Universal Journal of Mathematics and Applications, 7 (2) (2024) 93-101
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1409291

On Weighted Cauchy-Type Problem of Riemann-Liouville Fractional Differential Equations in Lebesgue Spaces with Variable Exponent

Mokhtar Mokhtari[1], Ahmed Refice[2], Mohammed Said Souid[1] and Ali Yakar[3*]

[1]*Department of Economic Sciences, University of Tiaret, Algeria*
[2]*Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, Algeria*
[3]*Department of Mathematics, Tokat Gaziosmanpasa University, Türkiye*
[*]*Corresponding author*

Article Info

Keywords: *Fixed point theorem, Fractional differential equations, Ulam-Hyers stability, Variable exponent Lebesgue spaces*
2010 AMS: *26A33, 34A12*
Received: *24 December 2023*
Accepted: *2 May 2024*
Available online: *16 May 2024*

Abstract

This paper aims to investigate the existence, uniqueness, and stability properties for a class of fractional weighted Cauchy-type problem in the variable exponent Lebesgue space $L^{p(\cdot)}$. The obtained results are set up by employing generalized intervals and piece-wise constant functions so that the $L^{p(\cdot)}$ is transformed into the classical Lebesgue spaces. Moreover, the usual Banach Contraction Principle is utilized, and the Ulam-Hyers (UH) stability is studied. At the final stage, we provide an example to support the accuracy of the obtained results.

1. Introduction

Lebesgue spaces with variable exponents were originally examined in Orlicz's work [1] in 1931 and then in Nakano's papers [2,3]. More specifically, [2] provides a precise characterization that describes Musielak-Orlicz spaces, however, it appears that Orlicz is mostly focused on the completeness of the function spaces. Afterward, a Russian researcher named Sharapudinov in [4] individually improved variable exponent Lebesgue spaces (VELS) on the real line. In the early 1900s, Kováčik and Rákosník in [5] detailed the essential characteristics of Lebesgue and Sobolev spaces with variable exponents. Actually, this paper has a major effect on subsequent papers and was accepted as the norm reference providing the current basic properties. The authors offered a suitable counterpart of the Lebesgue spaces L^p and of the Sobolev spaces $W^{k,p}$ and proposed the concept of $L^{p(x)}$ for functions p accepting the values on $[1,\infty]$. They also provide an application of generalized Sobolev spaces $W^{k,p(x)}$ to partial differential equations involving Dirichlet conditions with coefficients of a variable growth. A decade later, Fan and Zhao [6] deduced the same features by applying different techniques.
The basic idea behind the VELS $L^{p(\cdot)}$ is to substitute a variable exponent measurable function (VEMF) $p(.)$ into the traditional constant exponent p in classical Lebesgue spaces (CLS). As a result, we naturally expect that $L^{p(\cdot)}$ becomes a generalization for CLS L^p. Though the concept seems to be complex and challenging, it has substantial effects and implications that perfectly represent several phenomena in image processing, optimization, electrorheological(ER) fluids, etc. See [7–11] and the references therein.
A lot of papers have been published concerning the existence and uniqueness of solutions of fractional differential equations (FDEs) in the space of continuous functions $C(\Lambda,\mathscr{R})$, whereas relatively fewer articles exist studying the existence and uniqueness of solutions of FDEs in $L^p(\Lambda,\mathscr{R})$ space of integrable functions. For example; by using the well-known monotone technique combined with the method of upper and lower solutions, Derbazi et al. [12] find the existence and uniqueness of maximal and minimal solutions in $C(J,\mathscr{R})$ to an initial value problem involving ψ-Caputo fractional derivative:

$$\begin{cases} D_a^{\alpha,\psi}\kappa(s) = f(s,\kappa(s)), \ s\in J, \\ \kappa(a) = a^*. \end{cases}$$

Email addresses and ORCID numbers: **mokhtari12006@yahoo.fr, 0000-0002-2504-5278 (M. Mokhtari), ahmedrefice@gmail.com, 0000-0002-8906-6211 (A. Refice), souimed2008@yahoo.com, 0000-0002-4342-5231 (M. S. Souid) ali.yakar@gop.edu.tr, 0000-0003-1160-577X (A. Yakar)**
Cite as: M. Mokhtari, A. Refice, M. S. Souid, A. Yakar *On weighted Cauchy-Type problem of Riemann-Liouville fractional differential equations in Lebesgue spaces with variable exponent*, Univers. J. Math. Appl., 7(2) (2024), 93-101.

The existence and uniqueness of p-integrable solution in $L^p(\alpha,\beta)$ space has been discussed in [13] for Caputo FDE with a boundary condition having the form:

$$\begin{cases} D_a^{\varpi}\kappa(s) = \Phi(s,\kappa(s)), \ s\in[\alpha,\beta], \\ \gamma\kappa(\alpha)+\mu\kappa(\beta) = c. \end{cases}$$

