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Solvability of Quadratic Integral Equations of Urysohn Type Involving Hadamard Variable-Order Operator

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Abstract

This study investigates the existence of solutions to integral equations in the form of quadratic Urysohn type with Hadamard fractional variable order integral operator. Due to the lack of semigroup properties in variable-order fractional integrals, it becomes challenging to get the existence and uniqueness of corresponding integral equations, hence the problem is examined by employing the concepts of piecewise constant functions and generalized intervals to address this issue. In this context, the problem is reformulated as integral equations with constant orders to obtain the main results. Both the Schauder and Banach fixed point theorems are employed to prove the uniqueness results. In addition, an illustration is included in order to verify those results in the final step.

1. Introduction

As a subject of mathematical analysis, fractional calculus arose as a result of investigating the matter regarding whether it is feasible to employ the complex or real number powers for integral and differential operators. (see [1, 2]). Over the past three decades, the theory has led to numerous significant results in both pure and applied mathematics alongside other branches of sciences, for example: chemistry, ,signal and image processing, physics,control theory, biology, biophysics, economics etc. (see [3, 4, 5, 6, 7, 8, 9]).
The arbitrary order of integral and differentiation whose order depends on a function of certain variables, which corresponds to a more complicated category, are known as variable-order operators. Following its introduction in 1993 by Samko and Ross [10], the concept of fractional variable-order (FVO) differential and integral operators, along with its basic features, have naturally garnered significant interest from numerous researchers. The investigation of fractional variable models is still in its early stages since addressing the variable fractional order is certainly tough to study in some cases, whose features like the semi-group property are separated from the associated characteristics of systems with conventional fractional orders. However, for recent developments on the theory of fractional variational calculus and numerical methods dealing with fractional problems of variable order see [11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein.
There have been very recent publications dealing with fractional equations of variable order coupled with auxiliary conditions from a qualitative perspective. For example, in [20], the authors examined the existence of solutions to a boundary value problem for a class of fractional equations of Riemann–Liouville(R-L) variable order with finite delay by employing the Darbo type fixed-point as well as Kuratowski measure of noncompactness. It has been considered in [21] for the first time a Caputo FVO initial value problem(IVP) under impulsive conditions, and the uniqueness and existence of solutions have been examined. Two fixed point theories were employed to show the main results. In [22], monotone iterative technique together with upper and lower solutions have been applied to IVP for linear homogeneous and non-homogeneous diffusion equations involving the conformable operator of variable order to show the existence and uniqueness properties. In [23], existence and uniqueness of a boundary value problem with variable order operators of Hadamard type have been examined with the aid of Schauder,

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and Banach fixed point theorems and stability criteria have been obtained regarding Ulam–Hyers–Rassias(UHR). Suitable criteria ensuring the existence and uniqueness of a class of FVO Riemann-Liouville equations including fractional boundary conditions was discussed in [24] and suitable conditions providing the stability in the UHR sense were also established. For additional papers, one might consult the latest publications[25, 26, 27, 28, 29, 30] and the associated references therein.
The study of integral equations is an essential component of nonlinear analysis and investigated by many scholars in view of their wide range of scientific applications [31, 32]. There have also been a number of studies that examine the existence of solutions of functional integral and integro-differential equations of fractional conventional order [33]. However, only a limited number of papers have discussed the existence of solutions to such problems involving FVO operators [34, 35, 36, 37, 38].
In this work, we study the quadratic integral equation of Urysohn type with fractional variable order (**QIEUFVO**)

$$u(t)= \quad q(t)+(\Phi u)(t)\int_1^t \tfrac{1}{\Gamma(\omega(t))}(\log\tfrac{t}{\sigma})^{\omega(t)-1}\xi(t,\sigma,u(\sigma))\tfrac{d\sigma}{\sigma}, \tag{1.1}$$

where $t\in\Upsilon:=[1,K],\ q\in C(\Upsilon,\mathbb{R}),\ \xi:\Upsilon^2\times\mathbb{R}\to\mathbb{R}$ is given function, $1<\omega(t)\le 2$ and $\Phi: C(\Upsilon,\mathbb{R})\to C(\Upsilon,\mathbb{R})$ is an appropriate operator.
In order to find the existence and uniqueness of (1.1), we employ the notions of generalized interval, partition and piece-wise constant functions, hence converting the equation to fractional integral equations of constant order.

2. Preliminaries

This section provides several concepts and results that will be required throughout the subsequent sections.

Definition 2.1. *([40], [23]) Let $\omega(t):\Upsilon\to(1,2]$, then the left Hadamard fractional integral of variable order (HFIVO) for function ψ is*

$$({}^HI_{1^+}^{\omega(t)}\psi)(t)=\frac{1}{\Gamma(\omega(t))}\int_1^t(\log\frac{t}{\sigma})^{\omega(t)-1}\psi(\sigma)\frac{d\sigma}{\sigma},\ t>1, \tag{2.1}$$

As expected, when $\omega(t$ is constant, then HFIVO is corresponds to the standard Hadamard fractional integral operator.

