec{H}} = \chi \operatorname{Id} \), whereby \( \operatorname{Id} \) stands for the identity operator on \( TM \) and \( \chi \) is some real function on \( M \).

Let \( R \) denote the induced Riemann-Christoffel curvature corresponding to the induced Levi-Civita connection \( \nabla \) on \( M \). Then according to the equation of Gauss
\[ R(X, Y, Z, W) = \bar{g}(h(X, W)h(Y, Z) - h(X, Z)h(Y, W)) + \bar{k}(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)), \]
where \( X, Y, Z, W \) are tangent vector fields in \( M \).

Let \( \kappa \) be the scalar curvature function of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \); we have
\[ \kappa(p) = \sum_{i<j} K(p, E_i(p) \wedge E_j(p)) \]
where \( K(p, E_i(p) \wedge E_j(p)) \) is the sectional curvature of \( M \) at a point \( p \in M \) for the plane section \( \phi = E_i(p) \wedge E_j(p) \) in \( T_pM \) (where \( \{E_i(p), E_j(p)\} \) is independent). By \( \inf(K) \) we will further denote the function \( \inf(K) : M \to \mathbb{R} \) which attains to \( p \in M \) the minimal value \( \inf(K)(p) \) of all sectional curvatures of \( M \) at \( p \). The normalized scalar curvature of \( (M, g) \) is defined as follows
\[ \rho = \frac{2}{n(n-1)} \sum_{i<j} R(E_i, E_j, E_i, E_j), \]
whereby \( \{E_s\}_{s \in \{1, \ldots, n\}} \) is any local orthonormal tangent framefield on \( M \).

By the equation of Ricci, the normal curvature tensor \( R^\perp \) of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \), i.e., the curvature tensor of the normal connection \( \nabla^\perp \) of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \), is defined as follows
\[ R^\perp(X, Y; \xi, \eta) = \bar{g} \left( \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]} \xi, \eta \right) = g([A_\xi, A_\eta] X, Y), \]
whereby \( [A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi \).

The normalized scalar normal curvature \( \rho^\perp \) of \( (M, g) \) in \( \mathbb{R}^{n+m}(\vec{k}) \) is given as follows
\[ \rho^\perp = \frac{2}{n(n-1)} \sqrt{\sum_{i<j} \sum_{\alpha<\beta} R^\perp(E_i, E_j, \xi_\alpha, \xi_\beta)}, \]
whereby \( \{E_s\}_{s \in \{1, \ldots, n\}} \) is any local orthonormal tangent framefield on \( M \), and \( \{\xi_\alpha\}_{\alpha \in \{1, \ldots, m\}} \) is any local orthonormal normal framefield on \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \). One can remark that \( \rho^\perp = 0 \) if and only if \( R^\perp = 0 \), which means that the normal connection is flat. This follows from (1.6) and as it was already observed by E. Cartan (see [1]), is equivalent to the simultaneous diagonalisability of all shape operators \( A_\xi \) of \( M \) in \( \mathbb{R}^{n+m}(\vec{k}) \).
1.3. Wintgen ideal submanifolds satisfying some pseudo-symmetry type curvature conditions

In classical differential geometry, for a surface $M^2$ in a Euclidean 3-space $\mathbb{E}^3$, the well-known Euler inequality is given by

$$K \leq H^2,$$

whereby $K$ is the intrinsic Gaussian curvature of $M^2$ and $H^2$ is the extrinsic squared mean curvature of $M^2$ in $\mathbb{E}^3$, at once follows from the fact that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ whereby $k_1$ and $k_2$ denote the principal curvatures of $M^2$ in $\mathbb{E}^3$. Obviously, $K = H^2$ everywhere on $M^2$ if and only the surface $M^2$ is totally umbilical in $\mathbb{E}^3$, i.e. $k_1 = k_2$ at all points of $M^2$, or still, by a theorem of Meunier; it and only $M^2$ is a part of a plane $\mathbb{E}^2$ or of a round sphere $S^2$ in $\mathbb{E}^3$. P. Wintgen (see [54]), in the late 19 seventies, proved that the Gaussian curvature $K$ and the squared mean curvature $H^2$ of any surface $M^4$ in $\mathbb{E}^4$ always satisfy the inequality

$$K \leq H^2 - K^1,$$

and that actually the equality if and only if the curvature ellipse of $M^2$ in $\mathbb{E}^4$ is a circle. We recall that the ellipse of curvature at a point $p$ of $M^2$ is defined as

$$\Sigma_p = \{ h(X, X) \mid X \in T_p M \text{ and } \|X\| = 1 \}.$$

The ellipse of curvature is the analogue of the Dupin curvature of an ordinary surface in $\mathbb{E}^3$.

B. Rouxel in [43] and V. Guadalupe and L. Rodriguez in [36] extended Wintgen’s inequality to surfaces of arbitrary codimension in real space forms $\mathbb{R}^{2+ m}(c)$ with $m \geq 2$. Also, B.-Y. Chen extended Wintgen’s inequality in [6, 8] to surfaces in a pseudo-Euclidean 4-space $\mathbb{E}^4_2(c)$ with a neutral metric.


$$\rho \leq H^2 - \rho^1 + \bar{k},$$

for all submanifolds $M^n$ of codimension 2 in all real space forms, whereby $\rho$ is the normalised scalar curvature of the Riemannian manifold $M$, and whereby $H^2$ and $\rho^1$, are the squared mean curvature and the normalized normal scalar curvature of $M$ in the ambient space, respectively, characterizing the equality in terms of the shape operators of $M^n$ in $\mathbb{R}^{n+2}(\bar{k})$. And in [15] they proposed a conjecture of Wintgen inequality for general Riemannian submanifolds in real space forms, which was later well-known as the DDVV conjecture. This conjecture was proven to be true by Z. Lu (see [40]), and by J. Ge and Z. Tang (see [34, 35]) independently. In [10], B.-Y. Chen provided a comprehensive survey on developments in Wintgen inequality and Wintgen ideal submanifolds, we also refer to the recent article of G.-E. Vilcu [53, Chapter 7].

The main purpose of the present article is to study the so-called Wintgen ideal submanifolds. The following theorem of J. Gè and Z. Tang (see [34, 35]) and Lu (see [40]) states the Wintgen inequality for submanifolds $M = M^n$ of any codimension $m \geq 1$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, and characterizes its equality case. As it is stated in [34], it concerns a basic general optimal inequality between likely the most primitive scalar valued geometric quantities that can be defined on submanifolds as intrinsic invariant it involves the scalar curvature and as extrinsic invariants it involves the scalar normal curvature and the squared mean curvature.

**Theorem A.** (see [11, 34, 35, 40]) Let $M = M^n$ be a submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 2$. Then

$$(\ast) \quad \rho \leq H^2 - \rho^1 + \bar{k},$$

and in ($\ast$) actually the equality holds if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_a\}$ on $M$ in $\mathbb{R}^{n+m}(\bar{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} a & \mu & 0 & \cdots & 0 \\ \mu & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} b + \mu & 0 & 0 & \cdots & 0 \\ 0 & b - \mu & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \quad (1.7)$$

$$A_4 = \ldots = A_m = 0,$$

where $a, b, c$ and $\mu$ are real functions on $M$.$\square$
Definition 1.1. The submanifold \( M \) in \( \mathbb{R}^{n+m}(\bar{k}) \) which satisfy the equality

\[
(\bar{k}) \quad \rho = H^2 - \rho^+ + \bar{k}
\]

in the Wintgen’s general inequality (\( \bar{k} \)) is called a Wintgen ideal submanifold. In such case, the frames \( \{E_i, \xi_n\} \) in which the shape operators assume the forms of (1.7) will further be called the Choi-Lu frames, and the corresponding tangent \( E_1E_2 \)-planes will be called the Choi-Lu planes of \( M \) in \( \mathbb{R}^{n+m}(\bar{k}) \).

From [13] (see also [29, 42]) we recall the following results.

