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Moduli Space for Invariant Solutions of Seiberg-Witten Equations

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Abstract: In this work we study the G-invariant solutions of the Seiberg-Witten equations when G is a cyclic group acting on a manifold M, preserving the metric and the orientation. G is assumed to have a lift to principle $Spin^c$ bundle which gives rise to Seiberg-Witten equations in question. In this work, we prove that when the dimension b_+^G of the G-fixed points of harmonic two forms is positive, for a generic choice of an element in this fixed point set, the moduli space of invariant solutions of Seiberg-Witten equations is a compact, smooth and oriented manifold of dimension $d^G = ind\ D_A^G - b_+^G - 1$.

Key words: Gauge Theory, Equivariant Seiberg-Witten theory, Equivariant Seiberg-Witten moduli space.

1. Introduction,

In 1949 Whitehead [1] classified simply connected, closed, oriented 4-manifolds up to orientation-preserving homotopy equivalence by their intersection form. A proof of this theorem is given in [4], page 103. Later on M. Freedman in 1982 gave a homeomorphism classification of closed, simply connected 4-manifolds [3]. His results were expressed in terms of intersection forms. However, the classical tools, like intersection forms, were not enough to detect differential structures. Differential topology of 4-manifolds is intensively studied by Simon Donaldson during the years of 1980's. Using moduli space of connections on an $SU(2)$ bundle, he introduced an invariant which detects differential structures. However, as the Yang-Mills equations are nonlinear, to make explicit computations was not easy and substantial analysis was necessary. Sometimes, instead of using this invariant, mere use of moduli space of Gauge equivalence classes of connections on a $SU(2)$ or $SO(3)$ bundle itself led to important results. One of these was a well-known theorem of Donaldson [2],[8], which states that the only negative definite, unimodular form, represented by a closed, smooth, simply connected four manifold, is the negative of the standard (diagonal) form.

In the year 1994, a set of equations, namely Seiberg-Witten equations, were introduced by Edward Witten, and with them, most of the main results of Donaldson Theory are obtained in a much shorted and simpler work.

These equations were associated to a $Spin^c(4)$ structure on the manifold in question and they were invariant under the group of bundle automorphisms of the determinant line bundle associated to this $Spin^c(4)$ structure. This group is called Gauge group. As in Donaldson theory gauge equivalence classes of solutions of Seiberg-Witten equations form a moduli space and give important information about the differential topology of the manifold. In fact, a diffeomorphism invariant, called Seiberg-Witten invariant, was introduced using this moduli space (see [5], [7], [2]).

The moduli space of Gauge equivalence classes of the solutions of the perturbed Seiberg Witten equations is compact, and in some cases, for a generic perturbation, is a zero dimensional manifold and hence consists of finitely many points. In this case, Seiberg-Witten invariant is the algebraic sum of the points in the moduli space counted with the multiplicities according to the orientation.

In this work, we construct the moduli space of solutions of Seiberg-Witten equations that are invariant under certain cyclic group action. The manifold structure is stated and then proven. As a future work, we will concentrate on the special structure near singularities.

Let G be a cyclic group of order α. Suppose G acts to preserve orientation on a closed, oriented four dimensional manifold. Choose a G-invariant Riemannian metric and a characteristic G line bundle L. We denote the associated principal $U(1)$-bundle of L by P_L and the associated principal $SO(4)$-bundle of T^*M by $P_{SO(4)}$. Let $P_{Spin^c(4)}$ be the associated principal $Spin^c(4)$-bundle whose determinant bundle is L. Assume G action on $P_{SO(4)}\ x\ P_L$ lifts to a G action on $P_{Spin^c(4)}$. Let D_A denote the Dirac operator associated to this $Spin^c(4)$-structure. Since D_A is equivariant under the action of G, the map D^G which is the restriction to the G-fixed point set of the domain of Dirac operator D makes sense.

The main theorems of this work are following.

Main Theorem 1: If $\pi_1(M) = 0$, for every choice of G-invariant self dual form $\Phi \in \Omega_+^G$, the moduli space $\tilde{M}_\Phi^G$ is compact.