Agarwal et al. [14] proved the existence of L^p solutions of fractional order integral equations with abstract Volterra operators in separable Banach spaces. Arshad et al. [15] have studied local and global existence results by applying a compactness-type condition for L^p solutions for fractional integral equations in Banach spaces. The existence of the solutions of FDEs in $L^{p(.)}$ actually has received little attention since we are aware of several notable challenges in that space. Dongg et al. [16] employed the Riesz-Kolmogorov theorem to get the existence and uniqueness of solutions for a Cauchy problem involving FDEs in VELS. Some qualitative properties of a boundary value problem in [17] and a terminal value problem in [18] involving Riemann-Liouville(R-L) fractional operator were discussed in $L^{p(.)}$ space with variable exponent. In a very recent work [19], the results in [17, 18] have been generalized by discussing a multi-term fractional boundary value problem in VELS. See [20–22] for the most current works regarding the subject.
In this paper, we shall investigate the following problem involving weighted Cauchy type condition in order to obtain some qualitative properties in $L^{p(.)}(\Lambda,\mathscr{R})$:

$$\begin{cases} D_0^{\varpi}\kappa(s) = \vartheta(s,\kappa(\psi(s))), \ s\in\Lambda:=[0,1], \\ s^{1-\varpi}\kappa(s)|_{s=0} = \beta, \end{cases} \tag{1.1}$$

where $0<\varpi<1$, $\vartheta(.,\kappa(.))\in L^{p(.)}(\Lambda\times\mathscr{R},\mathscr{R})$, $\kappa\in L^{p(.)}(\Lambda,\mathscr{R})$ and $\psi:\Lambda\to\Lambda$, and D_0^{ϖ} denotes the left Riemann Liouville (R-L) FDE of order ϖ in $L^{p(.)}$ defined as (see [16, 23]):

$$(D_{0^+}^{\varpi}\kappa)(s) = \frac{1}{\Gamma(1-\varpi)}\frac{d}{ds}\int_0^t (s-\rho)^{-\varpi}\kappa(\rho)d\rho, \tag{1.2}$$

where $\Gamma(.)$ is the gamma function.
On the other hand, left-sided R-L FDE of order ϖ for function $\kappa(s)$ in $L^{p(.)}$ is given by

$$I_{0^+}^{\varpi}\kappa(s) = \frac{1}{\Gamma(\varpi)}\int_0^s (s-\rho)^{\varpi-1}\kappa(\rho)d\rho.$$

The outline of the paper is as follows: Fundamental concepts and helpful lemmas that are necessary for establishing the main results are introduced in Section 2. Critical results regarding the existence of solutions in the Lebesgue space of variable exponent for the problem (1.1), under certain conditions are established in the subsequent section. The UH stability of the solution is demonstrated in the following section. The last section is dedicated to a demonstrative case that supports the obtained results.

2. Mathematical Preliminaries

Definition 2.1 ([24], [23]). *By $L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$, we express the classical space of measurable functions $\Phi:[\alpha,\beta]\to\mathscr{R}$, provided with the norm*

$$\|\Phi\|_r = \left(\int_\alpha^\beta |\Phi(s)|^p ds\right)^{\frac{1}{p}} < \infty$$

and

$$\|\Phi\|_\infty = ess\ sup_{\alpha\le s\le\beta}|\Phi(s)| \quad if \quad r=\infty.$$

Lemma 2.2 ([23]). *Let $\Phi_1,\Phi_2\in L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$ and $\varpi,\beta>0$ then the following properties of the left RL fractional integral and R-L FDE are demonstrated.*
(1) $I_{\alpha^+}^{\varpi}I_{\alpha^+}^{\beta}f_1(s) = I_{\alpha^+}^{\varpi+\beta}\Phi_1(s)$
(2) $I_{\alpha^+}^{\varpi}[\Phi_1(s)+\Phi_2(s)] = I_{\alpha^+}^{\varpi}\Phi_1(s)+I_{\alpha^+}^{\varpi}\Phi_2(s)$
(3) $D_{\alpha^+}^{\varpi}I_{\alpha^+}^{\varpi}\Phi_1(s) = \Phi_1(s)$
(4) $\|I_{\alpha^+}^{\varpi}\Phi_1\|_p \le \frac{(\beta-\alpha)^{\varpi}}{\Gamma(\varpi+1)}\|\Phi_1\|_p$.

Lemma 2.3 ([23]). *If $\Phi\in L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$, $\varpi>0$, then $I_{\alpha^+}^{\varpi}\Phi\in L^p([\alpha,\beta],\mathscr{R})$.*

Lemma 2.4 ([23]). *Let $\varpi>0$, then the differential equation*

$$D_{\alpha^+}^{\varpi}\xi = 0$$

has a unique solution

$$\xi(s) = c_1(s-\alpha)^{\varpi-1}+c_2(s-\alpha)^{\varpi-2}+\ldots+c_n(s-\alpha)^{\varpi-n}$$

$c_\omega\in\mathscr{R}$, $1\le\omega\le n$, here $n=[\varpi]+1$.

Lemma 2.5 ([23]). *Let $\alpha>0$, $\xi\in L^1(\Lambda,\mathscr{R})$, $D_{\alpha^+}^{\varpi}\xi\in L^1(\Lambda,\mathscr{R})$, then*

$$I_{\alpha^+}^{\varpi}D_{\alpha^+}^{\varpi}\xi(s) = \xi(s)+c_1(s-a)^{\varpi-1}+c_2(s-\alpha)^{\varpi-2}+\ldots+c_n(s-\alpha)^{\varpi-n}$$

where $c_\omega\in\mathscr{R}$, $1\le\omega\le n$, here $n=[\varpi]+1$.

We now recall the known Hölder inequality for integrals.