Theorem B. A Wintgen ideal submanifold \( M = M^n \) of dimension \( n \geq 4 \) and of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \) (\( m \geq 2 \)) is a Deszcz symmetrical Riemannian manifold if and only if \( M \) is totally umbilical (with Deszcz sectional curvature \( L_R = 0 \)), or, \( M \) is a minimal or pseudo-umbilical submanifold (with \( L_R = \bar{k} + H^2 \)) of this space form \( \mathbb{R}^{n+m}(\bar{k}) \). \( \square \)

Theorem C. A Wintgen ideal submanifold \( M = M^n \) of dimension \( n \geq 4 \) and of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \) (\( m \geq 2 \)) is Deszcz symmetric if and only if \( M \) is Deszcz Ricci-symmetric. \( \square \)

Theorem D. Let \( M = M^n \) be a Wintgen ideal submanifold of dimension \( n \geq 4 \) and of codimension \( m \geq 2 \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \). Then \( M \) is a Riemannian manifold with a pseudo-symmetric conformal Weyl tensor \( C \). \( \square \)

Theorem E. Let \( M = M^n \) be a Wintgen ideal submanifold of dimension \( n \geq 4 \) and of codimension \( m \geq 2 \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \).

(i) \( M \) is conformally flat if and only if \( M \) is a totally umbilical submanifold in \( \mathbb{R}^{n+m}(\bar{k}) \) (and, hence, \( M \) is a real space form).

(ii) If \( M \) is not a conformally flat submanifold, then \( M \) has a pseudo-symmetric conformal Weyl tensor \( C \) and the corresponding function is given by

\[
L_C = \frac{n - 3}{(n-1)(n-2)} \left( \kappa - n(n-1) \inf K \right),
\]

where \( \kappa \) is the scalar curvature of \( M \). \( \square \)

We also refer to [12, 41, 44, 45, 46] for further results on Wintgen ideal submanifolds.

2. Main results on Wintgen ideal submanifolds

2.1. Main results

In the paper we investigate Wintgen ideal submanifolds \( M = M^n \) in real space forms \( \mathbb{R}^{n+m}(\bar{k}) \), \( n > 3 \), \( m \geq 1 \), satisfying the following pseudo-symmetry type curvature conditions:

(i) the tensor \( R \cdot C \) and the Tachibana tensor \( Q(g, R) \) (resp., the tensor \( Q(g, C) \), \( Q(g, g \wedge \text{Ricc}) \), \( Q(\text{Ricc}, R) \), or \( Q(\text{Ricc}, C) \)) are linearly dependent;

(ii) the tensor \( C \cdot R \) and the Tachibana tensor \( Q(g, R) \) (resp., the tensor \( Q(g, C) \), \( Q(g, g \wedge \text{Ricc}) \), \( Q(\text{Ricc}, R) \), or \( Q(\text{Ricc}, C) \)) are linearly dependent;

(iii) the derivation-commutator \( R \cdot C - C \cdot R \), formed by the tensors \( R \) and \( C \), and the Tachibana tensor \( Q(g, R) \) (resp., the tensor \( Q(g, C) \), \( Q(g, g \wedge \text{Ricc}) \), \( Q(\text{Ricc}, R) \), or \( Q(\text{Ricc}, C) \)) are linearly dependent.

In this subsection we present our main results.

Theorem 2.1. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \), \( n \geq 4 \) and \( m \geq 2 \). If \( \bar{k} > 0 \), then \( M \) is Weyl-semi-symmetric if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\bar{k}) \) (and hence \( M \) is conformally flat). \( \square \)

Theorem 2.2. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\bar{k}) \), \( n \geq 4 \) and \( m \geq 2 \). If \( \bar{k} \leq 0 \), then \( M \) is Weyl-semi-symmetric if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\bar{k}) \) (and hence
$M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on $M$ in $\mathbb{R}_n^{n+m}(\tilde{k})$, the shape operators are given by

$$
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_3 = \begin{pmatrix}
c & 0 & 0 & \cdots & 0 \\
0 & c & 0 & \cdots & 0 \\
0 & 0 & c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
$$

(2.1)

where $c, \mu$ are real functions on $M$ such that $\mu \neq 0$ and $c^2 = -\tilde{k}$. In this second case, $M$ is a minimal or pseudo-umbilical submanifold in $\mathbb{R}_n^{n+m}(\tilde{k})$.

**Corollary 2.1.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}_n^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 2$. If $\tilde{k} = 0$, then $M$ is Weyl-semi-symmetric if and only if (i) either $M$ is totally umbilical in $\mathbb{R}_n^{n+m}$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on $M$ in $\mathbb{R}_n^{n+m}$, the shape operators are given by

$$
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_3 = \begin{pmatrix}
c & 0 & 0 & \cdots & 0 \\
0 & c & 0 & \cdots & 0 \\
0 & 0 & c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
$$

(2.2)

where $\mu$ is a real function on $M$ such that $\mu \neq 0$. In this second case, $M$ is minimal in $\mathbb{R}_n^{n+m}$.

**Theorem 2.3.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}_n^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 2$. Then

$$
C \cdot R = 0
$$

if and only if $M$ is totally umbilical in $\mathbb{R}_n^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat).

**Theorem 2.4.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}_n^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 2$. Then

$$
R \cdot C - C \cdot R = 0
$$

if and only if $M$ is totally umbilical in $\mathbb{R}_n^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat).

**Theorem 2.5.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}_n^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 2$. If $\tilde{k} > 0$, then the tensors $R \cdot C$ and $Q(g, R)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}_n^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat).

**Theorem 2.6.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}_n^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 2$. If $\tilde{k} \leq 0$, then the tensors $R \cdot C$ and $Q(g, R)$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}_n^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_\alpha\}$ on $M$ in $\mathbb{R}_n^{n+m}(\tilde{k})$, the shape operators are given by

$$
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
A_3 = \begin{pmatrix}
c & 0 & 0 & \cdots & 0 \\
0 & c & 0 & \cdots & 0 \\
0 & 0 & c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
$$

(2.3)

where $c, \mu$ are real functions on $M$ such that $\mu \neq 0$ and $c^2 = -\tilde{k}$. Moreover, $M$ is Weyl-semi-symmetric, and, in the second case, $M$ is a minimal or pseudo-umbilical submanifold in $\mathbb{R}_n^{n+m}(\tilde{k})$. □
Corollary 2.2. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(k) \), \( n \geq 4 \) and \( m \geq 2 \). If \( k = 0 \), then the tensors \( R \cdot C \) and \( Q(g, R) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m} \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m} \), the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
A_3 = A_4 = \ldots = A_m = 0,
\]

where \( \mu \) is a real function on \( M \) such that \( \mu \neq 0 \). In this second case, \( M \) is minimal in \( \mathbb{R}^{n+m} \). □

Theorem 2.7. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(k) \), \( n \geq 4 \) and \( m \geq 1 \). Then the tensors \( C \cdot R \) and \( Q(g, R) \) are linearly dependent if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(k) \) (and hence \( M \) is conformally flat). □

Theorem 2.8. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(k) \), \( n \geq 4 \) and \( m \geq 2 \). If \( k > 0 \), then the tensors \( R \cdot C - C \cdot R \) and \( Q(g, R) \) are linearly dependent if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(k) \) (and hence \( M \) is conformally flat). □

Theorem 2.9. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(k) \), \( n \geq 4 \) and \( m \geq 2 \). If \( k \leq 0 \), then the tensors \( R \cdot C - C \cdot R \) and \( Q(g, R) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(k) \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m}(k) \), the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
A_3 = A_4 = \ldots = A_m = 0,
\]

where \( c, \mu \) are real functions on \( M \) such that \( \mu \neq 0 \) and \( c^2 = -k \). Moreover,

\[
R \cdot C - C \cdot R = -\frac{2(n-3)\mu^2}{(n-1)(n-2)} Q(g, R);
\]

and, in the second case, \( M \) is a minimal or pseudo-umbilical submanifold in \( \mathbb{R}^{n+m}(k) \). □

Corollary 2.3. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(k) \), \( n \geq 4 \) and \( m \geq 2 \). If \( k = 0 \), then the tensors \( R \cdot C - C \cdot R \) and \( Q(g, R) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m} \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m} \), the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
A_3 = A_4 = \ldots = A_m = 0,
\]

where \( \mu \) is a real function on \( M \) such that \( \mu \neq 0 \). In this second case, \( M \) is minimal in \( \mathbb{R}^{n+m} \). □
Theorem 2.10. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). Then the tensors \( R \cdot C \) and \( Q(g, C) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\kappa) \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m}(\kappa) \), the shape operators are given by

\[
A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & 0 & 0 & \cdots & c \\
\end{pmatrix},
\]