Main Theorem 2: If $b_+^G > 0$,then for a generic perturbation φ in Ω_+^G, the moduli space M_φ^G of Seiberg-Witten equations perturbed by φ is an oriented smooth manifold of dimension $d^G = ind\ D_A^G - b_+^G - 1$.

2. Material and Method

2.1. Bundle Theory

Definition 1: Let G be a Lie group. A principal G -bundle is a triple $P(M, G, \pi)$ where P is a smooth manifold on which G acts from the right freely, and around each point of the smooth manifold $M = P/G$ there exists a neighborhood U so that, for the projection $\pi: P \to P/G = M$, $P|_U = \pi^{-1}(U) \cong U \times G$ isomorphic as G-spaces. P is called the total space, M is called the base space and G is called the structure group.

Theorem 1: Isomorphism classes of principal G-bundles over M are in one-to-one correspondence with the elements of $H^1(M; G)$ and also with the elements of $[M, BG]$, that is, homotopy classes of the maps from M to the classifying space BG.

Definition 2: Let F be a smooth manifold on which G acts from left. Then given a principal G-bundle $P(M, G, \pi)$ over M, we define $P_F = (P \times F)/\sim$ where $(p, f) \sim (p \bullet g, g^{-1} \bullet f)$. The bundle $P_F \to M$ is called as a fiber bundle associated to P with fiber F.

Definition 3: As a special case of the fiber bundle, defined above, if we take F to be a vector space V and via a representation $\rho: G \to GL(V)$, define a left action of G by $(g, v) \to \rho(g)(v)$. Then the fiber bundle $(P \times V)/\sim$ we get is called a vector bundle modeled on V and denoted by $(P \times_\rho V)$.

Theorem 2: Again as a special case of fiber bundle, take $F = H$ another Lie group with a group homomorphism $\rho: G \to H$. Define a left action of G on H by : $g \bullet h = \rho(g)h$ Then $P_H = P \times_\rho H$ is a principal H-bundle over M.

Definition 4: Given two principal bundles $P_1(M_1, G_1, \pi_1)$, $P_2(M_2, G_2, \pi_2)$, and a Lie group homomorphism $\gamma: G_1 \to G_2$, a map $\varphi: P_1 \to P_2$ is called a bundle map if $\varphi(p \bullet g_1) = \varphi(p) \bullet \gamma(g_1)$. Note that φ induces a map on the base spaces $\varphi^{\sim}: M_1 \to M_2$, and we have $\varphi(p_1) \in \pi_2^{-1}(\varphi^{\sim}(\pi_1(p_1)))$ for all $p_1 \in P_1$.

Given $\gamma: G_1 \to G_2$ and a bundle map $\varphi: P_1 \to P_2$ consider the map $P_1 \times_\gamma G_2 \to P_2$ defined by $[p_1, g_2] \to \varphi(p_1) \bullet g_2$. Since $[p_1 \bullet h_1, \gamma(h_1)^{-1} g_2] \to \varphi(p_1 \bullet h_1) \bullet (\gamma(h_1)^{-1} \bullet g_2) = \varphi(p_1) \bullet \gamma(h_1) \bullet \gamma(h_1)^{-1} \bullet g_2 = \varphi(p_1) \bullet g_2$, the above bundle map is well defined and hence we have P_2 is isomorphic to $P_2 \times_\gamma G_2$.

Notation: $\Gamma(E)$ denotes the space of smooth sections of the bundle: $p : E \to M$. That is, a smooth map $\psi \in \Gamma(E)$ if $\psi: M \to E$ satisfies $\rho \circ \psi(x) = x$ for all $x \in M$. We usually write $\Gamma(M)$ for $\Gamma(TM)$.

2.2. Connection and Curvature

Definition 5: A connection on a vector bundle $p : E \to M$ is a map

$$\nabla : \Gamma(M) \times \Gamma(E) \to \Gamma(E)$$

$$(X, \sigma) \to \nabla_X \sigma = \nabla(X, \sigma)$$

which satisfies the following properties:

- $\nabla_X (f\sigma + \tau) = (Xf)(\sigma) + f \nabla_X \sigma + \nabla_X \tau$
- $\nabla_{fX+Y}(\sigma) = f \nabla_X(\sigma) + \nabla_Y(\sigma)$

where $(Xf)(p) = X(p)f$ is the directional derivative.

