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On Bertrand and Spherical PH-Curves in Euclidean 3 −Space

Burcu Gür SOĞAT[1*]
https://orcid.org/0009-0002-3189-9867

Mehmet GÜMÜŞ[1]
https://orcid.org/0000-0001-7938-2918

[1]ÇOMÜ Lapseki Meslek Yüksekokulu, Muhasebe Bölümü. 17100, Çanakkale.

[*]Sorumlu yazar: burcu_1734@hotmail.com

Özet

In this paper, the first part is related to the introduction. In the introduction, the works on Bertrand and PH-curves are presented. In the second section, basic concepts and theorems in Euclidean 3-space are given. In the third part, Spherical, Spherical PH-curves and Bertrand PH-curves in Euclidean 3-space are studied and related theorems are given.

Anahtar Kelimeler: Küresel Eğriler, Bertrand Eğriler, PH-Eğriler, Öklid Uzayı, Bertrand PH-Eğriler

Öklid-3 Uzayında Bertand ve Küresel PH-eğriler Üzerine

Abstract

Today, identity has become a concept of great interest and importance in our world. It has become an indicator of how an individual defines himself/herself internally as well as externally, in which groups and in what kind of social environment he/she lives and will live. In this way, individuals have started to adapt to their social environment through their identities. Identity is the subject of research in many social sciences ranging from social psychology to sociology, social anthropology to international relations, and the main research in these fields focuses on the characteristics of individuals, groups or larger nations and what distinguishes them from others. At this point, classification, comparison, the construction of the concepts of self and other, and historical traumas are of great importance.

Keywords: Spherical Curves, Bertrand Curves, PH-Curves, Euclidean Space, Bertrand PH-Curves.

Introduction

Curve theory is an important area of differential geometry and Bertrand curves are one of the important curves in this field. Bertrand curves are curves defined and studied by the French mathematician Joseph Bertrand in the 19th century. (Bertrand,1850).

Bertrand reached various results by classifying curves on regular surfaces. He defined a special class of curves called Bertrand curves. Bertrand curves are curves whose principal normal vectors are linear. This property implies that the curve has a certain symmetry and therefore Bertrand curves are an interesting topic in differential geometry and curve theory.

The study of the properties and character of Bertrand curves has various applications in differential geometry, mathematics, physics and mechanics. The properties and mathematical structures of these curves are of interest to mathematicians interested in the theory of curves and differential geometry in general. Bertrand curves have an important place in the study of differential geometry and theory of curves. Their properties and mathematical analysis provide more information about the shapes and behavior of curves. Bertrand curves are also used in the study of topics such as the tangent circles of curves and their relationships with each other.

The simplest example of a Bertrand curve is a circle or a straight line. However, there are also more complex and diverse Bertrand curves. For example, ellipses, parabolas, hyperbolas, Bernoulli curves and Cassini curves are examples of Bertrand curves.

Joseph Bertrand examined the properties of Bertrand curves in more detail and put forward the necessary and sufficient condition to characterize pairs of Bertrand curves. A pair of Bertrand curves consists of two curves that have a certain property (Bertrand, 1850).

In his work, the characterization obtained by Bertrand is as follows: The necessary and sufficient condition for a curve $\boldsymbol{\alpha}$ with curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$ in $\mathbf{3}$ −dimensional Euclidean space to be a pair of Bertrand curves is that the following equation holds with coefficients expressed in real numbers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$:

$$\boldsymbol{\lambda\kappa + \mu\tau = 1} \qquad (1.1)$$

Here $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ denote a linear relationship between the curvature ($\boldsymbol{\kappa}$) and torsion ($\boldsymbol{\tau}$) properties of the curve. That is, if a curve $\boldsymbol{\alpha}$ has $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfying equation (1.1), then the curve $\boldsymbol{\alpha}$ and the curve $\boldsymbol{\alpha}^*$ (the second curve generated by the principal normal vectors) form a pair of Bertrand curves (Hsiung, 1981).

Equation (1.1) helps us to better understand Bertrand curves by describing their properties and relationships. Bertrand's characterization provides an important tool for further study and analysis of the geometry and character of Bertrand curves.

Izumiya and Takeuchi expressed how Bertrand curves can be obtained from spherical curves in Euclidean $\mathbf{3}$ −space. In their work, Izumiya and Takeuchi showed that Bertrand curves can be obtained from spherical curves in $\mathbf{3}$ −dimensional Euclidean space. This approach shows that Bertrand curves can be obtained by transforming spherical curves with a certain rotational motion. The work of Izumiya and Takeuchi is an important resource for mathematicians and differential geometry researchers, especially those interested in the geometry and origins of Bertrand curves. This work helps us to understand the different types of Bertrand curves and to study how they can be transformed into this special class of spherical curves. More details and mathematical expressions of how Bertrand curves can be derived from spherical curves can be found in the original work by Izumiya and Takeuchi. This work is an interesting resource for those interested in a more in-depth study of Bertrand curves and differential geometry (Izumiya and Takeuchi, 2002).

Murat Babaarslan (2009), in his master's thesis, first obtained the Cartan framework and Cartan curvatures in the spaces $\mathbb{R}_1^5, \mathbb{R}_{2,}^4, \mathbb{R}_2^5$. Then he defined null Bertrand curves in these spaces and gave their characteristic properties (Babaarslan, 2009).