Lemma 2.6 ([25]). *Let p and ℓ satisfy $1<p<\infty, 1<\ell<\infty$, and $\frac{1}{p}+\frac{1}{\ell}=1$. If $\Phi_1 \in L^p(\Lambda,\mathscr{R})$ and $\Phi_2 \in L^{\ell}(\Lambda,\mathscr{R})$, then Φ_1,Φ_2 belongs to $L^1(\Lambda,\mathscr{R})$ and satisfies*

$$\int_\Lambda |\Phi_1\Phi_2|dx \leq \left[\int_\Lambda |\Phi_1|^p dx\right]^{\frac{1}{p}} \left[\int_\Lambda |\Phi_2|^{\ell} dx\right]^{\frac{1}{\ell}}.$$

Definition 2.7 ([26]). *Let $\Omega \subseteq \mathscr{R}^n$ be an open set in $\mathscr{R}^n$. By $L^{p(\cdot)}(\Omega)$ we denote all space of measurable functions Φ on Ω such that*

$$I_{p(.)}(\Phi)=\int_\Omega |\Phi(s)|^{p(s)}ds<\infty,$$

where $p(s)$ is a VEMF on Ω with values in $[1,\infty)$. This is a Banach space given with the norm

$$\|\Phi\|_{p(.)}=\inf\{\eta>0 : I_{p(.)}(\Phi/\eta)\leq 1\}.$$

We use the following notation:

$$p_-=\inf_{s\in\Omega} p(s), \quad p_+=\sup_{s\in\Omega} p(s)$$

$\ell(.)$ the conjugate exponent of $p(.)$:

$$\ell(.)=\frac{p(.)}{p(.)-1},$$

$\mathscr{P}(\Omega)$ is defined as the set of bounded measurable functions $p(s):\Omega\to[1,\infty)$ while $\mathscr{P}^{log}(\Omega)$ designates the set of exponents $p\in\mathscr{P}(\Omega)$ satisfying the local Log condition:

$$|p(s)-p(\rho)|\leq \frac{A_r}{-log|s-\rho|}, |s-\rho|\leq\frac{1}{2}, \quad s,\rho\in\Omega,$$

where $A_p>0$ is independent of t and ρ.
$\mathfrak{I}_{log}(\Omega)$ is the set off bounded exponents $\varpi:\Omega\to\mathscr{R}$ satisfying the local log condition.
$\mathbb{P}^{log}(\Omega)$ is a set consisting of exponents $p\in\mathscr{P}^{log}(\Omega)$ with $1<p_-\leq p_+<\infty$.
The following lemma is related to Hölder inequality in the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

Lemma 2.8 ([27]). *Let $\Omega\subseteq\mathscr{R}^n$ be an open set in $\mathscr{R}^n$ and $p(s),\ell(s)$ are two VEMF on Ω with values in $[1 \ \ \infty)$ where $1\leq p(s)\leq\infty$ and $\frac{1}{p(s)}+\frac{1}{\ell(s)}=1$. If $\Phi_1\in L^{p(\cdot)}(\Lambda)$ and $\Phi_2\in L^{\ell(\cdot)}(\Lambda)$, we have*

$$\int_\Omega |\Phi_1(s)\Phi_2(s)|ds\leq p\|\Phi_1\|_{p(.)}\|\Phi_2\|_{\ell(.)},$$

where $p=sup_{s\in\Omega}\frac{1}{p(s)}+sup_{s\in\Omega}\frac{1}{\ell(s)}$.

Theorem 2.9 ([16]). *Let $p(.)\in\mathscr{P}[0,M]$ and $0<\frac{1}{p_-}<\varpi<1$, then I_{0+}^{ϖ} is bounded in $L^{p(\cdot)}([0,M],\mathscr{R})$.*

Definition 2.10 ([28]). *Let $\Lambda\subset\mathscr{R}$, Λ is named as a generalized interval if it is either an interval or $\{a_1\}$ or $\emptyset$.*
A finite set $\mathscr{P}$ is called a partition of Λ if each x inn Λ lies in exactly one of the generalized intervals E in $\mathscr{P}$.
A function $p:\Lambda\to\mathscr{R}$ is named by piece-wise constant as regards to partition $\mathscr{P}$ of Λ if for any $E\in\mathscr{P}$, p is constant on E.

Definition 2.11 ([29]). *The problem (1.1) is Ulam-Hyers(UH) stable if there exists $c_\vartheta>0$, such that for any $\varepsilon>0$ and for each solution $y\in L^p(\Lambda,\mathscr{R})$ of the following inequality*

$$|D_{0+}^{\varpi}y(s)-\vartheta(s,y(s))|\leq\varepsilon, \ s\in\Lambda \tag{2.1}$$

there exists a solution $\kappa\in L^p(\Lambda,\mathscr{R})$ of problem (1.1) with

$$|y(s)-\kappa(s)|\leq c_\vartheta\varepsilon, \ s\in\Lambda.$$

3. Existence and Uniqueness of Solutions

Let us begin with the following assumption:

(H1) Let the finite sequence of points $\{M_\omega\}_{\omega=0}^n$ satisfy $0=M_0<M_\omega<M_n=1$, and Λ_ω be defined as $\Lambda_\omega=(M_{\omega-1}, \ M_\omega]$, $\omega=1,2,...,n$, $n\in\mathbb{N}$. Then $\mathscr{P}=\bigcup_{\omega=1}^n\Lambda_\omega$ would be a partition of the interval Λ.
For each $\omega=1,2,...,n$, the notation $\Upsilon_\omega=L^{p_\omega}(\Lambda_\omega,\mathscr{R})$ denotes the Banach space of VEMF from Λ_ω into $\mathscr{R}$ equipped with the norm:

$$\|\kappa\|_{\Upsilon_\omega}=\left(\int_{\Lambda_\omega}|\kappa|^{p_\omega}dx\right)^{\frac{1}{p_\omega}}<\infty,$$

where $1\leq\omega\leq n$.
Let $p(s):\Lambda\to[1,\infty)$ be a piece-wise constant function with regard to $\mathscr{P}$, i.e., $p(s)=\sum_{\omega=1}^n p_\omega I_\omega(s)$, where $1\leq p_\omega<\infty$ are constants and I_ω is the indicator of the interval Λ_ω, $\omega=1,2,...,n$

$$I_\omega(s)=\begin{cases}1, & for\ s\in\Lambda_\omega,\\ 0, & for\ elsewhere.\end{cases}$$

So, for any $s\in\Lambda_\omega$, $1\le\omega\le n$, the left R-L FDE for the function defined by (1.2), can be written as

$$(D_{0^+}^{\varpi}\kappa)(s)=\frac{1}{\Gamma(1-\varpi)}\frac{d}{ds}\int_0^s(s-\rho)^{-\varpi}\kappa(\rho)d\rho$$

$$=\frac{1}{\Gamma(1-\varpi)}\Big(\sum_{\iota=1}^{\omega-1}\frac{d}{ds}\int_{M_{\iota-1}}^{M_\iota}(s-\rho)^{-\varpi}\kappa(\rho)d\rho+\frac{d}{ds}\int_{M_{\omega-1}}^{s}(s-\rho)^{-\varpi}\kappa(\rho)d\rho\Big). \tag{3.1}$$

Thus, the problem (1.1) can be explained for any $s\in\Lambda_\omega, 1\le\omega\le n$ in the form:

$$\frac{1}{\Gamma(1-\varpi)}\Big(\sum_{\iota=1}^{\omega-1}\frac{d}{ds}\int_{M_{\iota-1}}^{M_\iota}(s-\rho)^{-\varpi}\kappa(\rho)d\rho+\frac{d}{ds}\int_{M_{\omega-1}}^{s}(s-\rho)^{-\varpi}\kappa(\rho)d\rho\Big)=\vartheta(s,\kappa(\psi(s))) \tag{3.2}$$

Let the function $\kappa\in L^{p_\omega}(\Lambda_\omega)$ with $\kappa\equiv 0$ on $s\in[0,M_{\omega-1}]$ and it solves integral equation (3.2).
Then, (3.2) is reduced to

$$(D_{M_{\omega-1}}^{\varpi}\kappa)(s)=\vartheta(s,\kappa(\psi(s))),\ \ s\in\Lambda_\omega.$$

For any $1\le\omega\le n$, we look at the following auxiliary weighted Cauchy type problem of constant order :

$$\begin{cases}D_{M_{\omega-1}}^{\varpi}\kappa(s)=\vartheta(s,\kappa(\psi(s))),\ s\in\Lambda_\omega,\\ s^{1-\varpi}\kappa(s)|_{s=M_{\omega-1}}=b.\end{cases} \tag{3.3}$$

Lemma 3.1. *Let $1\le\omega\le n$ be a natural number, $0<\varpi<1$, $\vartheta\in L^{p_\omega}(\Lambda_\omega\times\mathscr{R},\mathscr{R})$. A function $\kappa_\omega\in\Upsilon_\omega$ is a solution of (3.3) if and only if $\kappa_\omega\in\Upsilon_\omega$ solves the integral equation*

$$\kappa_\omega(s)=bs^{\varpi-1}+\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,\kappa_\omega(\psi(\rho)))d\rho \tag{3.4}$$

Proof. To show the necessity, we can write from (3.3)

$$s^{1-\varpi}\kappa_\omega(s)=b+s^{1-\varpi}I_{M_{\omega-1}}^{\varpi}\vartheta(s,\kappa_\omega(\psi(s))).$$

which implies

$$s^{1-\varpi}\kappa_\omega(s)|_{t=M_{\omega-1}}=b.$$

Also, applying $I_{M_{\omega-1}}^{1-\varpi}$ on both sides of (3.4), then

$$I_{M_{\omega-1}}^{1-\varpi}\kappa_\omega(s)=b_0+I_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))).$$

Differentiating both sides of order one, we achieve

$$D_{M_{\omega-1}}^{\varpi}\kappa_\omega(s)=\vartheta(s,\kappa_\omega(\psi(s)))$$

Inversely, let κ_ω be a solution of (3.3), by integrating both sides, then

$$I^{1-\varpi}\kappa_\omega(s)-I^{1-\varpi}\kappa_\omega(s)|_{t=0}=I_{M_{\omega-1}}^{1}\vartheta(s,\kappa_\omega(\psi(s))).$$

Operating by $I_{M_{\omega-1}}^{\varpi}$ on both sides of the last equation, we have

$$I\kappa_\omega(s)-I^{\varpi}C=I_{M_{\omega-1}}^{1+\varpi}\vartheta(s,\kappa_\omega(\psi(s))).$$

taking the ordinary derivative of the first order, it follows that

$$\kappa_\omega(s)-C_1s^{\varpi-1}=I_{M_{\omega-1}}^{\varpi}\vartheta(s,\kappa_\omega(\psi(s))),$$

By recalling the initial condition, we find that $C_1=b$, then we obtain (3.4), i.e., problem (3.3) and equation (3.4) are equivalent to each other. $\square$

Banach Contraction Principle (BCP) is implemented to arrive at the conclusion of the following result.