An equivalent way of defining a connection on a vector bundle $p : E \to M$ is using the isomorphism

$$\Gamma(T^*M \otimes E) \cong \Gamma(Hom(TM, E)) \cong Hom_{C^\infty M}(\Gamma(TM), \Gamma(E));$$

It is a map

$$d^E : \Gamma(E) \to \Gamma(T^*M \otimes E) \text{ such that}$$

$$d^E\left(f|_{U_\alpha} + \tau\right) = (df) \otimes \sigma + f d^E\sigma + d^E\tau.$$

Note that, after choosing a local trivialization $(U_\alpha, g_{\alpha\beta})$ such that over U_α the bundle is trivial, i.e. $E|_{U_\alpha} = U_\alpha \times R^m$, any connection restricted to U_α is of the form $d^E|_{U_\alpha}(\sigma_\alpha)$=d$\sigma_\alpha + w_\alpha\sigma_\alpha$ where σ_α is a section over U_α and w_α is a $m \times m$ matrix of one forms on M. That is

$$d^E\begin{pmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_m\end{pmatrix} = \begin{pmatrix}d\sigma_1\\ d\sigma_2\\ \vdots\\ d\sigma_m\end{pmatrix} + \begin{pmatrix}w_1^1 & w_2^1 & \cdots & w_m^1\\ w_1^2 & w_2^2 & \cdots & w_m^2\\ \vdots & \vdots & \ddots & \vdots\\ w_1^m & w_2^m & \cdots & w_m^m\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_m\end{pmatrix}$$

Notation: $\Omega^k(E) = \Gamma(\Lambda^k(T^*\mathrm{M})\otimes\mathrm{E})$

Above definition of connection d^E can be extended to a R-linear map

$$d^E\colon \Omega^k(E) \to \Omega^{k+1}(E)$$

by tensoring with deRham complex as in [6]. For, define

$$d^E(\sigma_1 \wedge \sigma_2) = d\sigma_1 \otimes \sigma_2 + (-1)^k\sigma_1 \wedge d^E\sigma_2$$

where $\sigma_1 \in \Omega^k, \sigma_2 \in \Omega^0(E)$.

Definition 6: Curvature of a connection $d^E\colon \Omega^0(E) \to \Omega^1(E)$ on E is defined to be the $C^\infty(M)$-linear tensor $d^E{\circ}d^E\colon \Omega^0(E) \to \Omega^2(E)$.

Again, over U_α we have $d^E{\circ}d^E(\sigma_\alpha) = (dw_\propto + w_\propto \wedge w_\propto)(\sigma_\alpha)$= $\Omega_\alpha\sigma_\alpha$ where Ω_α is a matrix of two forms.

One final remark about connection and its curvature is about how they transform from U_α to U_β. In order these locally defined connections and their curvature to be well defined globally, on $U_\alpha \cap U_\beta$ we must have:

$$w_\propto = g_{\alpha\beta}dg_{\alpha\beta}{}^{-1} + g_{\alpha\beta}w_{\alpha\beta}g_{\alpha\beta}{}^{-1}\text{and}$$
$$\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}{}^{-1}$$

Theorem 3: (Hodge's Theorem): Every deRham cohomology class on a compact oriented Riemannian manifold M possesses a unique harmonic representative. Thus

$$H^p(M; R) \cong H^p(M)$$

Moreover, $H^p(M; R)$ is finite dimensional and $\Omega^p(M)$ possesses direct sum decompositions

$$\Omega^p(M) = H^p(M) \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^p(M)).$$

2.3. The Groups $SO(4)$, $Spin(4)$ and $Spin^c(4)$

Following [7], we shall consider the quaternions H as 2×2 complex matrices of the form $= \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} = \begin{pmatrix} w & -\bar{v} \\ w & \bar{w} \end{pmatrix}$. With this identification, we have

$$\tilde{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \tilde{\imath} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};\ \tilde{\jmath} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\ \tilde{k} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$
$$Q = \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} = t\tilde{1} + z\tilde{\imath} + x\tilde{\jmath} - y\tilde{k}$$

and the matrix multiplication agrees with the quaternion multiplication.