Remark 2.2. *([40], [41]) In the general case, the semi-group property is not satisfied for the Integral operator of variable order, i.e.*

$${}^HI_{1^+}^{\omega(t)}{}^HI_{1^+}^{v(t)}\psi(t)\neq{}^HI_{1^+}^{\omega(t)+v(t)}\psi(t).$$

Lemma 2.3. *([41] Let $\omega:\Upsilon\to(1,2]$ be a continuous function, then for $\psi\in C_\sigma(\Upsilon,\mathbb{R})$ where*

$$C_\sigma(\Upsilon,\mathbb{R})=\{\psi(t)\in C(\Upsilon,\mathbb{R}),\ (logt)^\sigma\psi(s)\in C(\Upsilon,\mathbb{R}),0\le\sigma\le 1\},$$

the integral ${}^HI_{1^+}^{\omega(t)}\psi(t)$ exists for any $t\in\Upsilon$.

Lemma 2.4. *([41]) If $\omega\in C(\Upsilon,(1,2])$, then ${}^HI_{1^+}^{\omega(t)}\psi(t)\in C(\Upsilon,\mathbb{R})$ for any $\psi\in C(\Upsilon,\mathbb{R})$.*

Theorem 2.5. *(Schauder Fixed Point Theorem) ([42]) Let Λ be a convex subset of Banach Space Π and $\mathcal{F}:\Lambda\longrightarrow\Lambda$ be completely continuous map, then $\mathcal{F}$ has at least one fixed point in Λ.*

3. Existence and uniqueness results

We first state the underlying presumption:

(H1) Let

$$\mathcal{P}=\{\Upsilon_1:=[0,K_1],\Upsilon_2:=(K_1,K_2],\Upsilon_3:=(K_2,K_3]\ldots\Upsilon_n:=(K_{n-1},K]\}$$

be a partition of the interval Υ and let $\omega(s):\Upsilon\to(1,2]$ be a piece-wise continuous function with respect to $\mathcal{P}$, i.e.,

$$\omega(t)=\sum_{\vartheta=1}^{n}\omega_\vartheta I_\vartheta(t)=\begin{cases}\omega_1, & if\ t\in\Upsilon_1,\\ \omega_2, & if\ t\in\Upsilon_2,\\ \cdot\\ \cdot\\ \cdot\\ \omega_n, & if\ t\in\Upsilon_n,\end{cases}$$

where $1<\omega_\vartheta\le 2$ are constants, and

$$I_{\vartheta}(t)=\begin{cases}1, & for\, t\in\Upsilon_{\vartheta},\\ 0, & for\, elsewhere.\end{cases}$$

The notion $\Pi_{\vartheta}=C(\Upsilon_{\vartheta},\,\mathbb{R})$ denotes the Banach space of continuous functions from Υ_{ϑ} into $\mathbb{R}$ with the norm

$$\|u\|_{\Pi_{\vartheta}}=\sup_{t\in\Upsilon_{\vartheta}}|u(t)|,\vartheta\in\{1,2,...,n\}.$$

Then, for any $t\in\Upsilon_{\vartheta}$, $\vartheta=1,2,...,n$, the (HFIVO) for function $\xi(t,\sigma,u(\sigma))\in C(\Upsilon^2\times\mathbb{R},\mathbb{R})$, defined by (2.1), might then be stated as

$$^{H}I_{1^+}^{\omega(t)}\xi(t,\sigma,u(t)) \quad =\sum_{i=1}^{i=\vartheta-1}\int_{K_{i-1}}^{K_i}\frac{1}{\Gamma(\omega_i)}(\log\frac{t}{\sigma})^{\omega_i-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}+\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_{\vartheta})}(\log\frac{t}{\sigma})^{\omega_{\vartheta}-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}. \tag{3.1}$$

According to (3.1), for any $t\in\Upsilon_{\vartheta}$, $\vartheta\in\{1,2,...,n\}$, 1.1 can be stated in the following format:

$$u(t) \quad =q(t)+(\Phi u)(t)\Big(\sum_{i=1}^{i=\vartheta-1}\int_{K_{i-1}}^{K_i}\frac{1}{\Gamma(\omega_i)}(\log\frac{t}{\sigma})^{\omega_i-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}+\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_{\vartheta})}(\log\frac{t}{\sigma})^{\omega_{\vartheta}-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big). \tag{3.2}$$

Let $u\in C(\Upsilon_{\vartheta},\mathcal{R})$ be a solution of (3.2), such that $u(t)\equiv 0$ on $t\in[1,K_{\vartheta-1}]$. Then (3.2) is reduced to

$$u(t)=q(t)+(\Phi u)(t)\Big(\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_{\vartheta})}(\log\frac{t}{\sigma})^{\omega_{\vartheta}-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big),\; t\in\Upsilon_{\vartheta}. \tag{3.3}$$

We now impose the following assumptions:

(H2) There exists $\varpi_{\vartheta}>0$ such that

$$|(\Phi u)(t)-(\Phi\tilde{u})(t)|\leq\varpi_{\vartheta}|u(t)-\tilde{u}(t)|$$

for each $u,\tilde{u}\in\Pi_{\vartheta}$ and $t\in\Upsilon_{\vartheta}$.