In her master thesis, Gül Güner (2011) investigated how different curves can be transformed into Bertrand curves and the properties of Bertrand curves. In the thesis, it is first shown how to obtain cylindrical helices from planar curves and Bertrand curves from spherical curves in Euclidean 3-space. Using this method, the Bertrand curves corresponding to the spherical indicators of a curve are investigated. Also, the planar evolute of a cylindrical helix and the spherical evolute of a spherical curve are investigated. In addition, the hyperbolic evolute of a spherical curve in $\mathbb{E}^3$ space is also studied in this thesis. Gül Güner's thesis is an important contribution to the subject by investigating the relations between Bertrand curves and different curves. The methods and findings presented in the thesis can guide the research on Bertrand curves and the researchers working in the related field (Güner, 2011).

Pythagorean-hodograph (PH) curves were described by Farouki and Sakkalis in 1990 (Farouki and Sakkalis, 1990). These curves are known as curves whose length can be calculated explicitly. Farouki's work was aimed at determining the properties and descriptors of PH-curves. He also investigated their relationship with helix curves and proved that all helix curves are PH-curves, but the converse is not always true (Farouki and Sakkalis, 1992). Characterization studies for two and three dimensional PH-curves were carried out by Farouki using complex numbers and quartrenions (Farouki and Sakkalis, 1994).

In 2000, Moon defined Pythagorean-Hodograph (PH) curves according to the Minkowski metric and obtained the Minkowski Pythagorean-Hodograph (MPH) curves. The Minkowski metric is a metric used in the special theory of relativity in the four-dimensional Minkowski space of space and time. The MPH curves are versions of the PH curves defined in Minkowski space and are defined taking into account geometric properties in this space. Steographic projection is a method of projecting a curve in Minkowski space onto a plane and has been used to represent MPH curves. Moon's work has defined versions of the PH curves that are valid in Minkowski space and used steographic projection to represent these curves (Moon, 2000).

Çağla Ramis (2013) focused on PH-curves and their applications in her master thesis. The thesis focused on the study of PH-curves in both two-dimensional and three-dimensional Euclidean and Minkowski spaces. The properties of these curves are investigated to obtain results and formulas. Furthermore, the thesis emphasizes the close relationship between helix curves and PH-curves. Considering this relationship, a planar PH-curve is generated from a spatial PH-curve. This shows how PH-curves can have different properties in different spaces and how they can be transformed from one space to another. The characterizations obtained for Euclidean space are carried over to Minkowski space and supported with examples. In this way, it is understood how the properties of PH-curves in Euclidean space are valid in Minkowski space and how they can be used. This thesis shows how PH-curves can be used in different spaces and applications and how their properties can be studied. The research can be an important resource for those who want to make progress in the mathematical analysis of PH-curves and their applications (Ramis, 2013).

2. Preliminaries

In this part of the paper, basic concepts and theorems in Euclidean $\mathbf{3}$ −space was introduced. These concepts and theorems serve as a source for the third section of the paper.

Definition 2.1. Let $\boldsymbol{V}$ be a vector space and the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ be a subset of this vector space. If

$$\lambda_1 \boldsymbol{v_1} + \lambda_2 \boldsymbol{v_2} + \dots + \lambda_n \boldsymbol{v_n} = \mathbf{0} \tag{2.1}$$

equation (2.1) is satisfied when all scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ are zero, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ is called linearly independent (Hacısalihoğlu, 2000).

Definition 2.2. Let V be a vector space and the set $\{v_1, v_2, \dots, v_n\}$ be a subset of this vector space. If for all vectors $u \in V$ and the scalars $a_1, a_2, \dots, a_n \in \mathbb{R}$

$$u = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \tag{2.2}$$

is valid then the set $\{v_1, v_2, \dots, v_n\}$ spans the space V (Hacısalihoğlu, 2000).

Definition 2.3. Let V be a vector space and the set $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a subset of this vector space. If the set $\mathcal{B}$ satisfies the following conditions;

1. The set $\{v_1, v_2, \dots, v_n\}$ is linearly independent,
2. The set $\{v_1, v_2, \dots, v_n\}$ spans the space V,

then the set $\mathcal{B}$ is called a base of the space V (Hacısalihoğlu, 2000).

Definition 2.4. Let $\langle\, , \rangle$ be a function on n-dimensional Euclidean space $\mathbb{E}^n$. If we define this function for all vectors $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{E}^n$ as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

then the function $\langle\, , \rangle$ is called inner product (Hacısalihoğlu, 2000).

Definition 2.5. Let V be a real inner product space. The transformation $\|\ \|$ defined as

$$\|\ \| : V \to \mathbb{R}, \ \ \|u\| = \sqrt{\langle u, u \rangle}$$

specifies a norm on V. Specifically, if we take in the form $V = \mathbb{E}^n$ using the standard Euclidean inner product for $u = (u_1, u_2, \dots, u_n) \in \mathbb{E}^n$ then the following equality is given,

$$\|u\| = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}\,. \qquad (2.3)$$

The value $\|u\|$ is called the norm or length of the vector u (Hacısalihoğlu, 2000).

Definition 2.6. Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{E}^n$

$$d : \mathbb{E}^n x \mathbb{E}^n \to \mathbb{R}$$

$$(x, y) \longrightarrow d(x, y) = \|\overrightarrow{xy}\| = \sqrt{\textstyle\sum_{i=1}^{n} (x_i - y_i)^2} \qquad (2.4)$$

The function d is called the distance function in $\mathbb{E}^n$ and the real number $d(x, y)$ is called the distance between the points $x, y \in E^n$ (Hacısalihoğlu, 2000).