Since $\det(Q) = t^2 + x^2 + y^2 + z^2 = \langle Q, Q \rangle$ −Euclidean dot product, regarding $(t, z, x, y) \in R^4$as $t + iz + jx - ky \in H$, we can identify the unit sphere in $R^4$with the special unitary group

$$SU(2) = \{Q \in H; \langle Q, Q \rangle = 1\} = \left\{Q = \begin{pmatrix} w & -\bar{v} \\ w & \bar{w} \end{pmatrix}; \det(Q) = 1\right\}.$$

Definition 7: $Spin(4) = SU_+(2) \times SU_-(2)$, where $SU_+(2)$ and $SU_-(2)$ are copies of $SU(2)$.

Definition 8: $SO(4) = (SU_+(2) \times SU_-(2))/Z_2$.

A typical element of $Spin(4)$ will be represented by (A_+, A_-). We have special orthogonal representation

$$\rho: Spin(4) \to SO(4) = \frac{SU_+(2) \times SU_-(2)}{Z_2},$$
$$\rho(A_+, A_-)(Q) = [A_+, A_-](Q) = A_- Q A_+^{-1}.$$

In fact ρ is surjective and since both $SO(4)$ and $Spin(4)$ are compact Lie groups, it induces an isomorphism in the level of Lie algebras and hence $Spin(4) \to SO(4)$ is a covering space (double cover).

Elements of $Spin(4)$ can also be represented by the 4×4 matrices $\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$. This representation suggests that we can also consider $Spin(4)$ as a subgroup of $Spin^c(4)$where;

Definition 9: $Spin^c(4) = \left\{\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix};\ \lambda \in \mathrm{U}(1) = S^1\right\}$, which also can be defined as $Spin^c(4) = (\,Spin(4) \times \mathrm{U}(1)\,)\,/\,Z_2$ where Z_2acts diagonally.

We have representation

$$\rho^{\,c}: Spin^c(4) \to GL(H)$$

$$\rho\left(\begin{pmatrix}\lambda A_+ & 0\\ 0 & \lambda A_-\end{pmatrix}\right)(Q) = (\lambda A_-)Q(\lambda A_+)^{-1}.$$

We also have a group homomorphism:

$$\pi: Spin^c(4) \to U(1), \text{ given by}$$
$$\pi\begin{pmatrix}\lambda A_+ & 0\\ 0 & \lambda A_-\end{pmatrix}=\lambda^2.$$

2.4. $SO(4)$, $Spin(4)$ and $Spin^c(4)$ Structures on a Manifold M

Definition 10: $SO(4)$ structure is a collection $\{(U_\alpha, g_{\alpha\beta});\ \alpha, \beta \in \Lambda\}$ where U_α is an open cover of orientable 4 manifold M, $g_{\alpha\beta}: U_\alpha \cap U_\beta \to SO(4)$satisfying the cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. An alternative way of defining $SO(4)$ structure is specifying a map $f_0: M \to BSO(4)$.

Definition 11: An associated $Spin(4)$ structure to $SO(4)$ structure is a collection $\{(U_\alpha, \bar{g}_{\alpha\beta})\}$; where $\bar{g}_{\alpha\beta}: U_\alpha \cap U_\beta \to Spin(4)$ satisfying cocycle condition and $\rho \circ \bar{g}_{\alpha\beta} = g_{\alpha\beta}$, where $\rho: Spin(4) \to SO(4)$. Alternatively, an associated $Spin(4)$ structure to $SO(4)$ structure is a lifting of $f_0: M \to BSO(4)$ to $\bar{f}_0: M \to BSpin(4)$.

