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Some Identities Concerning Generalized Fibonacci and Lucas Numbers by Matrix Methods

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Abstract: In this study, we give some matrices whose powers consist of the terms of generalized Fibonacci and Lucas sequences. Then we give some identities concerning the terms of those sequences.

Matris Yöntemleriyle Genelleştirilmiş Fibonacci ve Lucas Sayıları ile İlgili Bazı Özellikler

Anahtar Kelimeler
Genelleştirilmiş Fibonacci dizisi,
Genelleştirilmiş Lucas dizisi

Öz: Bu çalışmada kuvvetleri genelleştirilmiş Fibonacci ve Lucas dizilerinin terimlerinden oluşan bazı matrisler verilecektir. Daha sonra bu dizilerin terimlerine ilişkin bazı özdeşlikler elde edilecektir.

1. INTRODUCTION

Let k, t be nonzero integers with $k^2+4t>0$ and $k>0$. Generalized Fibonacci and Lucas sequences $(U_n(k,t))$ and $(V_n(k,t))$ are defined by $U_0(k,t)=0$, $U_1(k,t)=1$, $V_0(k,t)=2$, $V_1(k,t)=k$ and $U_{n+1}(k,t)=kU_n(k,t)+tU_{n-1}(k,t)$, $V_{n+1}(k,t)=kV_n(k,t)+tV_{n-1}(k,t)$ for $n\geq 0$. These sequences are defined firstly by Lucas in [1].

For $n<0$, we define

$$U_n(k,t)=-(-t)^n U_{-n}$$

and

$$V_n(k,t)=(-t)^n V_{-n}.$$

Then it is well known that

$$U_n(k,t)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ V_n(k,t)=\alpha^n+\beta^n$$

for every integer n, where

$$\alpha=\frac{k+\sqrt{k^2+4t}}{2},\ \beta=\frac{k-\sqrt{k^2+4t}}{2}$$

are the roots of the characteristic equation

$$x^2-kx-t=0.$$

The above formulas are known as Binet's formulas. If $k=t=1$, we get Fibonacci sequence (F_n) and Lucas sequence (L_n) respectively. For $k=2$, $t=1$, we get Pell and Pell-Lucas sequences (P_n) and (Q_n), respectively.

For briefly, we will write U_n and V_n instead of $U_n(k,t)$ and $V_n(k,t)$. For more information and applications these sequences one can consult [2] and [5], respectively.

Many identities concerning the terms of these sequence can be proved by using Binet's formulas. Also, matrices can be used to obtain these identities. The most known matrix is $\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and it is well known that

$$\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}^n=\begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}$$

for every integer n (see [3,4]).

The most known identities are given below as (see [2] or [4])

$$V_n = U_{n+1} + tU_{n-1},$$
$$(k^2+4t)U_n = V_{n+1} + tV_{n-1},$$
$$U_n^2 - (k^2+4t)V_n^2 = 4(-t)^n.$$

In this study, we give some matrices whose powers consist of the terms of the above sequences. Then we will give some identities concerning the terms of these sequences. As we did not run into these identities in the literature, we think that these identities are new.

2. MAIN THEOREMS

Theorem 1. Let $A = \begin{pmatrix} k & -\alpha \\ \beta & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \alpha \\ -\beta & k \end{pmatrix}$. Then

$$A^n = \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix} \text{ and } B^n = \begin{pmatrix} tU_{n-1} & \alpha U_n \\ -\beta U_n & U_{n+1} \end{pmatrix}$$

for every integer n.

Proof: It can be seen easily that eigenvalues of the matrix A are α and β. Eigenvectors related to α are of the form $\begin{pmatrix} \alpha t \\ \beta t \end{pmatrix}$ for $t \neq 0$. And eigenvectors related to β are of the form $\begin{pmatrix} t \\ t \end{pmatrix}$ for $t \neq 0$. Let $P = \begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix}$. As $\det P \neq 0$, we can write

$$A = PDP^{-1}, \text{ where } D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Thus, it is easily seen that $A^n = PD^nP^{-1}$ and therefore

$$A^n = \frac{1}{\alpha-\beta}\begin{pmatrix} \alpha & 1 \\ \beta & 1 \end{pmatrix}\begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -\beta & \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & \dfrac{-\alpha(\alpha^n-\beta^n)}{\alpha-\beta} \\ \dfrac{\beta(\alpha^n-\beta^n)}{\alpha-\beta} & \dfrac{-\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta} \end{pmatrix}$$
$$= \begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix}.$$

The proof for the matrix B is similar with the matrix A. Therefore we omit the details.

Now we can give our main theorems.

Theorem 2. Let n be a natural number. Then

a) $U_{n+1} = t\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-1-j}$,

b) $U_n = -\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j}$,

c) $tU_{n-1} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n+1-j}$,

d) $V_n = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jV_{n-j}$

Proof. Let the matrices A and B as in Theorem 1. Then, since $A + B = \begin{pmatrix} k & -\alpha \\ \beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\beta & k \end{pmatrix} = kI$, we get $A = -B + kI$ and therefore

$$A^n = (-B+kI)^n = \sum_{j=0}^{n}\binom{n}{j}(-B)^{n-j}k^j = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jB^{n-j}.$$

Thus, it is seen that

$$\begin{pmatrix} U_{n+1} & -\alpha U_n \\ \beta U_n & tU_{n-1} \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n+1-j} & \alpha\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j} \\ -\beta\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j} & \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n+1-j} \end{pmatrix}$$

Then we get

$$U_{n+1} = t\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j-1}, \quad (1)$$
$$U_n = -\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j},$$
$$tU_{n-1} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jU_{n-j+1}. \quad (2)$$

If we add (1) to (2) and use $V_r = U_{r+1} + tU_{r-1}$, then we get

$$V_n = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}k^jV_{n-j}.$$

Theorem 3. Let n be a natural number. Then

a) $k^n = \sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-2j+1}$,

b) $k^n = t\sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-2j-1}$,

c) $0 = \sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-2j}$,

d) $2k^n = \sum_{j=0}^{n}\binom{n}{j}(-t)^jV_{n-2j}$

Proof. Let the matrices A and B as in Theorem 1. As $A + B = kI$ and $AB = BA = -tI$, we get

$$k^nI = (A+B)^n = \sum_{j=0}^{n}\binom{n}{j}A^{n-j}B^j$$
$$= \sum_{j=0}^{n}\binom{n}{j}A^{n-2j}A^jB^j$$
$$= \sum_{j=0}^{n}\binom{n}{j}A^{n-2j}(-tI)^j = \sum_{j=0}^{n}\binom{n}{j}(-t)^jA^{n-2j}.$$

Therefore

$$\begin{pmatrix} k^n & 0 \\ 0 & k^n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n+1-2j} & -\alpha\sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-2j} \\ \beta\sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-2j} & t\sum_{j=0}^{n}\binom{n}{j}(-t)^jU_{n-1-2j} \end{pmatrix}.$$

Then the proof follows.

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