Definition 2.7. In 3 −dimensional Euclidean space $\mathbb{E}^3$ the vector product is defined for all vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{E}^3$ as following.

$$u \times v = (u_2 v_3 - v_2 u_3,\ u_3 v_1 - v_3 u_1, u_1 v_2 - v_1 u_2) \qquad (2.5)$$

(Hacısalihoğlu, 2000).

Theorem 2.1. The distance function on $\mathbb{E}^n$ is a metric (Hacısalihoğlu, 2000).

Definition 2.8.

$$d : \mathbb{E}^n x \mathbb{E}^n \to \mathbb{R}$$

$$(x, y) \longrightarrow d(x, y) = \|\overrightarrow{xy}\|$$

The function d defined as above is called Euclidean metric function on $\mathbb{E}^n$ (Hacısalihoğlu, 2000).

Definition 2.9. Let $I \subseteq \mathbb{R}$ be an interval.

$$\alpha \colon I \longrightarrow \mathbb{E}^n$$

$$t \longrightarrow (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$$

If the function defined above α is differentiable α(I) is called a curve in $\mathbb{E}^n$ defined by the coordinate neighborhood (I,α) (Hacısalihoğlu, 2000).

Definition 2.10. Let α be a curve in $\mathbb{E}^n$ and defined by the coordinate functions (I,α) and (J, β). If the followings valid

$$h = \alpha^{-1} \circ \beta : J \longrightarrow \beta$$

$$s \longrightarrow h(s) = t$$

then the differentiable function h defined above is called a parameter change function (Hacısalihoğlu, 2000).

Definition 2.11. Let the curve α in $\mathbb{E}^n$ be parametric,

$$\alpha : \quad I \longrightarrow \mathbb{E}^n$$

$t: \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$

and for curve α ;

$$\alpha'(t) = \frac{d\alpha}{dt}$$

$= \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \dots, \frac{d\alpha_n}{dt}\right).$

Then the vector $(\alpha(t), \alpha'(t)) \in T_{\mathbb{E}^n}(p)$ is called the velocity vector or tangent vector of the curve α at $\alpha'(t)$ corresponding to the parameter value $t \in I$ (Hacısalihoğlu, 2000).

Definition 2.12. Let the α curve at $\mathbb{E}^n$ is defined parametrically,

$$\alpha : I \longrightarrow \mathbb{E}^n$$

$t = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$

The derivative of the curve α,

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \dots, \frac{d\alpha_n}{dt}\right)$$

and the norm is to be

$$\| \alpha'(t) \| : I \longrightarrow \mathbb{R}$$

$$t \longrightarrow \| \alpha'(t) \| = \sqrt{\sum_{i=1}^{n} \left(\frac{d\alpha_i}{dt}\right)^2}$$

scalar velocity function. The real number at the point $t = t_0$

$$\| \alpha'(t_0) \| = \sqrt{\sum_{i=1}^{n} \left(\frac{d\alpha_i}{dt}\right)^2} \qquad (2.6)$$

is called scalar velocity (Hacısalihoğlu, 2000).

Definition 2.13. Let the curve α in $\mathbb{E}^n$ be

$$\alpha : I \longrightarrow \mathbb{E}^n$$

$t : \alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)).$

For all $t_1, t_2 \in I$

$$s = \int_{t_1}^{t_2} \| \alpha'(t) \| dt \qquad (2.7)$$

the real number α is called the arc length of the curve α between the points $\alpha(t_1)$ and $\alpha(t_2)$ (Hacısalihoğlu, 2000).

Definition 2.14. Let α be a curve in $\mathbb{E}^n$. If the norm of the curve α satisfies

$\| \alpha'(s) \| = 1$

then the curve α is called the unit speed curve and the parameter s is called the arclength parameter (Hacısalihoğlu, 2000).

Definition 2.15. If the curve α in $\mathbb{E}^n$ satisfies the following

$$\| \alpha'(t)\| = \left\| \frac{d\alpha}{dt} \right\| \neq 0 \qquad (2.8)$$

then the curve is called a regular curve (Hacısalihoğlu, 2000).

3. PH-Curves in Euclidean 3-Space

Definition 3.1. Let α be a curve in $\mathbb{E}^n$ and $\alpha(t) = \big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\big)$. The hodograph of the polynominal curve α is defined by

$$\|\alpha'(t)\| = \alpha_1'(t)^2 + \alpha_2'(t)^2 + \cdots + \alpha_n'(t)^2 = \sigma(t)^2 \qquad (3.1)$$

and if there's a $\sigma(t)$ polynominal then the curve α is called Pythagorean Hodograph curve(PH-curve) (Farouki ve Sakkalis, 1994).

Definition 3.2. Let $n \in N_0$ and $a_i \in \mathbb{R}$ where $0 \le i \le n$,

$$\alpha(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \,, a_n \neq 0 \qquad (3.2)$$

in the from of t function and n is called the degree of the polynomial (Larson, 2012).

Definition 3.3. Let α be a curve in $\mathbb{E}^n$. If the curve α defined as,

$$\alpha: [a, b] \to E^n$$

$$\alpha(t) = \Big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\Big)$$

where the components $\alpha_i(t)$ for all $1 \le i \le n$ are polynomials then the curve α is called n −dimensional polynomial curve (Larson, 2012).