From the obstruction theory, it is known that the only obstruction for the existence of this lifting, that is, for the existence of $Spin(4)$ structure, i.e. a bundle with structure group

$Spin(4)$, associated to the given $SO(4)$ structure on the tangent bundle $T(M)$, is $w_2(TM) \in H^2(M, Z_2)$.

Let $W_+$and W_- be two copies of C^2. Consider the representations $\rho_+$and ρ_- given by

$$\rho_\mp\begin{pmatrix}A_+ & 0\\ 0 & A_-\end{pmatrix}(w_\mp)= A_\mp w_\mp.$$

Definition 12: Given a $Spin(4)$ structure, using the above representations ρ_+ and ρ_-, we can define new transition functions $\rho_\mp \circ \bar{g}_{\alpha\beta}: U_\alpha \cap U_\beta \to SU_\mp(2)$, to get two new complex bundles also denoted by $W_+$and W_-, called Spinor bundles and the relation between these bundles and TM is $TM \otimes \mathrm{C} \cong \mathrm{Hom}(W_+, W_-)$.

Therefore a $Spin$ structure determines $TM \otimes \mathrm{C} \cong \mathrm{Hom}(W_+, W_-)$. Moreover if we also have a line bundle L, $TM \otimes \mathrm{C} \cong \mathrm{Hom}(W_+ \otimes \mathrm{L}, W_- \otimes \mathrm{L})$, since $L \otimes L^*$ is the trivial bundle.

Given a $Spin(4)$ structure $\{(U_\alpha, \bar{g}_{\alpha\beta})\}$, if we have a line bundle L with transition functions $h_{\alpha\beta}: U_\alpha \cap U_\beta \to U(1)$ then we can define a $Spin^c(4)$ structure with the transition functions $h_{\alpha\beta} * \bar{g}_{\alpha\beta}: U_\alpha \cap U_\beta \to Spin^c(4)$, where for $x \in U_\alpha \cap U_\beta$ if $\bar{g}_{\alpha\beta}(x) = \begin{pmatrix}A_+ & 0\\ 0 & A_-\end{pmatrix}$ and if $h_{\alpha\beta}(x) = \lambda$ then $h_{\alpha\beta} * \bar{g}_{\alpha\beta}(x) = \begin{pmatrix}\lambda A_+ & 0\\ 0 & \lambda A_-\end{pmatrix}$. Note that these maps also satisfy the cocycle condition.

More generally, a $Spin^c(4)$ structure can be defined as $\bar{g}_{\alpha\beta}: U_\alpha \cap U_\beta \to Spin^c(4)$ with cocycle condition. That is one doesn't need to have a $Spin(4)$ structure or a line bundle in the first place. Combining this with π we get a complex line bundle, denoted by L^2. Finally, given a $Spin^c(4)$ structure, associated to it we can define two bundles $W^+ \otimes L$

and $W^- \otimes L$ although L may not exist. $W^{\mp} \otimes$L is the bundle whose transition data is $\rho^c_{\mp} \circ \bar{g}_{\mp}$ where $\rho^c_{\mp}\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix}(w_{\mp}) = \lambda A_{\mp} w_{\mp}$. Note that $TM \otimes \mathrm{C} \cong \mathrm{Hom}(W_+ \otimes L, W_- \otimes L)$.

Definition 13: Given an $SO(4)$ structure on $T(M)$ and $U(1)$ structure, that is, a complex structure, on line bundle L over M, an associated $Spin^c(4)$ structure is a principal $Spin^c(4)$ bundle $P \to M$ such that the associated frame bundles satisfy $P_{SO(4)}(TM) = P \times_{\rho^c} SO(4)$ and $P_{S^1}(L) = P \times_\pi S^1$, where $\rho^c\,[A_+, A_-, \lambda](Q) = [A_+, A_-](Q) = A_- Q {A_+}^{-1}$ and $\pi\,[A_+, A_-, \lambda] = \lambda^2$.

From the obstruction theory, we know that these liftings exist when L is a characteristic line bundle, in other words, when the first chern class of L is equivalent to the second Steifel Withney class of the tangent bundle (i.e, $c_1(L) \equiv w_2(TM)\ mod 2$), as the only obstruction for these liftings to exist is $w_2(TM \otimes L) \equiv c_1(L) + w_2(TM) \in H^2(M, Z_2)$.