(2.7)

where \( c \) and \( \mu \) are real functions on \( M \) such that \( \mu \neq 0 \). Moreover,

\( R \cdot C = (\kappa + \kappa^2)Q(g, C) \);

and, in the second case, \( M \) is a minimal or pseudo-umbilical submanifold in \( \mathbb{R}^{n+m}(\kappa) \). \( \square \)

Theorem 2.11. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). Then the tensors \( C \cdot R \) and \( Q(g, C) \) are linearly dependent if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\kappa) \). In this case, \( M \) is conformally flat. \( \square \)

Theorem 2.12. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). Then the tensors \( R \cdot C - C \cdot R \) and \( Q(g, C) \) are linearly dependent if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\kappa) \). In this case, \( M \) is conformally flat. \( \square \)

Theorem 2.13. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). If \( \kappa > 0 \), then the tensors \( R \cdot C \) and \( Q(g, g \wedge \text{Ricc}) \) are linearly dependent if and only if \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\kappa) \) (and hence \( M \) is conformally flat). \( \square \)

Theorem 2.14. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). If \( \kappa \leq 0 \), then the tensors \( R \cdot C \) and \( Q(g, g \wedge \text{Ricc}) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\kappa) \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m}(\kappa) \), the shape operators are given by

\[
A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & 0 & 0 & \cdots & c \\
\end{pmatrix},
\]

(2.8)

where \( c \) and \( \mu \) are real functions on \( M \) such that \( \mu \neq 0 \) and \( c^2 = -\kappa \). Moreover, \( M \) is Weyl-semi-symmetric ; and, in the second case, \( M \) is a minimal or pseudo-umbilical submanifold in \( \mathbb{R}^{n+m}(\kappa) \). \( \square \)

Corollary 2.4. Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\kappa) \), \( n \geq 4 \) and \( m \geq 1 \). If \( \kappa = 0 \), then the tensors \( R \cdot C \) and \( Q(g, g \wedge \text{Ricc}) \) are linearly dependent if and only if (i) either \( M \) is totally umbilical in \( \mathbb{R}^{n+m} \) (and hence \( M \) is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \( \{ E_i, \xi_\alpha \} \) on \( M \) in \( \mathbb{R}^{n+m} \), the shape operators are given by

\[
A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix},
\]

(2.9)

where \( \mu \) is a real function on \( M \) such that \( \mu \neq 0 \). In this second case, \( M \) is minimal in \( \mathbb{R}^{n+m} \). \( \square \)
Theorem 2.15. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. Then $C \cdot R$ and $Q(g, g \wedge \text{Ricc})$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.16. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(g, g \wedge \text{Ricc})$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.17. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. If $\bar{k} > 0$, then the tensors $R \cdot C$ and $Q(\text{Ricc}, R)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.18. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. If $\bar{k} \leq 0$, then the tensors $R \cdot C$ and $Q(\text{Ricc}, R)$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_n\}$ on $M$ in $\mathbb{R}^{n+m}(\bar{k})$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

$$A_4 = \ldots = A_m = 0,$$

where $c$ and $\mu$ are real functions on $M$ such that $\mu \neq 0$ and $\mu > 0$. Moreover, $M$ is Weyl-semi-symmetric ; and, in the second case, $M$ is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\bar{k})$. □

Corollary 2.5. Let $M$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. If $\bar{k} = 0$, then the tensors $R \cdot C$ and $Q(\text{Ricc}, R)$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}^{n+m}$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_n\}$ on $M$ in $\mathbb{R}^{n+m}$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

$$A_4 = \ldots = A_m = 0,$$

where $\mu$ is a real function on $M$ such that $\mu \neq 0$. In this second case, $M$ is minimal in $\mathbb{R}^{n+m}$. □

Theorem 2.19. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(\text{Ricc}, R)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.20. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(\text{Ricc}, R)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.21. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. If $\bar{k} > 0$, then the tensors $R \cdot C$ and $Q(\text{Ricc}, C)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\bar{k})$ (and hence $M$ is conformally flat). □

Theorem 2.22. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\bar{k})$, $n \geq 4$ and $m \geq 1$. If $\bar{k} \leq 0$, then the tensors $R \cdot C$ and $Q(\text{Ricc}, C)$ are linearly dependent if and only if (i) either $M$ is totally umbilical
umbilical in $\mathbb{R}^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \(\{E_i, \xi_\alpha\}\) on $M$ in $\mathbb{R}^{n+m}(\tilde{k})$, the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
\]

(2.12)

\[A_4 = \ldots = A_m = 0,\]

where $c$ and $\mu$ are real functions on $M$ such that $\mu \neq 0$ and $c^2 = -\tilde{k}$. Moreover, $M$ is Weyl-semi-symmetric in $\mathbb{R}^{n+m}(\tilde{k})$; and, in the second case, $M$ is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\tilde{k})$. □

**Corollary 2.6.** Let $M$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 1$. If $\tilde{k} = 0$, then the tensors $R \cdot C$ and $Q(R\text{ricc}, C)$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}^{n+m}$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \(\{E_i, \xi_\alpha\}\) on $M$ in $\mathbb{R}^{n+m}$, the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
\]

(2.13)

\[A_3 = A_4 = \ldots = A_m = 0,\]

where $\mu$ is a real function on $M$ such that $\mu \neq 0$. In this second case, $M$ is minimal in $\mathbb{R}^{n+m}$. □

**Theorem 2.23.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C$ and $Q(R\text{ricc}, C)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat).

**Theorem 2.24.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(R\text{ricc}, C)$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat).

**Theorem 2.25.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C$ and $Q(R\text{ricc}, g \wedge R\text{ricc})$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames \(\{E_i, \xi_\alpha\}\) on $M$ in $\mathbb{R}^{n+m}(\tilde{k})$, the shape operators are given by

\[
A_1 = \begin{pmatrix}
0 & \mu & 0 & \cdots & 0 \\
\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\mu & 0 & 0 & \cdots & 0 \\
0 & -\mu & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c
\end{pmatrix},
\]

(2.14)

\[A_4 = \ldots = A_m = 0,\]

where $c$ and $\mu$ are real functions on $M$ such that $\mu \neq 0$ and $c^2 + \tilde{k} \neq 0$. Moreover,

\[R \cdot C = \frac{\tilde{k} + c^2}{2\mu^2}Q(R\text{ricc}, g \wedge R\text{ricc});\]

and, in the second case, $M$ is a minimal or pseudo-umbilical submanifold in $\mathbb{R}^{n+m}(\tilde{k})$. □

**Theorem 2.26.** Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(\tilde{k})$, $n \geq 4$ and $m \geq 1$. Then the tensors $C \cdot R$ and $Q(R\text{ricc}, g \wedge R\text{ricc})$ are linearly dependent if and only if $M$ is totally umbilical in $\mathbb{R}^{n+m}(\tilde{k})$ (and hence $M$ is conformally flat). □
Theorem 2.27. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(k)$, $n \geq 4$ and $m \geq 1$. Then the tensors $R \cdot C - C \cdot R$ and $Q(Ricc, g \wedge Ricc)$ are linearly dependent if and only if (i) either $M$ is totally umbilical in $\mathbb{R}^{n+m}(k)$ (and hence $M$ is conformally flat), (ii) or with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_k\}$ on $M$ in $\mathbb{R}^{n+m}(k)$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2\mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

(2.15)

where $c$ and $\mu$ are real functions on $M$ such that $\mu \neq 0$; in this second case,

$$H^2 + k = \frac{2}{n-1} \mu^2 \quad \text{and} \quad R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)} Q(Ricc, g \wedge Ricc). \square$$

Corollary 2.7. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(k)$, $n \geq 4$ and $m \geq 1$. Suppose $k \leq 0$. The following conditions are both equivalent.

(I) With respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_k\}$ on $M$ in $\mathbb{R}^{n+m}(k)$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

(2.16)

where $c$ and $\mu$ are real functions on $M$ such that $\mu \neq 0$ and $c^2 = -k$.

(II) $M$ is Weyl-pseudo-symmetric, i.e., $R \cdot C$ and $Q(g, R)$ are linearly dependent.