Definition 3.4. Let α be a curve in $\mathbb{E}^n$ defined as,

$$\alpha: [a, b] \to E^n$$

$$\alpha(t) = \Big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\Big).$$

The degree of the polynomial curve α is the number $deg\alpha(t)$ defined by

$$deg\alpha(t) = max\{deg\big(\alpha_1(t)\big), deg\big(\alpha_2(t)\big), \dots, deg\big(\alpha_n(t)\big)\} \qquad (3.3)$$

(Larson, 2012).

Theorem 3.1. Let $a(t), b(t), c(t)$ be polynomials, The Pythagorean condition

$$a^2(t) + b^2(t) = c^2(t)$$

is satisfied by the polynomials $a(t), b(t), c(t)$ where

$$a(t) = [u^2(t) - v^2(t)]w(t)$$

$$b(t) = 2u(t)v(t)w(t)$$

$$c(t) = [u^2(t) + v^2(t)]w(t)$$

in the form of $u(t), v(t), w(t)$ polynomials (Ramis, 2013).

3.1. Spherical PH-Curves in Euclidean 3-Space

Theorem 3.1.1. There is no spherical PH-curve in $\mathbb{E}^3$.

Proof: Let $\gamma: I \to S^2$ be a spherical-PH curve in $\mathbb{E}^3$. Since γ is a polynomial curve in $\mathbb{E}^3$ for $\gamma_1(t), \gamma_2(t), \gamma_3(t)$ polynomials;

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

can be written and since γ is a PH-curve then

$$(\dot{\gamma}_1)^2 + (\dot{\gamma}_2)^2 + (\dot{\gamma}_3)^2 = \sigma^2 \quad (3.4)$$

the equality (3.4) must be satisfied for an arbitrary polynomial σ. Also γ lies on the sphere it must satisfy

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \quad (3.5)$$

In this case:

$$deg\{\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t)\} = max\{\deg(\gamma_1^2(t), \gamma_2^2(t), \gamma_3^2(t))\} = 0$$

As a result we get that γ is a polynomial where its degree is zero, This means that it is a point. Consequently, there is no spherical PH-curve in Euclidean space.

4. Bertrand Curves in Euclidean 3-Space

Definition 4.1. Let $\alpha : I \to \mathbb{E}^n$ and $\alpha^*: I \to \mathbb{E}^n$ be two differentiable curves, the Frenet frames of these curves are respectively $\{T, N_1, N_2 \ldots, N_{n-1}\}$ and $\{T^*, N_1{}^*, N_2{}^*, \ldots, N_{n-1}{}^*\}$ and $N_1(s)$ the principal normal vector of the curve α, of the curve $N_1{}^*(s)$ the principal normal vector of the curve α^*. If the principal normal vectors $N_1(s)$ and $N_1{}^*(s)$ are linearly dependent then the (α, α^*) is called Bertrand curves pair, α curve is also called a Bertrand curve (Hacısalihoğlu, 2000).

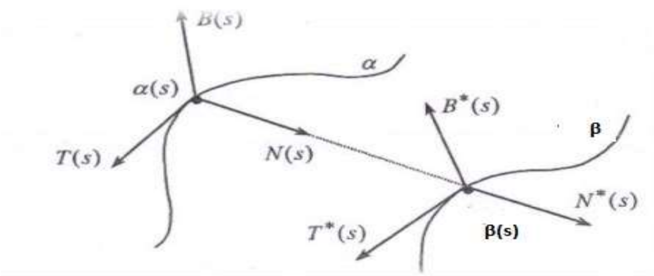

Figure 1. Bertrand Curve Pair

Definition 4.2. Let $\gamma: I \to S^2$ be a unit spherical curve with arc parameter σ. The unit tangent vector of γ at σ is given by $T(\sigma) = \dot{\gamma}(\sigma)$ where $\dot{\gamma} = \frac{d\gamma}{d\sigma}$. Together with the vector $S(\sigma)$, defined as $S(\sigma) = \gamma(\sigma) \times T(\sigma)$ we obtain an orthonormal frame $\{\gamma(\sigma), T(\sigma), S(\sigma)\}$ along γ. This frame is called the Sabban Frame of the γ curve (Izumiya ve Takeuchi, 2002).

Theorem 4.1. Let $\gamma: I \to S^2$ be a spherical curve. The spherical Frenet formulas for the unit spherical curve are as follows,

$$\dot{\gamma}(\sigma) = T(\sigma)$$
$$\dot{T}(\sigma) = -\gamma(\sigma) + K_g(\sigma)S(\sigma)$$
$$\dot{S}(\sigma) = -K_g(\sigma)T(\sigma)$$

Here $K_g(\sigma)$, is the geodesic curvature of γ in S^2 given by $K_g(\sigma) = \det(\gamma(\sigma), T(\sigma), \dot{T}(\sigma))$ (Izumiya ve Takeuchi, 2002).

Theorem 4.2. Given a spherical curve $\gamma(\sigma)$, with unit speed then

$$\tilde{\gamma}(\sigma) = a\int_{\sigma_0}^{\sigma} \gamma(v)dv + acot\theta \int_{\sigma_0}^{\sigma} S(v)dv + c \quad (3.6)$$

the space curve $\tilde{\gamma}(\sigma)$ defined by (3.7) is a Bertrand curve, and all Bertrand curves can be constructed by this method. Here a and θ are constant numbers and c is a constant vector (Izumiya ve Takeuchi, 2002).