Note that the assumption M is compact oriented smooth 4-manifold guarantees the existence of $Spin^c(4)$ structure. Moreover, the assumption that M is simply connected ensures that the liftings considered above are unique.

2.5. Gauge Group

Definition 14: A gauge transformation on a line bundle L is a bundle homomorphism $h : L \to L$ commuting with the action of the structure group $U(1)$. That is $h(g \bullet a) = g \bullet h(a)$ for all $g \in U(1)$.

The set of all gauge transformations of L forms a group, denoted by $\mathfrak{G}(L)$, under composition. This group can be considered as maps $f: M \to S^1$, see Section 1.7 of [5] for details. Hence we have $\mathfrak{G}(L) \cong \mathrm{Map}(\mathrm{M}, S^1)$.

We define an action of the gauge group $\mathfrak{G}(L)$ on $A(L)$ by $g \bullet d_A = d_A + g d g^{-1}$ which can also be expressed as $g \circ d_A \circ g^{-1}$. Action of $\mathfrak{G}(L)$ on $\Gamma(W^+)$ is just complex multiplication.

Note that if we regard $\mathfrak{G}(L)$ as $\mathrm{Map}(M, S^1)$ then, since M is simply connected, any $g \in \mathfrak{G} \cong Map(M, S^1)$ can be written as $g = e^{iu}$ for some $u: M \to R$. According to this representation, $g \bullet (\,d_A - ia, \psi) = (d_{A_0} - i(a + du), e^{iu}\,\psi)$.

Fix a base point $P_0 \in M$ and define $\mathfrak{G}_\circ(L) = \{g \in \mathfrak{G}\ (L); g(P_0\,) = 1\,\}$

We have the isomorphism $\mathfrak{G}(L) \to \mathfrak{G}_\circ(L) \times S^1$ defined by $h \to (s^{-1}h, s)$ where $s = h(P_0) \in S^1; h \in Map(M, S^1) = \mathfrak{G}(L)$.

Note that $\mathfrak{G}(L)$acts freely on $A(L)$ since $d_A + g d g^{-1}$=d_A means $g d g^{-1} = 0$ that $d g^{-1} = 0$ which holds if and only if g=constant. Elements of S^1 are constant functions $M \to S^1$. Hence S^1 acts trivially on $A(L)$, whereas it acts freely on $\Gamma(W^+ - 0)$ as complex multiplication.

Definition 15: The Dirac operator D_A: $\Gamma(W^+ \otimes L) \to \Gamma(W^+ \otimes L)$, is defined by $D_A(\psi) = \sum_{i=1}^4 e_i \bullet d_A\psi(e_i) = \sum_{i=1}^4 \rho(e_i)\,\nabla_A\,\psi(e_i)$, where d_A: $\Gamma(W \otimes L) \to \Gamma\big(T^*\mathrm{M} \otimes$

$(W \otimes L)) \cong Hom_{C^{\infty}(M)}(TM, W \otimes L)$, $e_i \in TM \otimes C \subseteq End(W \otimes L)$ and $e^i \in T^*M \otimes C$ are orthonormal basis, ∇_{e_i} is the covariant derivative along e_i.

2.6. *Seiberg – Witten Equations*

Let M be oriented, Riemannian 4-manifold with a $Spin^c(4)$ structure. We consider the pairs (d_A, ψ) where d_A is a connection on L^2 and $\psi \in \Gamma(W^+ \otimes L)$. Let $\{(d_{A_0} - ia, \psi)\}$ denote the configuration space. Seiberg-Witten equations are defined as

$$D_A^+ \psi = 0$$

$$F_A^+ = i\sigma(\psi)$$

where $F_A^+ \in \Gamma(\Omega^2(T^*M \otimes iR)) = \Omega^2(M)$.