Theorem 3.2. *Suppose that Lemma 3.1 is satisfied and we have a constant $M>0$ such that $|\vartheta(s,\kappa_1)-\vartheta(s,\kappa_2)|\le M|\kappa_1-\kappa_2|$, for any $\kappa_1,\ \kappa_2\in L^{p_\omega}(\Lambda_\omega)$ $s\in\Lambda_\omega$ and moreover the inequality*

$$W_{\varpi,M,p_\omega,M_{\omega-1},M_\omega}<1, \tag{3.5}$$

holds where

$$W_{\varpi,M,p_\omega,M_{\omega-1},M_\omega}=\Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}}.$$

Then, for every $1\le\omega\le n$ there exists a unique solution on Λ_ω for the problem (3.3).

Proof. We use a transformation for the problem (3.4) so that it returns to a fixed point problem. Let the operator

$$S : L^{p_\omega}(\Lambda_\omega,\mathscr{R}) \to L^{p_\omega}(\Lambda_\omega,\mathscr{R})$$

which is given by

$$(S\kappa_\omega)(s) = bs^{\varpi-1} + \frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,\kappa_\omega(\psi(\rho)))d\rho.$$

BCP is used as the main tool to determine that S has a unique fixed point. To do that, let $\kappa_\omega, x_\omega \in L^{p_\omega}(\Lambda_\omega)$, then we have

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\|^{p_\omega} &= \|\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho,x_\omega(\rho))\big)d\rho\|^{p_\omega}\\
&= \int_{M_{\omega-1}}^{M_\omega}\Big|\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\Big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho,x_\omega(\rho))\Big)d\rho\Big|^{p_\omega}ds\\
&\le \frac{1}{(\Gamma(\varpi))^{p_\omega}}\int_{M_{\omega-1}}^{M_\omega}\Big|\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\Big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho,x_\omega(\rho))\Big)d\rho\Big|^{p_\omega}ds\\
&\le \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}}\int_{M_{\omega-1}}^{M_\omega}\Big(\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\Big|\kappa_\omega(\rho)) - x_\omega(\rho))\Big|d\rho\Big)^{p_\omega}ds\\
&\le \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}}\int_{M_{\omega-1}}^{M_\omega}\Bigg[\Big(\int_{M_{\omega-1}}^{s}(s-\rho)^{\ell_\omega(\varpi-1)}d\rho\Big)^{\frac{1}{\ell_\omega}}\times\Big(\int_{M_{\omega-1}}^{s}\Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega}d\rho\Big)^{\frac{1}{p_\omega}}\Bigg]^{p_\omega}ds.
\end{aligned}$$

Observe that we have utilized the Hölder Inequality. If we proceed with calculations

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\|^{p_\omega} &\le \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}}\int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\Big(\int_{M_{\omega-1}}^{s}\Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega}d\rho\Big)ds\\
&\le \Big[\frac{M}{(\ell_\omega(\varpi-1)+1)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Big]^{p_\omega}\int_{M_{\omega-1}}^{M_\omega}(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}\times\Big(\int_{M_{\omega-1}}^{s}\Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega}d\rho\Big)ds.
\end{aligned}$$

After rearranging the integrals, we reach at

$$\int_{M_{\omega-1}}^{M_\omega}(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}\Big(\int_{M_{\omega-1}}^{s}\Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega}d\rho\Big)ds = \int_{M_{\omega-1}}^{M_\omega}(s-M_{\omega-1})^{\theta_\omega}\sigma_\omega(s)ds = I,$$

where

$$\theta_\omega = \frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega},\ \ \sigma_\omega(s) = \int_{M_{\omega-1}}^{s}\Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega}d\rho.$$

Integrating by parts formula yields

$$\begin{aligned}
I &= \frac{(M_\omega - M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma_\omega(M_\omega) - \int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds\\
&= \frac{(M_\omega - M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma_\omega(M_\omega) - \int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds.
\end{aligned}$$

Since the integral

$$\int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds \ge 0$$

then,

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\| &\le \Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega}\frac{(M_\omega - M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}}[\sigma_\omega(M_\omega)]^{\frac{1}{p_\omega}}\\
&= \Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega}\frac{(M_\omega - M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}}\|\kappa_\omega - x_\omega\|_{\Upsilon_\omega}.
\end{aligned}$$

As a result, by (3.5), the operator S is a contraction. Therefore, by BCP, S has a unique fixed point $\widetilde{\kappa_i} \in L^{p_\omega}(\Lambda_\omega)$, that yields the unique solution of the problem (3.3). □

We are now ready to prove the existence result for (1.1).
Let us consider the following condition:

(H2) There exist a constant $M > 0$ such that,
$|\vartheta(s,\kappa_1) - \vartheta(s,\kappa_2)| \le M|\kappa_1 - \kappa_2|$, for any $\kappa_1,\ \kappa_2 \in L^{p(.)}(\Lambda)$ and $s \in \Lambda$.