5. Bertrand PH-Curves in Euclidean 3-Space

Since there is no spherical PH-curve in Euclidean space as shown in Theorem 3.1.1, the curve obtained by the method in Theorem 4.2. cannot be a Bertrand PH-curve. When this study was carried to Minkowski space, the existence of a spherical PH-curve was seen. Thus, spherical PH-curves and Bertrand PH-curves were studied in Minkowski space.

6. Conclusion

This study considers the spherical PH-curves in 3-dimensional Euclidean space. We studied these curves in 3-dimensional Euclidean space and we proved that there is no spherical PH-curve in 3-dimensional Euclidean space. Afterwards we concluded that Bertrand PH-curves cannot characterized by the method given in Theorem 4.2. But spherical PH-curves can be studied in Minkowski space.

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Conflicts of Interest

The authors declare no conflict of interest.

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Burcu Gür Soğat [1*] **Mehmet Gümüş**[1]

https://orcid.org/0009-0002-3189-9867 https://orcid.org/0000-0001-7938-2918

[1]ÇOMÜ Lapseki Meslek Yüksekokulu, Muhasebe Bölümü. 17100, Çanakkale.

[*]Sorumlu yazar: burcu_1734@hotmail.com

Özet

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Abstract

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Introduction

Curve theory is an important area of differential geometry and Bertrand curves are one of the important curves in this field. Bertrand curves are curves defined and studied by the French mathematician Joseph Bertrand in the 19th century. (Bertrand,1850).

Bertrand reached various results by classifying curves on regular surfaces. He defined a special class of curves called Bertrand curves. Bertrand curves are curves whose principal normal vectors are linear. This property implies that the curve has a certain symmetry and therefore Bertrand curves are an interesting topic in differential geometry and curve theory.

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The simplest example of a Bertrand curve is a circle or a straight line. However, there are also more complex and diverse Bertrand curves. For example, ellipses, parabolas, hyperbolas, Bernoulli curves and Cassini curves are examples of Bertrand curves.

Joseph Bertrand examined the properties of Bertrand curves in more detail and put forward the necessary and sufficient condition to characterize pairs of Bertrand curves. A pair of Bertrand curves consists of two curves that have a certain property (Bertrand, 1850).

In his work, the characterization obtained by Bertrand is as follows: The necessary and sufficient condition for a curve $\boldsymbol{\alpha}$ with curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$ in $\mathbf{3}-$dimensional Euclidean space to be a pair of Bertrand curves is that the following equation holds with coefficients expressed in real numbers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$:

$$\boldsymbol{\lambda\kappa + \mu\tau = 1} \tag{1.1}$$

Here $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ denote a linear relationship between the curvature ($\boldsymbol{\kappa}$) and torsion ($\boldsymbol{\tau}$) properties of the curve. That is, if a curve $\boldsymbol{\alpha}$ has $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfying equation (1.1), then the curve $\boldsymbol{\alpha}$ and the curve $\boldsymbol{\alpha}^*$ (the second curve generated by the principal normal vectors) form a pair of Bertrand curves (Hsiung, 1981).

Equation (1.1) helps us to better understand Bertrand curves by describing their properties and relationships. Bertrand's characterization provides an important tool for further study and analysis of the geometry and character of Bertrand curves.

Izumiya and Takeuchi expressed how Bertrand curves can be obtained from spherical curves in Euclidean $\mathbf{3}-$space. In their work, Izumiya and Takeuchi showed that Bertrand curves can be obtained from spherical curves in $\mathbf{3}-$dimensional Euclidean space. This approach shows that Bertrand curves can be obtained by transforming spherical curves with a certain rotational motion. The work of Izumiya and Takeuchi is an important resource for mathematicians and differential geometry researchers, especially those interested in the geometry and origins of Bertrand curves. This work helps us to understand the different types of Bertrand curves and to study how they can be transformed into this special class of spherical curves. More details and mathematical expressions of how Bertrand curves can be derived from spherical curves can be found in the original work by Izumiya and Takeuchi. This work is an interesting resource for those interested in a more in-depth study of Bertrand curves and differential geometry (Izumiya and Takeuchi, 2002).

Murat Babaarslan (2009), in his master's thesis, first obtained the Cartan framework and Cartan curvatures in the spaces $\mathbb{R}^{\mathbf{5}}_{\mathbf{1}}, \mathbb{R}^{\mathbf{4}}_{\mathbf{2},}, \mathbb{R}^{\mathbf{5}}_{\mathbf{2}}$. Then he defined null Bertrand curves in these spaces and gave their characteristic properties (Babaarslan, 2009).

In her master thesis, Gül Güner (2011) investigated how different curves can be transformed into Bertrand curves and the properties of Bertrand curves. In the thesis, it is first shown how to obtain cylindrical helices from planar curves and Bertrand curves from spherical curves in Euclidean 3-space. Using this method, the Bertrand curves corresponding to the spherical indicators of a curve are investigated. Also, the planar evolute of a cylindrical helix and the spherical evolute of a spherical curve are investigated. In addition, the hyperbolic evolute of a spherical curve in $\mathbb{E}^3$ space is also studied in this thesis. Gül Güner's thesis is an important contribution to the subject by investigating the relations between Bertrand curves and different curves. The methods and findings presented in the thesis can guide the research on Bertrand curves and the researchers working in the related field (Güner, 2011).