(H3) There are non-negative constants η and γ such that

$$|(\Phi u)(t)|\leq\eta+\gamma|u(t)|$$

for each $u\in\Pi_{\vartheta}$ and $t\in\Upsilon_{\vartheta}$.

(H4) Let $\xi:\Upsilon_{\vartheta}^2\times\mathbb{R}\to\mathbb{R}$ be a continuous function and non-decreasing with respect to its three variables, separately, and there exists a constants $0\leq\sigma\leq 1$, $D_{\vartheta}>0$ such that

$$(logt)^{\sigma}|\xi(t,\sigma,u)-\xi(t,\sigma,\tilde{u})|\leq D_{\vartheta}|u-\tilde{u}|$$

for all $(t,\sigma)\in\Upsilon_{\vartheta}^2$ and $u,\tilde{u}\in\mathbb{R}$.

(H5) There exists a continuous non-decreasing function $g\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\bar{h}\in C(\Upsilon,\mathbb{R}_+)$ and a constant $0\leq\sigma\leq 1$ such that for each $(t,\sigma)\in\Upsilon_m^2$ and $u\in\mathbb{R}$ we have

$$(logt)^{\sigma}|\xi(t,\sigma,u)|\leq\bar{h}(\sigma)g(|u|),$$

Theorem 3.1. *Let $\vartheta\in\{1,2,...,n\}$, suppose that hypotheses $(H1)-(H5)$ hold, and there exists a constant r_{ϑ}, such that*

$$\frac{r_{\vartheta}}{q^{\star}+\frac{(logK_{\vartheta})^{1-\sigma}-(logK_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_{\vartheta})}(\log\frac{K_{\vartheta}}{K_{\vartheta-1}})^{\omega_{\vartheta}-1}(\eta+\gamma r_{\vartheta})g(r_{\vartheta})\bar{h}^{\star}}>1, \tag{3.4}$$

where $\bar{h}^{\star}=sup\{\bar{h}(\sigma):\sigma\in\Upsilon_{\vartheta}\}$ and $q^{\star}=sup\{q(t):t\in\Upsilon_{\vartheta}\}$.
Then, (3.3) has at least solution in Π_{ϑ}.

Proof. Let the operator

$$S:\Pi_{\vartheta}\to\Pi_{\vartheta}$$

given by

$$(Su)(t)=q(t)+(\Phi u)(t)\Big(\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_{\vartheta})}(\log\frac{t}{\sigma})^{\omega_{\vartheta}-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big).$$

Let the set

$$B_{r_{\vartheta}}=\{u\in\Pi_{\vartheta}:\|u\|_{\Pi_{\vartheta}}\leq r_{\vartheta}\}.$$

Clearly $B_{r_{\vartheta}}$ is nonempty, convex, closed and bounded.

Step 1: Claim: $S(B_{r_\vartheta}) \subseteq (B_{r_\vartheta})$.
For $u \in B_{r_\vartheta}$, we have

$$\begin{aligned}
|(Su)(t)| &\leq |q(t)| + |(\Phi u)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&\leq |q(t)| + (\eta + \gamma|u(t)|) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq |q(t)| + (\eta + \gamma|u(t)|)(\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq q^\star + \frac{(logK_\vartheta)^{1-\sigma} - (logK_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)} (\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} (\eta + \gamma\|u\|_{\Pi_\vartheta}) g(\|u\|_{\Pi_\vartheta}) \bar{h}^\star. \\
&\leq r_\vartheta.
\end{aligned}$$

Step 2: Claim: S is continuous.
Let (u_n) be a sequence such that $u_n \to u$ in Π_ϑ then

$$\begin{aligned}
|(Su_n)(t) - (Su)(t)| &= \Big|(\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u_n(\sigma)) \frac{d\sigma}{\sigma} \\
&\quad - (\Phi u)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&\leq \Big|(\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u_n(\sigma)) \frac{d\sigma}{\sigma} \\
&\quad - (\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&\quad + \Big|(\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \\
&\quad - (\Phi u)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&\leq |(\Phi u_n)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u_n(\sigma)) d\sigma - \xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&\quad + |(\Phi u_n)(t) - (\Phi u)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&\leq (\eta + \gamma|u_n(t)|) D_\vartheta \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} |u_n(\sigma) - u(\sigma)| \frac{d\sigma}{\sigma} \\
&\quad + \varpi_\vartheta |u_n(\sigma) - u(\sigma)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq \Big[\Big(D_\vartheta(\eta + \gamma r_\vartheta) + \bar{h}^\star \varpi_\vartheta g(r_\vartheta) \Big) \frac{(\log K_\vartheta)^{1-\sigma} - (\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)} (\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} \Big] \|u_n - u\|_{\Pi_\vartheta}
\end{aligned}$$

i.e., we obtain

$$\|(Su_n) - (Su)\|_{\Pi_\vartheta} \to 0 \ \ as \ \ n \to \infty.$$

As a consequence, the operator S is a continuous on Π_n.