Theorem 3.3. *Assume that (H1), (H2) and inequality (3.5) fulfill for all $1 \le \omega \le n$. Then, problem (1.1) has at most a solution in $L^{p(\cdot)}(\Lambda)$*

Proof. As mentioned in Theorem 3.2, for each $1 \le \omega \le n$, (3.3) possesses a unique solution $\widetilde{\kappa} \in \Upsilon_\omega$.
For any $1 \le \omega \le n$ we give the function as follow;

$$\kappa_\omega = \begin{cases} 0, & if\ s \in [0\ \ M_{\omega-1}], \\ \widetilde{\kappa}, & if\ s \in \Lambda_\omega. \end{cases}$$

Therefore, $\kappa_\omega \in L^p([0,\ \ M_{\omega-1}], \mathscr{R})$ is a solution for the integral equation (3.2) for $s \in \Lambda_\omega$ meaning that it solves (3.3) for $s \in \Lambda_\omega$.
Then the function:

$$\kappa(s) = \begin{cases} \kappa_1(s) \in L^{p_1}(\Lambda_1, \mathscr{R}), \\ \kappa_2(s) \in L^{p_2}(\Lambda_2, \mathscr{R}), \\ . \\ . \\ . \\ \kappa_n(s) \in L^{p_n}(\Lambda_n, \mathscr{R}). \end{cases}$$

is a unique solution of the problem (1.1) in $L^{p(\cdot)}(\Lambda)$. □

4. Ulam-Hyers Stability

Theorem 4.1. *Assume that $(H1), (H2)$, and inequality (3.5) hold. Then, (1.1) is* ***UH*** *stable .*

Proof. Take ε as an arbitrary positive number and the function $y(s)$ from $y \in L^{p_\omega}(\Lambda_\omega, \mathscr{R})$ satisfy inequality (2.1).
For any $\omega \in \{1, 2, ..., n\}$ we define the functions $y_1(s) \equiv y(s), s \in [0, M_1]$ and for $\omega = 2, 3, ..., n$

$$y_\omega(s) = \begin{cases} 0, & s \in [0, M_{\omega-1}], \\ \\ y(s), & s \in \Lambda_\omega. \end{cases} \tag{4.1}$$

According to equality (3.1) for any $\omega \in \{1, 2, ..., n\}$ and $t \in \Lambda_\omega$ we get

$$(D^{\varpi}_{0^+} y_\omega)(s) = \frac{1}{\Gamma(1-\varpi)} \frac{d}{ds} \int_{M_{\omega-1}}^{s} (s-\rho)^{-\varpi} y(\rho) d\rho.$$

Taking $I^{\varpi}_{M^+_{\omega-1}}$ of both sides of the inequality (2.1), we get

$$\left| y_\omega(s) - bs^{\varpi-1} - \frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1} \vartheta(\rho, y_\omega(\psi(\rho))) d\rho \right| \le \varepsilon \frac{(s-M_{\omega-1})^{\varpi}}{\Gamma(\varpi+1)} \le \varepsilon \frac{(M_\omega - M_{\omega-1})^{\varpi}}{\Gamma(\varpi+1)}.$$

According to Theorem 3.3, (1.1) has a unique solution $\kappa \in L^{p(\cdot)}(\Lambda)$ defined by $\kappa(s) = \kappa_\omega(s)$ for $s \in \Lambda_\omega,\ \omega = 1, 2, ..., n$, where

$$\kappa_\omega = \begin{cases} 0, & s \in [0, M_{\omega-1}], \\ \widetilde{\kappa}_\omega, & s \in \Lambda_\omega, \end{cases} \tag{4.2}$$

and $\widetilde{\kappa}_\omega \in \Upsilon_\omega$ is a unique solution of problem (3.3).
In view of Lemma 3.1, the integral equation

$$\widetilde{\kappa}_\omega(s) = bs^{\varpi-1} + \frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1} \vartheta(\rho, \widetilde{\kappa}_\omega(\psi(\rho))) d\rho \tag{4.3}$$

holds.
For $t \in \Lambda_\omega,\ \omega = 1, 2, ..., n$, by (4.1), (4.2) we have,

$$|y(s) - \kappa(s)| = |y(s) - \kappa_\omega(s)| = |y_\omega(s) - \widetilde{\kappa}_\omega(s)|$$

Then, by (4.3) we get

$$\begin{aligned}
\|y-\kappa\|_{\Upsilon_\omega}^{p_\omega} &= \|y-\kappa_\omega\|_{\Upsilon_\omega}^{p_\omega} = \|y_\omega-\widetilde{\kappa}_\omega\|_{\Upsilon_\omega}^{p_\omega}\\
&= \|y_\omega(s)-bs^{\varpi-1}-\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))d\rho\|_{\Upsilon\omega}^{p_\omega}\\
&\le \|y_\omega(s)-bs^{\varpi-1}-\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,y_\omega(\rho))d\rho\|_{\Upsilon_\omega}^{p_\omega}+\|\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}(\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho)))d\rho\|_{\Upsilon_\omega}^{p_\omega}\\
&\le \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}(\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))|d\rho)^{p_\omega}ds\\
&\le \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{r_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}\Big[\Big(\int_{M_{\omega-1}}^{s}(s-\rho)^{\ell_\omega(\varpi-1)}d\rho\Big)^{\frac{1}{\ell_\omega}}\times\Big(\int_{M_{\omega-1}}^{s}\Big|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))\Big|^{p_\omega}d\rho\Big)^{\frac{1}{p_\omega}}\Big]^{p_\omega}ds\\
&\le \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\Big(\int_{M_{\omega-1}}^{s}\Big|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))\Big|^{p_\omega}d\rho\Big)ds\\
&\le \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\Big[\frac{M^{p_\omega}}{\Gamma^{p_\omega}(\varpi)\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Big]\|y_\omega-\widetilde{\kappa}_\omega\|_{\Upsilon_\omega}^{p_\omega}\\
&\le \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\tau\|y-\kappa\|_{\Upsilon_\omega}^{p_\omega},
\end{aligned}$$