Notation: $M^{\sim}(L)$ denotes the moduli space of $\mathcal{G}_\circ(L)$ equivalence classes of the solutions of the Seiberg-Witten equations, $M(L)$ denotes the moduli space of gauge equivalence classes of the solutions of the Seiberg-Witten equations. That is

$$M(L) = \{(A, \psi) \in A(L) \times \Gamma(W^+ \otimes L);\ D_A^+ \psi = 0\ and F_A^+ = i\sigma(\psi)\}/\mathcal{G}$$

$$= M^{\sim}(L)/S^1$$

In a similar manner, one can define the perturbed Seiberg-Witten equations:

$$D_A^+ \psi = 0$$

$$F_A^+ = i\sigma(\psi) - \Phi$$

and the corresponding perturbed moduli space $M_\Phi(L)$.

3. Results

Topology of Moduli Space of Invariant Solutions of Seiberg-Witten Equations

In this section, using the fact that Seiberg Witten equations are invariant under G-action, compactness and the manifold structure on the moduli space, whenever this structure exists, will be discussed.

Given a smooth closed 4-manifold M with a Riemannian metric on it and a characteristic line bundle L over M. Let G be a compact Lie group acting on the base manifold M to preserve the inner product and orientation, also acting on the characteristic line bundle L, commuting with the base projection and mapping fibers directly to fibers as a complex linear map. That is, let L be a G-line bundle. We will also assume that the G-action on L lifts to the associated $Spin^c(4)$ bundle whose determinant line bundle is L. Furthermore, we will take G a cyclic group of order α we will also assume that M/G has a positive definite intersection form, and that $H^1(M/G; R) = 0$. Note that since G is finite and preserves the orientation, M/G is a real homology manifold, that is M/Gsatisfies Poincare duality with coefficients in R. Hence $\frac{M}{G}$ has a well defined intersection form over R. Moreover Seiberg-Witten equations are invariant under the action of G.

Main Theorem 1: If M is simply connected, then for every choice of G-invariant self dual form $\Phi \in \Omega_+^G$, the moduli space $\tilde{M}_\Phi^G$ is compact.

It is known that every sequence of $\mathcal{G}_\circ(L)$ classes of solutions to the perturbed Seiberg Witten equations has a convergent subsequence. A detailed proof is given in section 3.3 of [5]. Using the continuity of the G-action, $\tilde{M}_\Phi^G$ can be identified with a closed subspace of $\tilde{M}_\Phi(L)$. Being a closed subspace of a compact space, $\tilde{M}_\Phi^G$ is also compact.

Main Theorem 2: Let M be a closed, simply connected smooth 4-manifold with a $Spin^c(4)$-structure. If dimension of G-fixed self dual two forms, that is $b_+^G > 0$ then, $M(L)_\Phi^G$ is, for a generic choice of G-invariant self-dual two form Φ, an oriented smooth manifold of dimension $d^G = ind_R\, D_A^G - b_+^G - 1$, where $ind_R\, D_A^G$denotes the index of the dirac operator D_A^G.

The existence of a reducible solution in $M^\sim(L)_\Phi^G$, which causes singularity in $M(L)_\Phi^G$ depends on the condition that $c_1(L)$ contains a connection with $F_A^+ = 0$, in turn which occurs only if $\Phi \in \Pi^G$- a subspace of Ω_+^G of codimension b_+^G. Since, by the assumption $b_+^G > 0$, these singularities are avoidable. Hence $S^1 \subseteq G$ acts freely on $M^\sim(L)_\Phi^G$. Therefore $M(L)_\Phi^G$ is an oriented smooth manifold with $dim(M(L)_\Phi^G) = dim(M^\sim(L)_\Phi^G) - 1 = ind_R\, D_A^G - b_+^G - 1$. The orientation of $M(L)_\Phi^G$ is induced from the orientation of $M^\sim(L)_\Phi^G$.

4. Conclusion

In this work we prove that, under certain conditions on the given group action on the base manifold, the compactness of the moduli space and give manifold structure and compute the dimension of it. As a future work we will concentrate on the case where $b_+^G = 0$ and try to understand special structures near singularities

Authorship contribution statement

M. Uğuz: All work done in preparation of this paper.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

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