(III) $R \cdot C$ and $Q(g, g \wedge Ricc)$ are linearly dependent.

(IV) $R \cdot C$ and $Q(Ricc, R)$ are linearly dependent.

(V) $R \cdot C$ and $Q(Ricc, C)$ are linearly dependent.

(VI) $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent.

In these cases (I) to (V), $M$ is Weyl-semi-symmetric in $\mathbb{R}^{n+m}(k)$. In the case (VI),

$$R \cdot C - C \cdot R = -\frac{2(n-3)\mu^2}{(n-1)(n-2)} Q(g, R). \square$$

Corollary 2.8. Let $M = M^n$ be a Wintgen ideal submanifold of codimension $m$ in a real space form $\mathbb{R}^{n+m}(k)$, $n \geq 4$ and $m \geq 1$. Suppose $k = 0$. The following conditions are both equivalent.

(I) $M$ is minimal and with respect to some suitable adapted Choi-Lu frames $\{E_i, \xi_k\}$ on $M$ in $\mathbb{R}^{n+m}(k)$, the shape operators are given by

$$A_1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

(2.17)

$$A_3 = A_4 = \cdots = A_m = 0,$$

where $\mu$ is a real function on $M$ such that $\mu \neq 0$.  

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(II) \( M \) is Weyl-pseudo-symmetric, i.e. \( R \cdot C \) and \( Q(g, R) \) are linearly dependent.

(III) \( R \cdot C \) and \( Q(g, g \wedge \text{Ricc}) \) are linearly dependent.

(IV) \( R \cdot C \) and \( Q(\text{Ricc}, R) \) are linearly dependent.

(V) \( R \cdot C \) and \( Q(\text{Ricc}, C) \) are linearly dependent.

(VI) \( R \cdot C - C \cdot R \) and \( Q(g, R) \) are linearly dependent. □

**Corollary 2.9.** Let \( M = M^n \) be a Wintgen ideal submanifold of codimension \( m \) in a real space form \( \mathbb{R}^{n+m}(\tilde{k}) \), \( n \geq 4 \) and \( m \geq 1 \). Suppose \( \tilde{k} > 0 \). Then the following conditions are both equivalent.

(I) \( R \cdot C \) and \( Q(g, R) \) are linearly dependent.

(II) \( R \cdot C \) and \( Q(g, g \wedge \text{Ricc}) \) are linearly dependent.

(III) \( R \cdot C \) and \( Q(\text{Ricc}, R) \) are linearly dependent.

(IV) \( R \cdot C \) and \( Q(\text{Ricc}, C) \) are linearly dependent.

(V) \( M \) is totally umbilical in \( \mathbb{R}^{n+m}(\tilde{k}) \) (and hence \( M \) is conformally flat). □

2.2. Proofs of main results

2.2.1. Introduction We consider mainly the Wintgen ideal submanifold \( M = M^n \) of dimension \( n > 3 \) and of codimension \( m \geq 2 \) in a real space form \( \mathbb{R}^{n+m}(\tilde{k}) \). We note that the second fundamental tensor \( H \) satisfies:

\[
H^2 = a^2 + b^2 + c^2.
\]

As in Theorem A, the shape operator \( A \) satisfies the following relations, for all \( \beta \geq 4 \) and \( 3 \leq i < j \leq n \):

\[
\begin{align*}
A_1(E_1) \wedge A_1(E_2) &= (a^2 - \mu^2) E_1 \wedge g E_2, \\
A_2(E_1) \wedge g A_2(E_2) &= (b^2 - \mu^2) E_1 \wedge g E_2, \\
A_3(E_1) \wedge g A_3(E_2) &= c^2 E_1 \wedge g E_2, \\
A_\beta(E_1) \wedge g A_\beta(E_2) &= 0,
\end{align*}
\]

(2.18)

\[
\begin{align*}
A_1(E_1) \wedge g A_1(E_i) &= a^2 E_1 \wedge g E_i + a \mu E_2 \wedge g E_i, \\
A_2(E_1) \wedge g A_2(E_i) &= (b^2 + b \mu) E_1 \wedge g E_i, \\
A_3(E_1) \wedge g A_3(E_i) &= c^2 E_1 \wedge g E_i, \\
A_\beta(E_1) \wedge g A_\beta(E_i) &= 0,
\end{align*}
\]

(2.19)

\[
\begin{align*}
A_1(E_2) \wedge g A_1(E_i) &= a \mu E_1 \wedge g E_i + a^2 E_2 \wedge g E_i, \\
A_2(E_2) \wedge g A_2(E_i) &= (b^2 - b \mu) E_2 \wedge g E_i, \\
A_3(E_2) \wedge g A_3(E_i) &= c^2 E_2 \wedge g E_i, \\
A_\beta(E_2) \wedge g A_\beta(E_i) &= 0,
\end{align*}
\]

(2.20)

\[
\begin{align*}
A_1(E_i) \wedge g A_1(E_j) &= a^2 E_i \wedge g E_j, \\
A_2(E_i) \wedge g A_2(E_j) &= b^2 E_i \wedge g E_j, \\
A_3(E_i) \wedge g A_3(E_j) &= c^2 E_i \wedge g E_j, \\
A_\beta(E_i) \wedge g A_\beta(E_j) &= 0,
\end{align*}
\]

(2.21)
Due to the relations (2.18) to (2.21), the Riemannian-Christoffel curvature operator $\mathcal{R}$ satisfies the following equalities:

\[
\begin{aligned}
\mathcal{R}(E_1, E_2) &= \left( H^2 + \tilde{k} - 2\mu^2 \right) E_1 \wedge g E_2, \\
\mathcal{R}(E_1, E_i) &= \left( H^2 + \tilde{k} + b\mu \right) E_1 \wedge g E_i + a\mu E_2 \wedge g E_i, \\
\mathcal{R}(E_2, E_i) &= a\mu E_1 \wedge g E_i + \left( H^2 + \tilde{k} - b\mu \right) E_2 \wedge g E_i, \\
\mathcal{R}(E_i, E_j) &= (H^2 + \tilde{k})E_i \wedge g E_j.
\end{aligned}
\] (2.22)

The local components $R_{uvwt} = R(E_u, E_v, E_w, E_t)$ of the Riemannian-Christoffel curvature $(0, 4)$-tensor $\mathcal{R}$ are given by:

\[
\begin{aligned}
R_{1221} &= H^2 + \tilde{k} - 2\mu^2, \\
R_{11i1} &= H^2 + \tilde{k} + b\mu \quad \text{for } i \geq 3, \\
R_{2i12} &= H^2 + \tilde{k} - b\mu \quad \text{for } i \geq 3, \\
R_{ijji} &= H^2 + \tilde{k} \quad \text{for } 3 \leq i < j \leq n, \\
R_{11i2} &= a\mu \quad \text{for } 3 \leq i \leq n,
\end{aligned}
\] (2.23)

the other values of $R_{uvwt}$ being null.

We denote Ricc the Ricci tensor. We set $S_{uv} = \text{Ricc}(E_u, E_v)$. Then

\[
\begin{aligned}
S_{11} &= (n - 1) \left( H^2 + \tilde{k} \right) - 2\mu^2 + (n - 2)b\mu, \\
S_{12} &= (n - 2)a\mu, \\
S_{22} &= (n - 1) \left( H^2 + \tilde{k} \right) - 2\mu^2 - (n - 2)b\mu \quad \text{for } i \geq 3, \\
S_{ii} &= (n - 1) \left( H^2 + \tilde{k} \right) \quad \text{for } 3 \leq i \leq n,
\end{aligned}
\] (2.24)

the other values of $S_{uv}$ being null. Next, computing the Ricci operator $\mathcal{S}$ (associated to the Ricci tensor Ricc) by using the above equalities, we obtain:

\[
\begin{aligned}
\mathcal{S}(E_1) &= \text{Ricc}(E_1, E_1) E_1 + \text{Ricc}(E_1, E_2) E_2 \\
&= \left[ (n - 1) \left( H^2 + \tilde{k} \right) - 2\mu^2 + (n - 2)b\mu \right] E_1 + [(n - 2)a\mu] E_2, \\
\mathcal{S}(E_2) &= \text{Ricc}(E_2, E_1) E_1 + \text{Ricc}(E_2, E_2) E_2 \\
&= [(n - 2)a\mu] E_1 + \left[ (n - 1) \left( H^2 + \tilde{k} \right) - 2\mu^2 - (n - 2)b\mu \right] E_2, \\
\mathcal{S}(E_i) &= \text{Ricc}(E_i, E_1) E_1 + \text{Ricc}(E_i, E_2) E_2 \\
&= \left[ (n - 1) \left( H^2 + \tilde{k} \right) \right] E_i \quad \text{for } i \geq 3.
\end{aligned}
\] (2.25)