where

$$\tau=\max_{\omega=1,2,\ldots,n}\Big[\frac{M^{p_\omega}}{\Gamma^{p_\omega}(\varpi)\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Big].$$

Then,

$$\|y-\kappa\|_{\Upsilon_\omega}\le\frac{(M_\omega-M_{\omega-1})^{\frac{\varpi p_\omega+1}{p_\omega}}}{(1-\tau)^{\frac{1}{p_\omega}}\Gamma(\varpi+1)}\varepsilon.$$

We obtain,

$$\|y-\kappa\|_p\le\frac{1}{\Gamma(\varpi+1)}\Big(\sum_{\omega=1}^{\omega=n}\frac{(M_\omega-M_{\omega-1})^{\frac{\varpi p_\omega+1}{p_\omega}}}{(1-\tau)^{\frac{1}{r_\omega}}}\Big)\varepsilon:=c_\vartheta\varepsilon.$$

Therefore, the (1.1) is **UH** stable. □

5. Example

Consider the flowing fractional weighted Cauchy type problem:

$$\begin{cases} D^{0.5}\kappa(s)=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))},\ s\in\Lambda:=[0,1],\\ s^{0.5}\kappa(s)=0.\end{cases} \tag{5.1}$$

Let

$$\vartheta(s,\psi(\kappa))=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))},\ s\in[0,1].$$

Then, we have

$$\begin{aligned}
|\vartheta(s,\psi(x))-\vartheta(s,\psi(\kappa))| &= \frac{1}{(2+e^s)}\Big|\frac{x}{1+x}-\frac{\kappa}{1+\kappa}\Big|\\
&= \frac{|x-\kappa|}{(2+e^s)(1+x)(1+\kappa)}\\
&\le \frac{|x-\kappa|}{2+e^s}\\
&\le \frac{1}{2}|x-\kappa|.
\end{aligned}$$

Thus the condition (H2) is satisfied with $M=\frac{1}{2}$.
Let

$$p(s)=\begin{cases} p_1=4,\ \ if\ s\in[0,0.5],\\ p_2=5,\ \ if\ s\in]0.5,1].\end{cases} \tag{5.2}$$

According to (3.3), we consider two auxiliary 3.3, the problem (5.1) is equivalent to the followings problems:

$$\begin{cases} D^{0.5}\kappa(s) = \frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))}, \ s \in \Lambda_1 := [0, 0.5], \\ s^{0.5}\kappa(s) = 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} D^{0.5}\kappa(s) = \frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))}, \ s \in \Lambda_2 :=]0.5, 1], \\ s^{0.5}\kappa(s) = 0. \end{cases} \tag{5.4}$$

Next, we demonstrate that (3.5) is satisfied for $\omega = 1, p_1 = 4$. Indeed,

$$W_{\varpi,M,p_1,M_0,M_1} = 0,172681927 < 1.$$

As a consequence, the inequality (3.5) is satisfied.
Thus, in light of Theorem (3.2), the (5.3) has a unique solution $\widetilde{\kappa}_1 \in L^4(\Lambda_1, \mathscr{R})$.
We have revealed that the inequality (3.5) is valid for $\omega = 2, p_2 = 5$. Indeed,

$$W_{\varpi,M,p_2,M_1,M_2} = 0,202489255 < 1.$$

Then, the inequality (3.5) is fulfilled.
Taking into account Theorem 3.2, the (5.4) provides a unique solution. $\widetilde{\kappa}_2 \in L^5(\Lambda_2, \mathscr{R})$.
Hence, in view of Theorem (3.3), the (5.1) possesses a unique solution.

$$\kappa(s) = \begin{cases} \widetilde{\kappa}_1(s) \in L^4(\Lambda_1, \mathscr{R}), \\ \kappa_2(s) \in L^5(\Lambda_2, \mathscr{R}), \end{cases}$$

where

$$\kappa_2(s) = \begin{cases} 0, \quad s \in \Lambda_1, \\ \widetilde{\kappa}_2(s), \quad s \in \Lambda_2. \end{cases}$$

Clearly, one can show that by Theorem 4.1, solution of problem (5.1) is **UH** stable.

6. Conclusion

We investigate some qualitative properties of a weighted Cauchy problem (1.1) in Lebesgue spaces with variable exponent $L^{p(\cdot)}$. Our main proofs are based on exploiting the generalized intervals and piece-wise constant functions that transform $L^{p(\cdot)}$ to the classical Lebesgue spaces. Additionally, we support the theoretical results by constructing a numerical example.
There have been only a few investigations conducted in this area due to the complex structure of the variable exponent Lebesgue spaces. As a result, the fundamental results provided in this paper offer several opportunities for further investigations.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private, or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

Availability of Data and Materials: Not applicable.