Step 3: Claim: S is compact
Step 1 leads to the outcome $\|S(u)\|_{\Pi_\vartheta} \leq r_\vartheta$ for each $u \in B_{r_\vartheta}$, yielding the boundedness of $S(B_{r_\vartheta})$. We shall now demonstrate the equicontinuity of $S(B_{r_\vartheta})$.

For $t_1, t_2 \in \Upsilon_\vartheta,\ t_1 < t_2$ and $u \in B_{r_\vartheta}$, estimate

$$
\begin{aligned}
|(Su)(t_2)-(Su)(t_1)| &\le |q(t_2)-q(t_1)| + \Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma} \\
&\quad -(\Phi u)(t_1)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_n-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\le |q(t_2)-q(t_1)| + \Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma} \\
&\quad -\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\le |q(t_2)-q(t_1)| + \Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\Big(\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma)) \\
&\quad -\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\Big)\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|(\Phi u)(t_2)\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\le |q(t_2)-q(t_1)| + \Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big(\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big)\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\frac{\xi(t_1,\sigma,u(\sigma))}{\Gamma(\omega_\vartheta)}((\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}-(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1})\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|(\Phi u)(t_2)\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\quad +\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big| \\
&\le |q(t_2)-q(t_1)| + |(\Phi u)(t_2)|\Big|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big|\frac{d\sigma}{\sigma} \\
&\quad +|(\Phi u)(t_2)||\xi(t_1,t_1,r_\vartheta)|\Big|\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}-(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big| \\
&\quad +|(\Phi u)(t_2)||\xi(t_2,s_2,r_\vartheta)|\Big|\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big| \\
&\quad +|(\Phi u)(t_2)-(\Phi u)(t_1)||\xi(t_1,t_1,r_\vartheta)|\Big|\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big| \\
&\le |q(t_2)-q(t_1)| + |(\Phi u)(t_2)|\Big|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big|\frac{d\sigma}{\sigma} \\
&\quad +|(\Phi u)(t_2)|\frac{|\xi(t_1,t_1,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}\Big|(\log\frac{t_2}{t_1})^{\omega_\vartheta}-(\log\frac{t_2}{K_{\vartheta-1}})^{\omega_\vartheta}+(\log\frac{t_1}{K_{\vartheta-1}})^{\omega_\vartheta}\Big| \\
&\quad +|(\Phi u)(t_2)|\frac{|\xi(t_2,t_2,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}(\log\frac{t_2}{t_1})^{\omega_\vartheta}+|(\Phi u)(t_2)-(\Phi u)(t_1)|\frac{|\xi(t_1,t_1,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}(\log\frac{t_1}{K_{\vartheta-1}})^{\omega_\vartheta}
\end{aligned}
$$

Owing to $(H4)$, we know that the function $\xi(t,\sigma,u)$ is uniformly continuous on $\Upsilon_\vartheta^2 \times B_{r_\vartheta}$, then we have

$$\lim_{t_2\to t_1}|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))| = 0$$

uniformly in $\sigma \in \Upsilon_\vartheta$ and $u \in B_{r_\vartheta}$. Hence, we have

$$\Big|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}\frac{\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))}{(t_2-\sigma)^{1-\omega_\vartheta}}d\sigma\Big| \le \sup_{\sigma\in\Upsilon_\vartheta,\ u\in B_{r_\vartheta}}\frac{(t_2-K_{\vartheta-1})^{1-\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))| \to 0 \quad (3.5)$$

as $t_2 \to t_1$.

So, $\|(Su)(t_2)-(Su)(t_1)\|_{\Pi_\vartheta} \to 0$ as $|t_2-t_1| \to 0$. It confirms $S(B_{r_\vartheta})$ is equicontinuous.
Thereby, considering Theorem 2.5, (3.3) has at least one solution $\widetilde{u}_\vartheta \in B_{r_\vartheta}$.

□

The subsequent result regarding uniqueness relates to the Banach Contradiction Principle.

Theorem 3.2. *Assume that given conditions in Theorem 3.1 hold, and moreover*

$$\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\le 1. \tag{3.6}$$

is satisfied.
Then, (3.3) has a unique solution in Π_ϑ.