Pythagorean-hodograph (PH) curves were described by Farouki and Sakkalis in 1990 (Farouki and Sakkalis, 1990). These curves are known as curves whose length can be calculated explicitly. Farouki's work was aimed at determining the properties and descriptors of PH-curves. He also investigated their relationship with helix curves and proved that all helix curves are PH-curves, but the converse is not always true (Farouki and Sakkalis, 1992). Characterization studies for two and three dimensional PH-curves were carried out by Farouki using complex numbers and quartrenions (Farouki and Sakkalis, 1994).

In 2000, Moon defined Pythagorean-Hodograph (PH) curves according to the Minkowski metric and obtained the Minkowski Pythagorean-Hodograph (MPH) curves. The Minkowski metric is a metric used in the special theory of relativity in the four-dimensional Minkowski space of space and time. The MPH curves are versions of the PH curves defined in Minkowski space and are defined taking into account geometric properties in this space. Steographic projection is a method of projecting a curve in Minkowski space onto a plane and has been used to represent MPH curves. Moon's work has defined versions of the PH curves that are valid in Minkowski space and used steographic projection to represent these curves (Moon, 2000).

Çağla Ramis (2013) focused on PH-curves and their applications in her master thesis. The thesis focused on the study of PH-curves in both two-dimensional and three-dimensional Euclidean and Minkowski spaces. The properties of these curves are investigated to obtain results and formulas. Furthermore, the thesis emphasizes the close relationship between helix curves and PH-curves. Considering this relationship, a planar PH-curve is generated from a spatial PH-curve. This shows how PH-curves can have different properties in different spaces and how they can be transformed from one space to another. The characterizations obtained for Euclidean space are carried over to Minkowski space and supported with examples. In this way, it is understood how the properties of PH-curves in Euclidean space are valid in Minkowski space and how they can be used. This thesis shows how PH-curves can be used in different spaces and applications and how their properties can be studied. The research can be an important resource for those who want to make progress in the mathematical analysis of PH-curves and their applications (Ramis, 2013).

2. Preliminaries

In this part of the paper, basic concepts and theorems in Euclidean $\mathbf{3}-$space was introduced. These concepts and theorems serve as a source for the third section of the paper.

Definition 2.1. Let $\boldsymbol{V}$ be a vector space and the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ be a subset of this vector space. If

$$\lambda_1 \boldsymbol{v_1} + \lambda_2 \boldsymbol{v_2} + \cdots + \lambda_n \boldsymbol{v_n} = \boldsymbol{0} \qquad (2.1)$$

equation (2.1) is satisfied when all scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ are zero, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ is called linearly independent (Hacısalihoğlu, 2000).

Definition 2.2. Let V be a vector space and the set $\{v_1, v_2, \dots, v_n\}$ be a subset of this vector space. If for all vectors $u \in V$ and the scalars $a_1, a_2, \dots, a_n \in \mathbb{R}$

$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \qquad (2.2)$$

is valid then the set $\{v_1, v_2, \dots, v_n\}$ spans the space V (Hacısalihoğlu, 2000).

Definition 2.3. Let V be a vector space and the set $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a subset of this vector space. If the set $\mathcal{B}$ satisfies the following conditions;

1. The set $\{v_1, v_2, \dots, v_n\}$ is linearly independent,
2. The set $\{v_1, v_2, \dots, v_n\}$ spans the space V,

then the set $\mathcal{B}$ is called a base of the space V (Hacısalihoğlu, 2000).

Definition 2.4. Let $\langle\, , \rangle$ be a function on n-dimensional Euclidean space $\mathbb{E}^n$. If we define this function for all vectors $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{E}^n$ as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

then the function $\langle\, , \rangle$ is called inner product (Hacısalihoğlu, 2000).

Definition 2.5. Let V be a real inner product space. The transformation $\|\ \|$ defined as

$$\|\ \| : V \to \mathbb{R}, \ \|u\| = \sqrt{\langle u, u\rangle}$$

specifies a norm on V. Specifically, if we take in the form $V = \mathbb{E}^n$ using the standard Euclidean inner product for $u = (u_1, u_2, \dots, u_n) \in \mathbb{E}^n$ then the following equality is given,

$$\|u\| = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}\,. \tag{2.3}$$

The value $\|u\|$ is called the norm or length of the vector u (Hacısalihoğlu, 2000).

Definition 2.6. Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{E}^n$

$$d : \mathbb{E}^n x \mathbb{E}^n \to \mathbb{R}$$

$$(x, y) \longrightarrow d(x, y) = \|\overrightarrow{xy}\| = \sqrt{\textstyle\sum_{i=1}^{n}(x_i - y_i)^2} \tag{2.4}$$

The function d is called the distance function in $\mathbb{E}^n$ and the real number $d(x, y)$ is called the distance between the points $x, y \in E^n$ (Hacısalihoğlu, 2000).

Definition 2.7. In 3 −dimensional Euclidean space $\mathbb{E}^3$ the vector product is defined for all vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{E}^3$ as following.

$$u \times v = (u_2 v_3 - v_2 u_3,\ u_3 v_1 - v_3 u_1, u_1 v_2 - v_1 u_2) \tag{2.5}$$

(Hacısalihoğlu, 2000).

Theorem 2.1. The distance function on $\mathbb{E}^n$ is a metric (Hacısalihoğlu, 2000).

Definition 2.8.

$$d : \mathbb{E}^n x \mathbb{E}^n \to \mathbb{R}$$

$$(x, y) \longrightarrow d(x, y) = \|\overrightarrow{xy}\|$$

The function d defined as above is called Euclidean metric function on $\mathbb{E}^n$ (Hacısalihoğlu, 2000).