The scalar curvature $\tau$ is given by

\[
\tau = \sum_{i=1}^{n} \text{Ricc}(E_i, E_i) = n(n - 1) \left( H^2 + \tilde{k} \right) - 4\mu^2,
\] (2.26)

so that the normalized scalar of $M$ is

\[
\rho = \frac{\tau}{n(n - 1)} = \left( H^2 + \tilde{k} \right) - \frac{4\mu^2}{n(n - 1)}.
\] (2.27)

We recall that for any tangent vector fields $X, Y, Z, W$ we have

\[
(g \wedge \text{Ricc})(X, Y; Z, W) = g(X, W)\text{Ricc}(Y, Z) + \text{Ricc}(X, W)g(Y, Z) \\
- g(X, Z)\text{Ricc}(Y, W) - \text{Ricc}(X, Z)g(Y, W) = g((X \wedge g S(Y) + S(X) \wedge g Y)(Z), W).
\]
If we set \((g \wedge \text{Ricc})_{uvst} = (g \wedge \text{Ricc}) (E_u, E_v, E_s, E_t)\), for any \(u \geq 1, v \geq 1, s \geq 1\) and \(t \geq 1\), then

\[
\begin{align*}
(g \wedge \text{Ricc})_{1221} &= 2(n-1) \left( H^2 + \tilde{k} \right) - 4\mu^2, \\
(g \wedge \text{Ricc})_{1i1} &= 2(n-1) \left( H^2 + \tilde{k} \right) - 2\mu^2 + (n-2)b\mu, \\
(g \wedge \text{Ricc})_{2i2} &= 2(n-1)(\tilde{k} + H^2) - 2\mu^2 - (n-2)b\mu \quad \text{for} \quad i \geq 3, \\
(g \wedge \text{Ricc})_{ijji} &= 2(n-1) \left( H^2 + \tilde{k} \right) \quad \text{for} \quad 3 \leq i \leq n,
\end{align*}
\]

the other values of \((g \wedge \text{Ricc})_{uvst}\) being null.

Let \(C\) be the Weyl conformal curvature tensor. Setting \(C_{uvst} = C (E_u, E_v, E_w, E_t)\), we get

\[
\begin{align*}
C_{1221} &= -\frac{2(n-3)\mu^2}{n-1}, \\
C_{1i1} &= -\frac{2(n-3)\mu^2}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3, \\
C_{2i2} &= -\frac{2(n-3)\mu^2}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3, \\
C_{ijji} &= -4\mu^2, \\
\end{align*}
\]

the other values of \(C_{uvst}\) being null.

2.2.2. Proofs of theorems 2.1, 2.2, 2.3, 2.4 and corollary 2.1 Now we compute the local components of the tensors \(R \cdot C, C \cdot R\) and \(R \cdot C - C \cdot R\) of a Wintgen ideal submanifold. Let \(Z, W\) be tangent vector fields.

Firstly, we compute the local components of the tensor \(R \cdot C\). For any index \(i \in \{3, \ldots, n\}\),

\[
\begin{align*}
(R \cdot C) (E_1, E_2, Z, W; E_1, E_i) &= \frac{2(n-3)\mu^2a}{n-2} \langle (E_1 \wedge_g E_i) (Z), W \rangle \\
&\quad - \frac{2(n-3)\mu^2}{n-2} \left( H^2 + \tilde{k} + \mu b \right) \langle (E_2 \wedge_g E_i) (Z), W \rangle, \\
(R \cdot C) (E_1, E_2, Z, W; E_2, E_i) &= \frac{2(n-3)\mu^2}{n-2} \left( H^2 + \tilde{k} - \mu b \right) \langle (E_1 \wedge_g E_i) (Z), W \rangle \\
&\quad - \frac{2(n-3)\mu^2a}{n-2} \langle (E_2 \wedge_g E_i) (Z), W \rangle.
\end{align*}
\]

For any indexes \(i, j \in \{3, \ldots, n\}\) such that \(i \neq j\),

\[
\begin{align*}
(R \cdot C) (E_1, E_i, Z, W; E_1, E_2) &= 0, \\
(R \cdot C) (E_1, E_i, Z, W; E_1, E_j) &= \frac{2\mu^2}{n-2} \left( H^2 + \tilde{k} + \mu b \right) \langle (E_i \wedge_g E_j) (Z), W \rangle, \\
(R \cdot C) (E_1, E_i, Z, W; E_2, E_i) &= \frac{2(n-3)\mu^2}{n-2} \left( H^2 + \tilde{k} - \mu b \right) \langle (E_1 \wedge_g E_2) (Z), W \rangle, \\
(R \cdot C) (E_1, E_i, Z, W; E_i, E_j) &= 0.
\end{align*}
\]

For any indexes \(i, j \in \{3, \ldots, n\}\) such that \(i \neq j\),

\[
\begin{align*}
(R \cdot C) (E_2, E_i, Z, W; E_2, E_1) &= 0, \\
(R \cdot C) (E_2, E_i, Z, W; E_2, E_j) &= \frac{2\mu^2}{n-2} \left( H^2 + \tilde{k} - \mu b \right) \langle (E_i \wedge_g E_j) (Z), W \rangle, \\
(R \cdot C) (E_2, E_i, Z, W; E_1, E_j) &= \frac{2(n-3)\mu^2}{n-2} \left( H^2 + \tilde{k} + \mu b \right) \langle (E_1 \wedge_g E_2) (Z), W \rangle, \\
(R \cdot C) (E_2, E_i, Z, W; E_i, E_j) &= 0.
\end{align*}
\]
For any indexes $i, j, k \in \{3, \ldots, n\}$ such that $i \neq j, i \neq k$ and $j \neq k$,

\[
\begin{aligned}
(R \cdot C)(E_i, E_j, Z, W; E_i, E_1) &= \frac{2\mu^2}{n-2} \left( H^2 + \tilde{k} + \mu b \right) ((E_1 \wedge_g E_j)(Z), W) \\
&\quad + \frac{2\mu^2 a}{n-2} ((E_2 \wedge_g E_j)(Z), W), \\
(R \cdot C)(E_i, E_j, Z, W; E_i, E_2) &= \frac{2\mu^3 a}{n-2} ((E_1 \wedge_g E_j)(Z), W) \\
&\quad + \frac{2\mu^2}{n-2} \left( H^2 + \tilde{k} - \mu b \right) ((E_2 \wedge_g E_j)(Z), W), \\
(R \cdot C)(E_i, E_j, Z, W; E_i, E_k) &= 0, \\
(R \cdot C)(E_i, E_j, Z, W; E_2, E_k) &= 0.
\end{aligned}
\]  

(2.33)

Secondly, we compute the local components of the tensor $C \cdot R$. For any index $i \in \{3, \ldots, n\}$,

\[
\begin{aligned}
(C \cdot R)(E_1, E_2, Z, W; E_1, E_i) &= \frac{2(n-3)\mu^3 a}{(n-1)(n-2)} ((E_1 \wedge_g E_i)(Z), W) \\
&\quad + \frac{2(n-3)\mu^2}{(n-1)(n-2)} (2\mu^2 - \mu b) ((E_2 \wedge_g E_i)(Z), W), \\
(C \cdot R)(E_1, E_2, Z, W; E_2, E_i) &= \frac{2(n-3)\mu^3 a}{(n-1)(n-2)} (2\mu^2 + \mu b) ((E_1 \wedge_g E_i)(Z), W) \\
&\quad + \frac{2(n-3)\mu^2}{(n-1)(n-2)} ((E_2 \wedge_g E_i)(Z), W).
\end{aligned}
\]  

(2.34)