Proof. Let $u,\tilde{u}\in B_{r_\vartheta}$, and $t\in\Upsilon_\vartheta$, we have

$$\begin{aligned}
&|(Su)(t)-(S\tilde{u})(t)|\\
=\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}-(\Phi\tilde{u})(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
\le\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}-(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
+\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}-(\Phi\tilde{u})(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
\le\;&|(\Phi u)(t)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}|\xi(t,\sigma,u(\sigma))d\sigma-\xi(t,\sigma,\tilde{u}(\sigma))|\frac{d\sigma}{\sigma}\\
+\;&|(\Phi u)(t)-(\Phi\tilde{u})(t)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}|\xi(t,\sigma,\tilde{u}(\sigma))|\frac{d\sigma}{\sigma}\\
\le\;&(\eta+\gamma|u(t)|)D_\vartheta\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}(log\sigma)^{-\sigma}|z(\sigma)-\tilde{u}(\sigma)|\frac{d\sigma}{\sigma}\\
+\;&\varpi_\vartheta|u(\sigma)-\tilde{u}(\sigma)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}(log\sigma)^{-\sigma}\bar{h}(\sigma)g(|\tilde{u}(\sigma)|)\frac{d\sigma}{\sigma}\\
\le\;&\Big[\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\Big]\|u-\tilde{u}\|_{\Pi_n}
\end{aligned}$$

Therefore,

$$\|(Su)(t)-(S\tilde{u})(t)\|_{\Pi_\vartheta}\le\Big[\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\Big]\|u-\tilde{u}\|_{\Pi_\vartheta}$$

Thus, according to equation (3.6), the operator S is a contraction mapping. Therefore, S has a unique fixed point referring to the uniqueness of solution of (3.3). □

The subsequent result discusses uniqueness property of (1.1).

Theorem 3.3. *Assume that hypotheses $(H1)-(H5)$ and inequality (3.6) hold for all $\vartheta\in\{1,2,\dots,n\}$. Then, the problem (1.1) has a unique solution in $C(\Upsilon,\mathfrak{R})$.*

Proof. In light of the proof previously mentioned, we may conclude that (3.3) has a unique solution.Based on the above proofs, we know that (3.3) possesses a unique solution $\widetilde{u}_\vartheta\in\Pi_\vartheta$, $\vartheta\in\{1,2,\dots,n\}$. This is in accordance with Theorem 3.2.

Let us define the solution function for any $\vartheta\in\{1,2,\dots,n\}$ as

$$u_\vartheta=\begin{cases}0, & t\in[1,K_{\vartheta-1}],\\ \widetilde{u}_\vartheta, & t\in\Upsilon_\vartheta,\end{cases} \tag{3.7}$$

Thus, $u_\vartheta\in C([1,K_\vartheta],\mathbb{R})$ solves the integral equation (3.2) for $t\in\Upsilon_\vartheta$.
Then, the function

$$u(t)=\begin{cases}u_1(t), \quad t\in\Upsilon_1,\\ u_2(t)=\begin{cases}0, & t\in\Upsilon_1,\\ \widetilde{u}_2, & t\in\Upsilon_2\end{cases}\\ .\\ .\\ .\\ u_n(t)=\begin{cases}0, & t\in[1,K_{n-1}],\\ \widetilde{u}_n, & t\in\Upsilon_n\end{cases}\end{cases}.$$

is a unique solution of (1.1) in $C(\Upsilon,\mathfrak{R})$.

□

4. Example

We shall examine the following problem

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega(t))}(\log\frac{t}{\sigma})^{\omega(t)-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon:=[1,e], \tag{4.1}$$

where

$$\omega(t)=\begin{cases}\frac{3}{2}, & t\in\Upsilon_1:=[1,2],\\ \frac{6}{5}, & t\in\Upsilon_2:=]2,e].\end{cases} \tag{4.2}$$

Let

$$\begin{aligned}K_0&=1,\ K_1=2,\ K_2=K=e,\\ q(t)&=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}},\ t\in\Upsilon\\ (\Phi u)(t)&=\frac{u(t)}{1+u(t)},\ t\in\Upsilon\ and\ u\in C(\Upsilon,\mathcal{R}_+),\\ \xi(t,\sigma,u)&=\frac{(logt)^{-\frac{1}{10}}}{t+2}.\frac{1}{\sigma+10}u,\ (t,\sigma,u)\in\Upsilon^2\times\mathcal{R}_+,\end{aligned}$$

and $u\in C(\Upsilon,\mathcal{R}_+)$. It is clear that (4.1) can be written as (1.1).
By using (4.2), according to (3.3) we take into consideration the subsequent auxiliary equations:

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega_1)}(\log\frac{t}{\sigma})^{\omega_1-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon_1, \tag{4.3}$$

and

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega_2)}(\log\frac{t}{\sigma})^{\omega_2-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon_2. \tag{4.4}$$

Let us show that conditions $(H1)-(H5)$ and inequalities (3.4),(3.6) hold.
For $\vartheta=1$, we have

$$|(\Phi u)(t)-(\Phi\tilde{u})(t)|=|\frac{u(t)}{1+u(t)}-\frac{\tilde{u}(t)}{1+\tilde{u}(t)}|=|\frac{u(t)-\tilde{u}(t)}{(1+u(t))(1+\tilde{u}(t))}|\le\frac{1}{2}|u(t)-\tilde{u}(t)|.$$

It is obvious that (H2) is satisfied with $\varpi_1=\frac{1}{2}$. for each $u,\tilde{u}\in\Pi_1$ and $t\in\Upsilon_1$.