References

[1] W. Orlicz, *Über konjugierte exponentenfolgen,* Studia Mathematica, **3**(1) (1931), 200-211.
[2] H Nakano, *Modular Semi-Ordered Spaces*, Maruzen Co. Ltd., Tokyo, Japan, 1950.
[3] H. Nakano, *Topology and Topological Linear Spaces,* Maruzen Co., Ltd., Tokyo, 1951.
[4] I. I. Sharapudinov, *Topology of the space* $L^{p(t)}(0,1)$, Matematicheskie Zametki, **26**(4) (1979), 613-632.
[5] O. Kovavcik, J. Rakosnik, *On spaces* $l^{p(x)}$ *and* $w^{k,p(x)}$, Czechoslovak Math. J., **41**(116) (1991), 592-618.
[6] X.L. Fan, D. Zhao, *On the spaces* $l^{p(x)}(\omega)$ *and* $w^{k,p(x)}(\omega)$, J. Math. Anal. Appl., **263**(2) (2001), 424-446.
[7] R. Aboulaich, D. Meskine, A. Souissi, *New diffusion models in image processing,* Comput. Math. Appl., **56**(4) (2008), 874-882.

[8] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, MOS-SIAM series on optimization, 2014.

[9] E. M. Bollt, R. Chartrand, S. Esedoğlu, P. Schultz, K. R. Vixie, *Graduated adaptive image denoising: Local compromise between total variation and isotropic diffusion,* Adv. Comput. Math., **31** (2009), 61-85.

[10] Y. Chen, W. Guo, Q. Zeng, Y. Liu, *A nonstandard smoothing in reconstruction of apparent diffusion coefficient profiles from diffusion weighted images*, Inverse Probl. Imaging, **2**(2) (2008), 205-224.

[11] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66**(4) (2006), 1383-1406.

[12] C. Derbazi, Z. Baitiche, M. Benchohra, A. Cabada, *Initial value problem for nonlinear fractional differential equations with ψ-caputo derivative via monotone iterative technique*, Axioms, **9**(2) (2020), 57.

[13] J. G. Abdulahad, S. A. Murad, *Local existence theorem of fractional differential equations in Lp space,* Raf. J. Comp. Maths, **9**(2) (2012), 71-78.

[14] R. P. Agarwal, V. Lupulescu, D. O'Regan, *L^p solutions for a class of fractional integral equations*, J. Integral Equ. Appl., **29**(2) (2017), 251-270.

[15] S. Arshad, V. Lupulescu, D. O'Regan, *L^p solutions for fractional integral equations,*, Fract. Calc. Appl., **17**(1) (2014), 259-276.

[16] B. Dong, Z. Fu, and J. Xu, *Riesz-Kolmogorov theorem in variable exponent Lebesgue spaces and its applications to Riemann-Liouville fractional differential equations,*, Sci. China Math., **61**(10) (2018), 1807-1824.

[17] A. Refice, M. Inc, M. S. Hashemi, M. S. Souid, *Boundary value problem of Riemann-Liouville fractional differential equations in the variable exponent Lebesgue spaces Lp (.)*, J. Geom. Phys., *178* (2022), 104554.

[18] A. Refice, M. S. Souid, Juan L.G. Guirao, H. Günerhan, *Terminal value problem for Riemann-Liouville fractional differential equation in the variable exponent Lebesgue space $L^{p(\cdot)}$* , Math. Meth. Appl. Sci., (2023), 1–19.

[19] M. S. Souid, A. Refice, K. Sitthithakerngkiet, *Stability of p (.)-integrable solutions for fractional boundary value problem via piecewise constant functions,* Fractal Fract., **7**(2) (2023), 198.

[20] K. Benia, M.S. Souid, F. Jarad, M. Alqudah, T. Abdeljawad, *Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order*, J. Inequal. Appl., **2023**(2023), 127.

[21] D. O'Regan, R. Agarwal, S. Hristova, M. Abbas, *Existence and stability results for differential equations with a variable-order generalized proportional Caputo fractional derivative*, Mathematics, **12**(2) (2024), 233.

[22] D. O'Regan, S. Hristova, R. Agarwal, *Ulam-type stability results for variable order Ψ-tempered Caputo fractional differential equations*, Fractal Fract., **8**(1) (2023), 11.

[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, Elsevier Science Tech, 2006.

[24] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis*, Springer Basel, 2013.

[25] H. Royden, P. Fitzpatrick, *Real Analysis*. Pearson Modern Classics for Advanced Mathematics Series, Pearson, 2017.

[26] V. S. Guliyev, S. G. Samko, *Maximal, potential, and singular operators in the generalized variable exponent Morrey spaces on unbounded sets*, J. Math. Sci., **193**(2) (2013), 228-248.

[27] H. Rafeiro, S. Samko, *Variable exponent campanato spaces*, J. Math. Sci., **172**(1) (2011), 143-164.

[28] A. Benkerrouche, M. S. Souid, K. Sitthithakerngkiet, A. Hakem, *Implicit nonlinear fractional differential equations of variable order,* Bound. Value Probl., **2021** (2021), 64.

[29] M. Benchohra, J. E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative,* Stud. Univ. Babes-Bolyai Math., **62**(1) (2017), 27-38.