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

\[
\begin{aligned}
(C \cdot R)(E_1, E_i, Z, W; E_1, E_2) &= -\frac{4(n-3)\mu^3 a}{n-1} ((E_1 \wedge_g E_i)(Z), W) \\
&\quad + \frac{4(n-3)\mu^3 b}{n-1} ((E_2 \wedge_g E_i)(Z), W), \\
(C \cdot R)(E_1, E_i, Z, W; E_1, E_j) &= \frac{2(n-3)\mu^3 b}{(n-1)(n-2)} ((E_1 \wedge_g E_j)(Z), W), \\
(C \cdot R)(E_1, E_i, Z, W; E_2, E_2) &= \frac{2(n-3)\mu^2}{(n-1)(n-2)} (2\mu^2 + \mu b) ((E_1 \wedge_g E_2)(Z), W), \\
(C \cdot R)(E_1, E_i, Z, W; E_1, E_j) &= 0.
\end{aligned}
\]  

(2.35)

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,

\[
\begin{aligned}
(C \cdot R)(E_2, E_i, Z, W; E_1, E_2) &= -\frac{4(n-3)\mu^3 b}{n-1} ((E_1 \wedge_g E_i)(Z), W) \\
&\quad - \frac{4(n-3)\mu^3 a}{n-1} ((E_2 \wedge_g E_i)(Z), W), \\
(C \cdot R)(E_2, E_i, Z, W; E_2, E_2) &= \frac{2(n-3)\mu^3 b}{(n-1)(n-2)} ((E_1 \wedge_g E_j)(Z), W), \\
(C \cdot R)(E_2, E_i, Z, W; E_2, E_1) &= \frac{2(n-3)\mu^2}{(n-1)(n-2)} (2\mu^2 - \mu b) ((E_1 \wedge_g E_2)(Z), W), \\
(C \cdot R)(E_1, E_i, Z, W; E_1, E_j) &= 0.
\end{aligned}
\]  

(2.36)
For any indexes \( i, j, k \in \{3, \ldots, n\} \) such that \( i \neq j, i \neq k \) and \( j \neq k \),

\[
\begin{cases}
(C \cdot R) (E_i, E_j, Z, W; E_i, E_1) = \frac{2(n-3)\mu^3 b}{(n-1)(n-2)} \langle (E_1 \wedge_g E_i) (Z), W \rangle \\
+ \frac{2(n-3)\mu^3 a}{(n-1)(n-2)} \langle (E_2 \wedge_g E_j) (Z), W \rangle,
\end{cases}
\]

\[
(C \cdot R) (E_i, E_j, Z, W; E_i, E_2) = \frac{2(n-3)\mu^3 a}{(n-1)(n-2)} \langle (E_1 \wedge_g E_i) (Z), W \rangle
\]

\[
(C \cdot R) (E_i, E_j, Z, W; E_k, E_k) = 0.
\] (2.37)

Thirdly, we compute the local components of the tensor \( R \cdot C - C \cdot R \) on a Wintgen ideal submanifold. For any index \( i \in \{3, \ldots, n\} \),

\[
\begin{cases}
(R \cdot C - C \cdot R) (E_1, E_2, Z, W; E_1, E_1) = \frac{2(n-3)\mu^3 a}{n-1} \langle (E_1 \wedge_g E_i) (Z), W \rangle \\
- \frac{2(n-3)\mu^2}{(n-1)(n-2)} \left( (n-1) \left( H^2 + \tilde{\kappa} \right) + (n-2)\mu b + 2\mu^2 \right) \langle (E_2 \wedge_g E_i) (Z), W \rangle,
\end{cases}
\]

\[
(R \cdot C - C \cdot R) (E_1, E_2, Z, W; E_2, E_1) = -\frac{2(n-3)\mu^3 a}{(n-1)(n-2)} \langle (E_2 \wedge_g E_i) (Z), W \rangle
\]

\[
+ \frac{2(n-3)\mu^2}{(n-1)(n-2)} \left( (n-1) \left( H^2 + \tilde{\kappa} \right) - n\mu b - 2\mu^2 \right) \langle (E_1 \wedge_g E_i) (Z), W \rangle.
\] (2.38)

For any different indexes \( i, j \in \{3, \ldots, n\} \),

\[
\begin{cases}
(R \cdot C - C \cdot R) (E_1, E_i, Z, W; E_1, E_2) = \frac{4(n-3)\mu^3 a}{n-1} \langle (E_1 \wedge_g E_i) (Z), W \rangle - \frac{4(n-3)\mu^3 b}{n-1} \langle (E_2 \wedge_g E_i) (Z), W \rangle,
\end{cases}
\]

\[
(R \cdot C - C \cdot R) (E_1, E_i, Z, W; E_1, E_j) = \frac{4(n-3)\mu^3 a}{n-1} \langle (E_2 \wedge_g E_i) (Z), W \rangle.
\] (2.39)

For any different indexes \( i, j \in \{3, \ldots, n\} \),

\[
\begin{cases}
(R \cdot C - C \cdot R) (E_2, E_i, Z, W; E_2, E_1) = \frac{4(n-3)\mu^3 b}{n-1} \langle (E_1 \wedge_g E_i) (Z), W \rangle + \frac{4(n-3)\mu^3 a}{n-1} \langle (E_2 \wedge_g E_i) (Z), W \rangle,
\end{cases}
\]

\[
(R \cdot C - C \cdot R) (E_2, E_i, Z, W; E_2, E_j) = \frac{2\mu^2}{(n-1)(n-2)} \left( (n-1) \left( H^2 + \tilde{\kappa} \right) - 2\mu b \right) \langle (E_1 \wedge_g E_j) (Z), W \rangle,
\]

\[
(R \cdot C - C \cdot R) (E_2, E_i, Z, W; E_i, E_1) = \frac{2(n-3)\mu^2}{(n-1)(n-2)} \left( (n-1)(H^2 + \tilde{\kappa}) + n\mu b \right) \langle (E_1 \wedge_g E_2) (Z), W \rangle
\]

\[
(R \cdot C - C \cdot R) (E_2, E_i, Z, W; E_i, E_j) = 0.
\] (2.40)
For any different indexes \(i, j, k \in \{3, \ldots, n\}\),

\[
\begin{align*}
(R \cdot C - C \cdot R) (E_i, E_j, Z, W ; E_i, E_1) &= \frac{2\mu^2}{(n-1)(n-2)} \left( (n-1)(H^2 + \tilde{k}) + 2\mu b \right) \langle (E_1 \wedge \eta E_i) (Z), W \rangle \\
&\quad + \frac{4\mu^2}{(n-1)(n-2)} \langle (E_2 \wedge \eta E_i) (Z), W \rangle, \\
(R \cdot C - C \cdot R) (E_i, E_j, Z, W ; E_i, E_2) &= \frac{4\mu^2}{(n-1)(n-2)} \langle (E_1 \wedge \eta E_i) (Z), W \rangle \\
&\quad + \frac{2\mu^2}{(n-1)(n-2)} \left( (n-1)(H^2 + \tilde{k}) - 2\mu b \right) \langle (E_2 \wedge \eta E_i) (Z), W \rangle, \\
(R \cdot C - C \cdot R) (E_i, E_j, Z, W ; E_i, E_k) &= 0, \\
(R \cdot C - C \cdot R) (E_i, E_j, Z, W ; E_j, E_k) &= 0.
\end{align*}
\]

In terms of the equalities (2.30), (2.31), (2.32), (2.34), (2.35), (2.36), (2.37), and (2.38), (2.39), (2.40), (2.41), we prove theorems 2.1, 2.2, 2.3 and corollary 2.1 above.