$$|(\Phi u)(t)|=|\frac{u(t)}{1+u(t)}|\le|u(t)|$$

Moreover (H3) holds with $\eta=0$ and $\gamma=1$. for each $u\in\Pi_1$ and $t\in\Upsilon_1$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)-\xi(t,\sigma,\tilde{u})|=|\frac{1}{t+2}.\frac{1}{\sigma+10}u-\frac{1}{t+2}.\frac{1}{\sigma+10}.\tilde{u}|=\frac{1}{t+2}.\frac{1}{\sigma+10}|u-\tilde{u}|\le\frac{1}{33}|z-\tilde{u}|$$

Hence, (H4) is satisfied with $\sigma=\frac{1}{10}$ and $D_1=\frac{1}{33}$. for all $(t,\sigma)\in\Upsilon_1^2$ and $u,\tilde{u}\in\Pi_1$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)|=\frac{1}{t+2}.\frac{1}{\sigma+10}.|u|\le\bar{h}(\sigma)g(|u|),$$

Then, (H5) holds with $\sigma=\frac{1}{10}$, $g(|u|)=|u|$ and $\bar{h}(\sigma)=\frac{1}{3(\sigma+10)}$ which means that $\bar{h}^\star=\frac{1}{33}$,
and the inequality

$$\frac{r_1}{q^\star+\frac{(logK_1)^{1-\sigma}-(logK_0)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_1)}(\log\frac{K_1}{K_0})^{\omega_1-1}(\eta+\gamma r_1)g(r_1)\bar{h}^\star}>1,$$

is satisfied for each $r_1\in(0.2533,\ 43.8014)$ which means that condition (3.4) holds, and the inequality

$$\Big(D_1(\eta+\gamma r_1)+\bar{h}^\star\varpi_1 g(r_1)\Big)\frac{(\log K_1)^{1-\sigma}-(\log K_0)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_1)}(\log\frac{K_1}{K_0})^{\omega_1-1}\le 1,$$

is satisfied for each $r_1 \in (0, > 29.3685)$ which means that condition (3.6) holds.
Consequently, by Theorem 3.2, (4.3) has a unique solution $\widetilde{u_1}$ in Π_1.

For $\vartheta = 2$, we have

$$|(\Phi u)(t) - (\Phi\tilde{u})(t)| = |\frac{u(t)}{1+u(t)} - \frac{\tilde{u}(t)}{1+\tilde{u}(t)}| = |\frac{u(t)-\tilde{u}(t)}{(1+u(t))(1+\tilde{u}(t))}| \leq \frac{1}{2}|u(t)-\tilde{u}(t)|$$

Then, (H2) is satisfied with $\varpi_2 = \frac{1}{2}$.
for each $u, \tilde{u} \in \Pi_2$ and $t \in \Upsilon_2$.

$$|(\Phi u)(t)| = |\frac{u(t)}{1+u(t)}| \leq |u(t)$$

Then, (H3) holds with $\eta = 0$ and $\gamma = 1$. for each $u \in \Pi_2$ and $t \in \Upsilon_2$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u) - \xi(t,\sigma,\tilde{u})| = |\frac{1}{t+2}.\frac{1}{\sigma+10}.u - \frac{1}{t+2}.\frac{1}{\sigma+10}.\tilde{u}| = \frac{1}{t+2}.\frac{1}{\sigma+10}|u-\tilde{u}| \leq \frac{1}{48}|u-\tilde{u}|$$

Hence, (H4) is satisfied with $\sigma = \frac{1}{10}$ and $D_2 = \frac{1}{48}$. for all $(t,\sigma) \in \Upsilon_2^2$ and $u, \tilde{u} \in \Pi_2$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)| = \frac{1}{t+2}.\frac{1}{\sigma+10}.|u| \leq \bar{h}(\sigma)g(|u|),$$

Then, (H5) holds with $\sigma = \frac{1}{10}$, $g(|u|) = |u|$ and $\bar{h}(\sigma) = \frac{1}{4(\sigma+10)}$, $\bar{h}^\star = \frac{1}{48}$,
and the inequality

$$\frac{r_2}{q^\star + \frac{(logK_2)^{1-\sigma}-(logK_1)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_2)}(\log\frac{K_2}{K_1})^{\omega_2-1}(\eta+\gamma r_2)g(r_2)\bar{h}^\star} > 1,$$

is satisfied for each $r_2 \in (0.2589,\ 178.3125)$ which means that condition (3.4) holds, and the inequality

$$\Big(D_2(\eta+\gamma r_2) + \bar{h}^\star\varpi_2 g(r_2)\Big)\frac{(\log K_2)^{1-\sigma}-(\log K_1)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_2)}(\log\frac{K_2}{K_1})^{\omega_2-1} \leq 1,$$

is satisfied for each $r_2 \in (0,\ 119.0476)$ which implies that condition (3.6) holds.
Consequently, by Theorem 3.2, (4.4) has a unique solution $\widetilde{u_2}$ in Π_2.
Then, according to Theorem 3.3, problem (4.1) has a unique solution

$$u(t) = \begin{cases} \widetilde{u_1}(t), & t \in \Upsilon_1, \\ u_2(t), & t \in \Upsilon_2. \end{cases}$$

where

$$u_2(t) = \begin{cases} 0, & t \in [1, K_1], \\ \widetilde{u_2}, & t \in \Upsilon_2, \end{cases}$$