Definition 2.9. Let $I \subseteq \mathbb{R}$ be an interval.

$$\alpha \colon I \longrightarrow \mathbb{E}^n$$

$$t \longrightarrow (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$$

If the function defined above α is differentiable α(I) is called a curve in $\mathbb{E}^n$ defined by the coordinate neighborhood (I,α) (Hacısalihoğlu, 2000).

Definition 2.10. Let α be a curve in $\mathbb{E}^n$ and defined by the coordinate functions (I,α) and (J, β). If the followings valid

$$h = \alpha^{-1} \circ \beta : J \longrightarrow \beta$$

$$s \longrightarrow h(s) = t$$

then the differentiable function h defined above is called a parameter change function (Hacısalihoğlu, 2000).

Definition 2.11. Let the curve α in $\mathbb{E}^n$ be parametric,

$$\alpha : \quad I \longrightarrow \mathbb{E}^n$$

$t: \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$

and for curve α ;

$$\alpha'(t) = \frac{d\alpha}{dt}$$

$= \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \ldots, \frac{d\alpha_n}{dt}\right).$

Then the vector $(\alpha(t), \alpha'(t)) \in T_{\mathbb{E}^n}(p)$ is called the velocity vector or tangent vector of the curve α at $\alpha'(t)$ corresponding to the parameter value $t \in I$ (Hacısalihoğlu, 2000).

Definition 2.12. Let the α curve at $\mathbb{E}^n$ is defined parametrically,

$$\alpha : I \longrightarrow \mathbb{E}^n$$

$t = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$

The derivative of the curve α,

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \ldots, \frac{d\alpha_n}{dt}\right)$$

and the norm is to be

$$\| \alpha'(t) \| : I \longrightarrow \mathbb{R}$$

$$t \longrightarrow \| \alpha'(t) \| = \sqrt{\sum_{i=1}^{n} \left(\frac{d\alpha_i}{dt}\right)^2}$$

scalar velocity function. The real number at the point $t = t_0$

$\| \alpha'(t_0) \| = \sqrt{\sum_{i=1}^{n} (\frac{d\alpha_i}{dt})^2}$ (2.6)

is called scalar velocity (Hacısalihoğlu, 2000).

Definition 2.13. Let the curve α in $\mathbb{E}^n$ be

$$\alpha : I \longrightarrow \mathbb{E}^n$$

$t : \alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)).$

For all $t_1, t_2 \in I$

$s = \int_{t_1}^{t_2} \| \alpha'(t) \| dt$ (2.7)

the real number α is called the arc length of the curve α between the points $\alpha(t_1)$ and $\alpha(t_2)$ (Hacısalihoğlu, 2000).

Definition 2.14. Let α be a curve in $\mathbb{E}^n$. If the norm of the curve α satisfies

$\| \alpha'(s) \| = 1$

then the curve α is called the unit speed curve and the parameter s is called the arclength parameter (Hacısalihoğlu, 2000).

Definition 2.15. If the curve α in $\mathbb{E}^n$ satisfies the following

$$\| \alpha'(t)\| = \left\| \frac{d\alpha}{dt} \right\| \neq 0 \qquad (2.8)$$

then the curve is called a regular curve (Hacısalihoğlu, 2000).

3. PH-Curves in Euclidean 3-Space

Definition 3.1. Let α be a curve in $\mathbb{E}^n$ and $\alpha(t) = \big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\big)$. The hodograph of the polynominal curve α is defined by

$$\|\alpha'(t)\| = \alpha_1'(t)^2 + \alpha_2'(t)^2 + \cdots + \alpha_n'(t)^2 = \sigma(t)^2 \qquad (3.1)$$

and if there's a $\sigma(t)$ polynominal then the curve α is called Pythagorean Hodograph curve(PH-curve) (Farouki ve Sakkalis, 1994).

Definition 3.2. Let $n \in N_0$ and $a_i \in \mathbb{R}$ where $0 \leq i \leq n$,

$$\alpha(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \, , a_n \neq 0 \qquad (3.2)$$

in the from of t function and n is called the degree of the polynomial (Larson, 2012).

Definition 3.3. Let α be a curve in $\mathbb{E}^n$. If the curve α defined as,

$$\alpha : [a, b] \to E^n$$

$$\alpha(t) = \Big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\Big)$$

where the components $\alpha_i(t)$ for all $1 \leq i \leq n$ are polynomials then the curve α is called n −dimensional polynomial curve (Larson, 2012).

Definition 3.4. Let α be a curve in $\mathbb{E}^n$ defined as,

$$\alpha : [a, b] \to E^n$$

$$\alpha(t) = \Big(\alpha_1(t), \alpha_{2,}(t), \dots \alpha_n(t)\Big).$$

The degree of the polynomial curve α is the number $deg\alpha(t)$ defined by

$$deg\alpha(t) = max\{deg(\alpha_1(t)), deg(\alpha_2(t)), \dots, deg(\alpha_n(t))\} \qquad (3.3)$$

(Larson, 2012).

Theorem 3.1. Let $a(t), b(t), c(t)$ be polynomials, The Pythagorean condition

$$a^2(t) + b^2(t) = c^2(t)$$

is satisfied by the polynomials $a(t), b(t), c(t)$ where

$$a(t) = [u^2(t) - v^2(t)]w(t)$$

$$b(t) = 2u(t)v(t)w(t)$$

$$c(t) = [u^2(t) + v^2(t)]w(t)$$

in the form of $u(t), v(t), w(t)$ polynomials (Ramis, 2013).