2.2.3. Proofs of theorems 2.5, 2.6, 2.7, 2.8, 2.9 and corollaries 2.2, 2.3 We compute the local components of the tensor \(Q(g, R)\) of a Wintgen ideal submanifold. Let \(Z, W\) be tangent vector fields of \(M\). For any index \(i \in \{3, \ldots, n\}\),

\[
\begin{align*}
Q (g, R) (E_1, E_2, Z, W ; E_1, E_i) &= (b\mu - 2\mu^2) \langle (E_1 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, R) (E_1, E_2, Z, W ; E_2, E_i) &= (b\mu + 2\mu^2) \langle (E_1 \wedge \eta E_i) (Z), W \rangle - a\mu \langle (E_2 \wedge \eta E_i) (Z), W \rangle.
\end{align*}
\]

For any indexes \(i, j \in \{3, \ldots, n\}\) such that \(i \neq j\),

\[
\begin{align*}
Q (g, R) (E_1, E_1, Z, W ; E_1, E_j) &= a\mu \langle (E_1 \wedge \eta E_i) (Z), W \rangle - 2b\mu \langle (E_2 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, R) (E_1, E_1, Z, W ; E_1, E_j) &= b\mu \langle (E_1 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, R) (E_1, E_1, Z, W ; E_2, E_i) &= (b\mu + 2\mu^2) \langle (E_1 \wedge \eta E_2) (Z), W \rangle, \\
Q (g, R) (E_1, E_1, Z, W ; E_j, E_j) &= 0.
\end{align*}
\]

For any indexes \(i, j \in \{3, \ldots, n\}\) such that \(i \neq j\),

\[
\begin{align*}
Q (g, R) (E_2, E_1, Z, W ; E_1, E_2) &= 2b\mu \langle (E_1 \wedge \eta E_i) (Z), W \rangle - a\mu \langle (E_2 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, R) (E_2, E_1, Z, W ; E_2, E_1) &= -b\mu \langle (E_1 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, R) (E_2, E_1, Z, W ; E_2, E_1) &= (b\mu + 2\mu^2) \langle (E_1 \wedge \eta E_2) (Z), W \rangle, \\
Q (g, R) (E_2, E_1, Z, W ; E_j, E_j) &= 0.
\end{align*}
\]

For any indexes \(i, j, k \in \{3, \ldots, n\}\) such that \(i \neq j, i \neq k\) and \(j \neq k\),

\[
\begin{align*}
Q (g, R) (E_1, E_j, Z, W ; E_1, E_1) &= b\mu \langle (E_1 \wedge \eta E_j) (Z), W \rangle, \\
Q (g, R) (E_1, E_j, Z, W ; E_1, E_2) &= a\mu \langle (E_1 \wedge \eta E_j) (Z), W \rangle - b\mu \langle (E_2 \wedge \eta E_j) (Z), W \rangle, \\
Q (g, R) (E_1, E_j, Z, W ; E_j, E_k) &= 0, \\
Q (g, R) (E_1, E_j, Z, W ; E_j, E_k) &= 0.
\end{align*}
\]

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.42), (2.43), (2.44), (2.45), we prove theorems 2.5, 2.6, 2.7, 2.8, 2.9 and corollaries 2.2, 2.3 above.

2.2.4. Proofs of theorems 2.10, 2.11, 2.12 Now we compute the local components of the tensor \(Q(g, C)\) for the considered Wintgen ideal submanifold. Let \(X, Y, Z, W\) be tangent vector fields. For any index \(i \in \{3, \ldots, n\}\),

\[
\begin{align*}
Q (g, C) (E_1, E_2, Z, W ; E_1, E_i) &= -\frac{2(n-3)\mu^2}{n-2} \langle (E_2 \wedge \eta E_i) (Z), W \rangle, \\
Q (g, C) (E_1, E_2, Z, W ; E_2, E_i) &= \frac{2(n-3)\mu^2}{n-2} \langle (E_1 \wedge \eta E_i) (Z), W \rangle.
\end{align*}
\]
For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,
\[
\begin{align*}
Q(g, C)(E_1, E_i, Z; E_2, E_j) &= 0, \\
Q(g, C)(E_1, E_i, Z; E_j, E_3) &= \frac{2\mu^2}{n-2} \langle (E_1 \wedge g E_j)(Z), W \rangle, \\
Q(g, C)(E_1, E_i, Z; E_j, E_3) &= \frac{(n-3)\mu^2}{n-2} \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, C)(E_1, E_i, Z; E_j, E_3) &= 0.
\end{align*}
\] (2.47)

For any indexes $i, j \in \{3, \ldots, n\}$ such that $i \neq j$,
\[
\begin{align*}
Q(g, C)(E_2, E_i, Z; E_2, E_1) &= 0, \\
Q(g, C)(E_2, E_i, Z; E_j, E_3) &= \frac{2\mu^2}{n-2} \langle (E_2 \wedge g E_j)(Z), W \rangle, \\
Q(g, C)(E_2, E_i, Z; E_j, E_3) &= \frac{(n-3)\mu^2}{n-2} \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, C)(E_2, E_i, Z; E_j, E_3) &= 0.
\end{align*}
\] (2.48)

For any indexes $i, j, k \in \{3, \ldots, n\}$ such that $i \neq j$, $i \neq k$ and $j \neq k$,
\[
\begin{align*}
Q(g, C)(E_1, E_j, Z; E_3, E_1) &= \frac{2\mu^2}{n-2} \langle (E_1 \wedge g E_j)(Z), W \rangle, \\
Q(g, C)(E_1, E_j, Z; E_3, E_2) &= \frac{2\mu^2}{n-2} \langle (E_2 \wedge g E_j)(Z), W \rangle, \\
Q(g, C)(X, Y, Z; E_j, E_k) &= 0, \\
Q(g, C)(X, Y, Z; E_j, E_k) &= 0.
\end{align*}
\] (2.49)

In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.46), (2.47), (2.48), (2.49), we prove theorems 2.10, 2.11, 2.12.

2.2.5. Proofs of theorems 2.13, 2.14, 2.15, 2.16 and corollary 2.4 We compute the local components of the tensor $Q(g, g \wedge \text{Ricc})$ of a Wintern ideal submanifold. Let $X, Y, Z, W$ be tangent vector fields of $M$. For any index $i \in \{3, \ldots, n\}$,
\[
\begin{align*}
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= ((n-2)b\mu - 2\mu^2) \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= ((n-2)b\mu + 2\mu^2) \langle (E_1 \wedge g E_2)(Z), W \rangle.
\end{align*}
\] (2.50)

For any different indexes $i, j \in \{3, \ldots, n\}$,
\[
\begin{align*}
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_2) &= -2(n-2)b\mu \langle (E_2 \wedge g E_1)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_2) &= -(n-2)b\mu + 2\mu^2 \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_2) &= ((n-2)b\mu + 2\mu^2) \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_2) &= 0.
\end{align*}
\] (2.51)

For any different indexes $i, j \in \{3, \ldots, n\}$,
\[
\begin{align*}
Q(g, g \wedge \text{Ricc})(E_2, E_1, Z; E_2, E_1) &= 2(n-2)b\mu \langle (E_2 \wedge g E_1)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_2, E_1, Z; E_2, E_1) &= -(n-2)b\mu - 2\mu^2 \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_2, E_1, Z; E_2, E_1) &= -(n-2)b\mu + 2\mu^2 \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_2, E_1, Z; E_2, E_1) &= 0.
\end{align*}
\] (2.52)

For any different indexes $i, j, k \in \{3, \ldots, n\}$,
\[
\begin{align*}
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= ((n-2)b\mu - 2\mu^2) \langle (E_1 \wedge g E_2)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= ((n-2)b\mu - 2\mu^2) \langle (E_2 \wedge g E_1)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= (n-2)b\mu - 2\mu^2 \langle (E_2 \wedge g E_1)(Z), W \rangle, \\
Q(g, g \wedge \text{Ricc})(E_1, E_2, Z; E_1, E_i) &= 0.
\end{align*}
\] (2.53)
In terms of the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.50), (2.51), (2.52), (2.53), we proved theorems 2.13, 2.14, 2.15, 2.16 and corollary 2.4 above.