5. Conclusion

In this work, we deal with integral equations in the form of quadratic Urysohn type involving Hadamard fractional variable order integral operator. A thorough study and an effective mathematical framework for fractional calculus of variable order has been presented recently. The literature contains surveys of the sorts of variable-order derivatives and integrals, along with some physical applications. Regrettably, when applied to variables of order, this attribute does not possess the semi-group property. Because of this, we are unable to simply transform the FVO differential equations into a corresponding integral equation, unlike with constant-order fractional equations. Based on our understanding of this challenge, we have utilized piece-wise constant functions to establish existence and uniqueness results. In the final stage, we apply the results of our method by constructing a numerical example.
As a future study, these findings on the Urysohn integral equation could be applied to other spaces, such as Frechet space combined with different fractional integral operators.

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References

[1] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science Tech, (2006). [CrossRef]
[2] V. Lakshmikantham, S. Leela and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientfic Publishers, Cambridge, (2009). [Web]
[3] R.K. Saxena and S.L. Kalla, *On a fractional generalization of the free electron laser equation*, Appl. Math. Comput., **143**(1) (2003), 89-97. [CrossRef] [Scopus] [Web of Science]
[4] A. Tepljakov, *Fractional-order Modeling and Control of Dynamic Systems*, Springer, (2017). [CrossRef] [Web of Science]
[5] S. Chakraverty, R.M. Jena and S.K. Jena, *Time-Fractional Order Biological Systems with Uncertain Parameters*, Synthesis Lectures on Mathematics & Statistics, Springer International Publishing, (2022). [CrossRef]
[6] H. Singh, D. Kumar, D. Baleanu, *Methods of Mathematical Modelling: Fractional Differential Equations*, Mathematics and its Applications, CRC Press, (2019). [CrossRef]
[7] J. Singh, J.Y. Hristov, Z. Hammouch, *New Trends in Fractional Differential Equations with Real-World Applications in Physics*, Frontiers Research Topics, Frontiers Media SA, 2020. [Web]
[8] V.E. Tarasov, *Mathematical economics: application of fractional calculus*, Mathematics, **8**(5) 660 (2020), 1-3. [CrossRef] [Scopus] [Web of Science]
[9] M. Yağmurlu, Y. Uçar and A. Esen, *Collocation method for the KdV-Burgers-Kuramoto equation with Caputo fractional derivative*, Fundam. Contemp. Math. Sci., **1**(1) (2020), 1-13. [Web]
[10] S.G. Samko and B. Ross, *Integration and differentiation to a variable fractional order*, Integral Transforms Spec. Funct., **1**(4) (1993), 277-300. [CrossRef] [Scopus]
[11] R. Almeida, D. Tavares and D.F.M. Torres, *The Variable-Order Fractional Calculus of Variations*, Springer, (2018). [CrossRef]
[12] H.G. Sun, A. Chang, Y. Zhang and W. Chen, *A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications*, Fract. Calc. Appl. Anal., **22**(1) (2019), 27-59. [CrossRef] [Scopus] [Web of Science]
[13] D. Sierociuk, W. Malesza and M. Macias, *Derivation, interpretation, and analog modelling of fractional variable order derivative definition*, Appl. Math. Model., **39**(13) (2015), 3876-3888. [CrossRef] [Scopus] [Web of Science]
[14] C.J. Zúñiga-Aguilar, H.M. Romero-Ugalde, J.F. Gómez-Aguilar, R.F. Escobar-Jiménez and M. Valtierra-Rodríguez, *Solving fractional differential equations of variable-order involving operators with Mittag-Leffler kernel using artificial neural networks*, Chaos Solitons Fract., **103** (2017), 382-403. [CrossRef] [Scopus] [Web of Science]
[15] J.F. Gómez-Aguilar, *Analytical and numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations*, Physica A Stat. Mech. Appl., **494** (2018), 52-75. [CrossRef] [Scopus] [Web of Science]
[16] J. Yang, H. Yao, and B. Wu, *An efficient numerical method for variable order fractional functional differential equation*, Appl. Math. Lett., **76** (2018), 221-226. [CrossRef] [Scopus] [Web of Science]
[17] D. Valério and J.S. Da Costa, *Variable-order fractional derivatives and their numerical approximations*, Signal Process., **91**(3) (2011), 470-483. [CrossRef] [Scopus] [Web of Science]
[18] K. Benia, M.S. Souid, F. Jarad, M.A. Alqudah and T. Abdeljawad, *Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order*, J. Inequal. Appl., **2023**(1) (2023), 127. [CrossRef] [Scopus] [Web of Science]
[19] D. O'Regan, S. Hristova and R.P. Agarwal, *Ulam-type stability results for variable order Ψ-tempered Caputo fractional differential equations*, Fractal Fract., **8**(1) 11 (2024), 1-15. [CrossRef] [Scopus] [Web of Science]
[20] B. Telli, M.S. Souid and I. Stamova, *Boundary-value problem for nonlinear fractional differential equations of variable order with finite delay via Kuratowski measure of noncompactness*, Axioms, **12**(1) 80 (2023), 1-23. [CrossRef] [Scopus] [Web of Science]
[21] A. Benkerrouche, S. Etemad, M.S. Souid, S. Rezapour, H. Ahmad and T. Botmart, *Fractional variable order differential equations with impulses: A study on the stability and existence properties*, AIMS Math., **8**(1) (2022), 775-791. [CrossRef] [Scopus] [Web of Science]
[22] S. Zhang, S. Li and L. Hu, *The existeness and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable derivative*, Rev. Real Acad. Cienc. Exactas Fis. Nat. - A: Mat., **113**(2) (2019), 1601-1623. [CrossRef] [Scopus] [Web of Science]