3.1. Spherical PH-Curves in Euclidean 3-Space

Theorem 3.1.1. There is no spherical PH-curve in $\mathbb{E}^3$.

Proof: Let $\gamma : I \to S^2$ be a spherical-PH curve in $\mathbb{E}^3$. Since γ is a polynomial curve in $\mathbb{E}^3$ for $\gamma_1(t), \gamma_2(t), \gamma_3(t)$ polynomials;

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

can be written and since γ is a PH-curve then

$$(\dot{\gamma}_1)^2 + (\dot{\gamma}_2)^2 + (\dot{\gamma}_3)^2 = \sigma^2 \quad (3.4)$$

the equality (3.4) must be satisfied for an arbitrary polynomial σ. Also γ lies on the sphere it must satisfy

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \quad (3.5)$$

In this case:

$$deg\{\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t)\} = max\{\deg(\gamma_1^2(t), \gamma_2^2(t), \gamma_3^2(t))\} = 0$$

As a result we get that γ is a polynomial where its degree is zero, This means that it is a point. Consequently, there is no spherical PH-curve in Euclidean space.

4. Bertrand Curves in Euclidean 3-Space

Definition 4.1. Let $\alpha : I \rightarrow \mathbb{E}^n$ and $\alpha^*: I \rightarrow \mathbb{E}^n$ be two differentiable curves, the Frenet frames of these curves are respectively $\{T, N_1, N_2 \dots, N_{n-1}\}$ and $\{T^*, N_1^*, N_2^*, \dots, N_{n-1}^*\}$ and $N_1(s)$ the principal normal vector of the curve α, of the curve $N_1^*(s)$ the principal normal vector of the curve α^*. If the principal normal vectors $N_1(s)$ and $N_1^*(s)$ are linearly dependent then the (α, α^*) is called Bertrand curves pair, α curve is also called a Bertrand curve (Hacısalihoğlu, 2000).

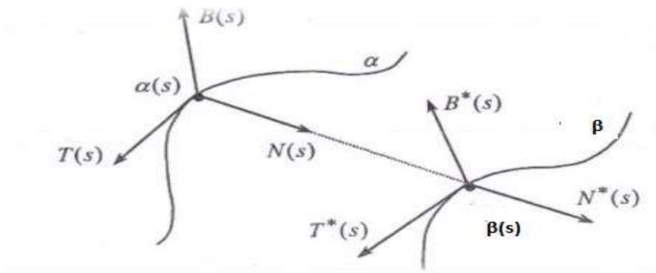

Figure 1. Bertrand Curve Pair

Definition 4.2. Let $\gamma: I \rightarrow S^2$ be a unit spherical curve with arc parameter σ. The unit tangent vector of γ at σ is given by $T(\sigma) = \dot{\gamma}(\sigma)$ where $\dot{\gamma} = \frac{d\gamma}{d\sigma}$. Together with the vector $S(\sigma)$, defined as $S(\sigma) = \gamma(\sigma) \times T(\sigma)$ we obtain an orthonormal frame $\{\gamma(\sigma), T(\sigma), S(\sigma)\}$ along γ. This frame is called the Sabban Frame of the γ curve (Izumiya ve Takeuchi, 2002).

Theorem 4.1. Let $\gamma: I \rightarrow S^2$ be a spherical curve. The spherical Frenet formulas for the unit spherical curve are as follows,

$$\dot{\gamma}(\sigma) = T(\sigma)$$
$$\dot{T}(\sigma) = -\gamma(\sigma) + K_g(\sigma)S(\sigma)$$
$$\dot{S}(\sigma) = -K_g(\sigma)T(\sigma)$$

Here $K_g(\sigma)$, is the geodesic curvature of γ in S^2 given by $K_g(\sigma) = \det(\gamma(\sigma), T(\sigma), \dot{T}(\sigma))$ (Izumiya ve Takeuchi, 2002).

Theorem 4.2. Given a spherical curve $\gamma(\sigma)$, with unit speed then

$$\tilde{\gamma}(\sigma) = a\int_{\sigma_0}^{\sigma} \gamma(v)dv + acot\theta \int_{\sigma_0}^{\sigma} S(v)dv + c \quad (3.6)$$

the space curve $\tilde{\gamma}(\sigma)$ defined by (3.7) is a Bertrand curve, and all Bertrand curves can be constructed by this method. Here a and θ are constant numbers and c is a constant vector (Izumiya ve Takeuchi, 2002).

5. Bertrand PH-Curves in Euclidean 3-Space

Since there is no spherical PH-curve in Euclidean space as shown in Theorem 3.1.1, the curve obtained by the method in Theorem 4.2. cannot be a Bertrand PH-curve. When this study was carried to Minkowski space, the existence of a spherical PH-curve was seen. Thus, spherical PH-curves and Bertrand PH-curves were studied in Minkowski space.

6. Conclusion

This study considers the spherical PH-curves in 3-dimensional Euclidean space. We studied these curves in 3-dimensional Euclidean space and we proved that there is no spherical PH-curve in 3-dimensional Euclidean space. Afterwards we concluded that Bertrand PH-curves cannot characterized by the method given in Theorem 4.2. But spherical PH-curves can be studied in Minkowski space.

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Conflicts of Interest

The authors declare no conflict of interest.

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