2.2.6. Proofs of theorems 2.17, 2.18, 2.19, 2.20 and corollary 2.5 We compute the local components of the tensor $Q(Ric, R)$ for a Wintgen ideal submanifold. Let $X, Y, Z, W$ be the tangent vector fields of $M$. For any index $i \in \{3, \ldots, n\}$,

$$
\begin{align*}
Q (Ric, R) (E_1, E_2, Z, W; E_1, E_i) &= a_{\mu} \left( - \left( H^2 + \tilde{k} \right) + 2\mu^2 \right) \langle (E_1 \wedge g E_i) (Z), W \rangle \\
&+ \left( (n-2)(a^2 + b^2)\mu^2 - 2\mu_3^2 - (2n-4) \mu^2 \left( H^2 + \tilde{k} \right) + b_\mu \left( H^2 + \tilde{k} \right) \right) \langle (E_2 \wedge g E_i) (Z), W \rangle,
\end{align*}
$$

(2.54)

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$
\begin{align*}
Q (Ric, R) (E_1, E_i, Z, W; E_1, E_2) &= 2a_{\mu} \left( \left( H^2 + \tilde{k} \right) - 2\mu^2 \right) \langle (E_1 \wedge g E_2) (Z), W \rangle \\
&+ 2b_\mu \left( - \left( H^2 + \tilde{k} \right) + 2\mu^2 \right) \langle (E_2 \wedge g E_1) (Z), W \rangle,
\end{align*}
$$

(2.55)

$$
\begin{align*}
Q (Ric, R) (E_1, E_i, Z, W; E_2, E_i) &= \left( H^2 + \tilde{k} \right) \left( b_{\mu} + 2\mu^2 \right) \langle (E_1 \wedge g E_j) (Z), W \rangle,
\end{align*}
$$

(2.56)

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$
\begin{align*}
Q (Ric, R) (E_2, E_i, Z, W; E_2, E_1) &= 2b_{\mu} \left( H^2 + \tilde{k} \right) - 4\mu_3^2 \langle (E_1 \wedge g E_i) (Z), W \rangle \\
&+ 2a_{\mu} \left( H^2 + \tilde{k} \right) - 4\mu^3 \langle (E_2 \wedge g E_i) (Z), W \rangle.
\end{align*}
$$

(2.57)

$$
\begin{align*}
Q (Ric, R) (E_2, E_i, Z, W; E_2, E_j) &= -b_{\mu} + 2\mu^2 \left( H^2 + \tilde{k} \right) \langle (E_1 \wedge g E_j) (Z), W \rangle,
\end{align*}
$$

(2.56)

For any different indexes $i, j, k \in \{3, \ldots, n\}$,

$$
\begin{align*}
Q (Ric, R) (E_i, E_j, Z, W; E_i, E_1) &= (b_{\mu} + 2\mu^2) \left( H^2 + \tilde{k} \right) \langle (E_1 \wedge g E_j) (Z), W \rangle \\
&+ a_{\mu} \left( H^2 + \tilde{k} \right) \langle (E_2 \wedge g E_1) (Z), W \rangle,
\end{align*}
$$

(2.57)

$$
\begin{align*}
Q (Ric, R) (E_i, E_j, Z, W; E_2, E_1) &= a_{\mu} \left( H^2 + \tilde{k} \right) \langle (E_2 \wedge g E_1) (Z), W \rangle \\
&+ (b_{\mu} + 2\mu^2) \left( H^2 + \tilde{k} \right) \langle (E_2 \wedge g E_1) (Z), W \rangle,
\end{align*}
$$

(2.57)

$$
\begin{align*}
Q (Ric, R) (E_i, E_j, Z, W; E_2, E_k) &= 0,
\end{align*}
$$

From the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.54), (2.55), (2.56), (2.57), we proved theorems 2.17, 2.18, 2.19, 2.20 and corollary 2.5 above.
2.7. Proofs of theorems 2.21, 2.22, 2.23, 2.24 and corollary 2.6 We compute the local components of the tensor $Q(R, C)$ for a Wintgen ideal submanifold. Let $X, Y, Z, W$ be the tangent vector fields of $M$. For any index $i \in \{3, \ldots, n\}$,

$$Q(R, C)(E_1, E_2, Z, W; E_1, E_i) = 2(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle - \frac{2(n - 3)\alpha^3}{n - 1} \left\langle (E_2 \wedge g E_i)(Z), W \right\rangle,$$

$$Q(R, C)(E_2, E_1, Z, W; E_1, E_i) = -2(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle + \frac{2(n - 3)\alpha^3}{n - 1} \left\langle (E_2 \wedge g E_i)(Z), W \right\rangle,$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$Q(R, C)(E_1, E_i, Z, W; E_1, E_j) = 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle - 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_2 \wedge g E_i)(Z), W \right\rangle,$$

$$Q(R, C)(E_1, E_i, Z, W; E_2, E_j) = 2(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle - 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_2 \wedge g E_i)(Z), W \right\rangle,$$

For any different indexes $i, j \in \{3, \ldots, n\}$,

$$Q(R, C)(E_1, E_i, Z, W; E_j, E_1) = 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle - 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_2 \wedge g E_j)(Z), W \right\rangle,$$

$$Q(R, C)(E_1, E_i, Z, W; E_j, E_2) = 2(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_1 \wedge g E_i)(Z), W \right\rangle - 4(n - 3)\alpha^3 \frac{b_{\mu}}{n - 1} \left\langle (E_2 \wedge g E_j)(Z), W \right\rangle,$$

From the equalities (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.58), (2.59), (2.60), (2.61), we proved theorems 2.21, 2.22, 2.23, 2.24 and corollary 2.6 above.
2.2.8. Proofs of theorems 2.25, 2.26, 2.27 and corollaries 2.7, 2.8, 2.9 We compute the local components of the tensor \( Q(\text{Ricc}, g \land \text{Ricc}) \) for a Wintgen ideal submanifold. Let \( X, Y, Z, W \) be the tangent vector fields of \( M \). For any index \( i \in \{3, \ldots, n\}, \)

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_2, Z, W; E_1, E_i \right) = (n-1)\alpha \mu \left[ 2(n-1) \left( H^2 + \bar{k} \right) + (n-2)\beta \mu - 2\mu^2 \right] \langle (E_1 \land g, E_i) (Z), W \rangle + 4\mu^4 + (n-2)\beta \mu^2 + 2(n-1)\beta \mu \left( H^2 + \bar{k} \right) - (n-1)(n-2)\beta \mu \left( H^2 + \bar{k} \right) \rangle \langle (E_2 \land g, E_i) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_2, Z, W; E_2, E_i \right) = \left[ 4\mu^4 - (n-2)^2\beta \mu^2 - 2(n-1)\beta \mu \left( H^2 + \bar{k} \right) - (n-1)(n-2)\beta \mu \left( H^2 + \bar{k} \right) \rangle \langle (E_1 \land g, E_i) (Z), W \rangle ,
\]

For any different indexes \( i, j \in \{3, \ldots, n\}, \)

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_i, Z, W; E_1, E_j \right) = -2(n-2)\alpha \mu \left[ 2(n-1) \left( H^2 + \bar{k} \right) + (n-2)\beta \mu - 2\mu^2 \right] \langle (E_1 \land g, E_i) (Z), W \rangle + 2(n-1)(n-2)\beta \mu \left( H^2 + \bar{k} \right) \langle (E_2 \land g, E_i) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_i, Z, W; E_2, E_j \right) = \langle (E_2 \land g, E_j) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_i, Z, W; E_j, E_j \right) = 0.
\]

For any different indexes \( i, j \in \{3, \ldots, n\}, \)

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_2, E_i, Z, W; E_2, E_j \right) = -2(n-1)(n-2)\beta \mu \left( H^2 + \bar{k} \right) \langle (E_1 \land g, E_i) (Z), W \rangle + 2(n-2)\alpha \mu \left[ 2(n-1) \left( H^2 + \bar{k} \right) - (n-2)\beta \mu - 2\mu^2 \right] \langle (E_2 \land g, E_i) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_2, E_i, Z, W; E_1, E_j \right) = \langle (E_1 \land g, E_j) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_2, E_i, Z, W; E_j, E_j \right) = 0.
\]

For any different indexes \( i, j, k \in \{3, \ldots, n\}, \)

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_j, Z, W; E_1, E_k \right) = \left[ (n-1) \left( H^2 + \bar{k} \right) - (n-2)\beta \mu + 2\mu^2 \right] \langle (E_1 \land g, E_j) (Z), W \rangle - 2(n-1)(n-2)\alpha \mu \left( H^2 + \bar{k} \right) \langle (E_2 \land g, E_j) (Z), W \rangle ,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_j, Z, W; E_i, E_k \right) = 0,
\]

\[
Q(\text{Ricc}, g \land \text{Ricc}) \left( E_1, E_j, Z, W; E_j, E_k \right) = 0.
\]
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Using (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40), (2.41) and (2.62), (2.63), (2.64), (2.65), we can prove theorems 2.25, 2.26, 2.27 and corollaries 2.7, 2.8, 2.9 above.

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