[23] A. Benkerrouche, M.S. Souid, S. Etemad, A. Hakem, P. Agarwal, S. Rezapour, S.K. Ntouyas and J. Tariboon, *Qualitative study on solutions of a Hadamard variable order boundary problem via the Ulam–Hyers–Rassias stability*, Fractal Fract., **5**(3) 108 (2021), 1-20. [CrossRef] [Scopus] [Web of Science]

[24] M.S. Souid, Z. Bouazza and A. Yakar, *Existence, uniqueness, and stability of solutions to variable fractional order boundary value problems*, J. New Theo., **41** (2022), 82-93. [CrossRef]

[25] S. Zhang and L. Hu, *The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order*, AIMS Math., **5**(4) (2020), 2923-2943. [CrossRef] [Scopus] [Web of Science]

[26] A. Refice, M.S. Souid and A. Yakar, *Some qualitative properties of nonlinear fractional integro-differential equations of variable order*, Int. J. Optim. Control: Theor. Appl., **11**(3) (2021), 68-78. [CrossRef] [Scopus] [Web of Science]

[27] S. Hristova, A. Benkerrouche, M.S. Souid and A. Hakem, *Boundary value problems of Hadamard fractional differential equations of variable order*, Symmetry **13**(5) 896 (2021), 1-16. [CrossRef] [Scopus] [Web of Science]

[28] A. Refice, M.S. Souid and I. Stamova, *On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique*, Mathematics, **9**(10) 1134 (2021), 1-16. [CrossRef] [Scopus] [Web of Science]

[29] S. Zhang, S. Sun and L. Hu, *Approximate solutions to initial value problem for differential equation of variable order*, J. Fract. Calc. Appl, **9**(2) (2018), 93-112. [CrossRef]

[30] A. Benkerrouche, D. Baleanu, M.S. Souid, A. Hakem and M. Inc, *Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique*, Adv. Differ. Equ., **2021**(1) 365 (2021). [CrossRef] [Scopus] [Web of Science]

[31] T.A. Burton, *Volterra Integral and Differential Equations*, Elsevier Science, (2005). [CrossRef]

[32] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, (2008). [CrossRef]

[33] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter Series in Nonlinear Analysis and Applications. De Gruyter, (2018). [Web of Science]

[34] E.H. Doha, M.A. Abdelkawy, A.Z.M. Amin and A.M. Lopes, *On spectral methods for solving variable-order fractional integro-differential equations*, Comput. Appl. Math., **37**(3) (2018), 3937–3950. [CrossRef] [Scopus] [Web of Science]

[35] R.M. Ganji, H. Jafari and S. Nemati, *A new approach for solving integro-differential equations of variable order*, J. Comput. Appl. Math., **379** (2020) 112946. [CrossRef] [Scopus] [Web of Science]

[36] A. Benkerrouche, M.S. Souid, G. Stamov and I. Stamova, *On the solutions of a quadratic integral equation of the Urysohn type of fractional variable order*, Entropy, **24**(7) 886 (2022), 1-14. [CrossRef] [Scopus] [Web of Science]

[37] M.H. Heydari, *Chebyshev cardinal wavelets for nonlinear variable-order fractional quadratic integral equations*, Appl. Numer. Math., **144** (2019), 190-203. [CrossRef] [Scopus] [Web of Science]

[38] M.H. Heydari, *A computational method for a class of systems of nonlinear variable-order fractional quadratic integral equations*, Appl. Numer. Math., **153** (2020), 164-178. [CrossRef] [Scopus] [Web of Science]

[39] A. Wazwaz, *First Course In Integral Equations, (A Second Edition)*, World Scientific Publishing Company, (2015). [CrossRef] [Scopus]

[40] S. Samko, Fractional integration and differentiation of variable order: an overview, Nonlinear Dyn., **71**(4) (2013) 653-662. [CrossRef] [Scopus] [Web of Science]

[41] S. Zhang, *Existence of solutions for two point boundary value problems with singular differential equations of variable order*, Electron. J. Differ. Equ., **245** (2013), 1-16. [Scopus] [Web of Science]

[42] D.R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, Cambridge University Press, (1980). [Web]

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