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RESEARCH ARTICLE

ON SEMI-EXHAUSTIVENESS, SEMI-UNIFORM CONVERGENCE AND KOROVKIN-TYPE THEOREMS

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Abstract

In this study, we scrutinize the Korovkin-type theorems based on various forms of convergence, such as almost uniform convergence, semi-uniform convergence, and the concept of semi-exhaustiveness. Since it is known that the convergence types mentioned above are between point-wise and uniform convergence, it will be noticed that the circumstances can be mitigated in the Korovkin theorem.

Keywords

Function sequences,
Convergence,
Exhaustiveness,
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1. INTRODUCTION

Let (X,d) and (Y,ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. In this study, there are two main types of convergence of sequences of functions: "exhaustiveness" and "almost uniform convergence". In 2008, V. Gregoriades and N. Papanastassiou introduced the notion of exhaustiveness which is closely connected to the notion of equicontinuity as follows:

Definition 1.1. [16] The sequence (f_n) is called *exhaustive* at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in B_d(x_0,\delta)$ and all $n \geq n_0$ we have that $\rho(f_n(x), f_n(x_0)) < \varepsilon$, where $B_d(x_0,\delta)$ is the ball with radius δ centered at x_0 according to the metric d.

The concept of exhaustiveness allows us to understand the convergence of a sequence of functions based on properties of the sequence itself, rather than properties of individual functions within the sequence [16]. In the following years, many generalizations of this concept were carried out. Z.H. Toyganozu and S. Pehlivan introduced the concept of exhaustiveness in the context of asymmetric metric spaces and examined several of its properties [21]. A. Caserta and Lj.D.R. Kočinac defined statistical versions of notions, exhaustiveness and weak exhaustiveness. Moreover, they presented several findings regarding the continuity of the statistical pointwise limit of a sequence of functions and elucidated the relationships between st-exhaustiveness and other forms of st-convergence [7]. E. Athanassiadou et al. introduced

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and examined the fundamental properties of I-exhaustiveness and I-convergence in sequences of real-valued functions, providing certain characterizations [4]. Subsequently, H. Albayrak and S. Pehlivan [1] introduced the concepts of $\mathcal{F}$-exhaustiveness, where $\mathcal{F}$ represents a filter on $\mathbb{N}$. See also [15].

The concept "almost uniform convergence" was defined by J. Ewert in 1993 [13]:

Definition 1.2. The sequence (f_n) is called *almost uniformly convergent* at x_0 to a function f and denoted by "$f_n \overset{a.u.}{\to} f$ at x_0" if for every $\varepsilon > 0$ there exists $\delta(\varepsilon, x_0) > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in B_d(x_0, \delta)$ implies $\rho(f_n(x), f(x)) < \varepsilon$.
Ewert provided instances of this form of convergence that exist between the concepts of uniform convergence and quasi-uniform convergence. Ewert also proved in which cases these are equivalent concepts [13]. R. Drozdowski et al. discussed Ewert's concept of "almost uniform convergence" with the same name but with a different approach [11].

Korovkin's Theorem stands as one of the fundamental theorems in constructive approximation theory [18]. While the original theorem was given according to the concept of uniform convergence, in recent years it has been given according to many different concepts of convergence and summability methods. A seminal paper discussing Korovkin-type theorems in the context of statistical convergence can be found in [14]. In the paper by K. Demirci et al. [9], the concept of relative uniform convergence of a sequence of functions at a specific point was introduced and they utilized this new form of convergence to prove a Korovkin-type approximation theorem. Additionally, they delved into the investigation of convergence rates in their study. Numerous studies have also been conducted on Korovkin-type theorems that are closely linked to convergence connected with summability methods, statistical convergence and filter convergence ([3,5,6,10,12,14,17,19,22,23]).

As of 2020, with the work of N. Papanastassiou [20], in addition to the ones mentioned above, semi-types of many convergence types for function sequences have been defined and their relationships with each other have been examined. See for example [8].

This paper focuses on dealing the Korovkin-type theorems that are contingent upon the semi-types of "exhaustiveness" and "almost uniform convergence". Since it is known that the convergence types mentioned above are between point-wise and uniform convergence, it will be noticed that the circumstances can be mitigated in the classical Korovkin's Theorem.

2. DEFINITIONS AND AUXILIARY RESULTS

Let (X, d) and (Y, ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. For $x_0 \in X$ and $\delta > 0$, $B_d(x_0, \delta)$ denotes the ball with radius δ centered at x_0 according to the metric d. Let us recall the definitions of exhaustiveness, semi-exhaustiveness, almost uniform convergence and semi-uniformly convergence.

Definition 2.1. [20] The sequence (f_n) is called *semi-exhaustive* at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that for all $x \in B_d(x_0, \delta)$ we have that $\rho(f_m(x), f_m(x_0)) < \varepsilon$.

From the definition, an exhaustive sequence of functions is semi-exhaustive, although the reverse implication may not hold. In [20] (Remark 4.2 (2) (Example 3.3)), the given example is accidentally overlooked. For correction, one can use the following example:

Example 2.2. Let $f_n: (-1,1) \to \mathbb{R}$, $f_n(x) = \begin{cases} nx, & n \text{ is odd} \\ x/n, & n \text{ is even} \end{cases}$. It is evident that while (f_n) is semi-exhaustive at $x = 0$ but lacks exhaustiveness at the same point.

Proposition 2.3. Let $x_0 \in X$. The sequence (f_n) is semi-exhaustive at x_0 iff there exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is exhaustive at x_0.

Proof.
Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ are given. Let's assume we have a sequence of positive integers, denoted by (n_k), which strictly increases such that (f_{n_k}) is exhaustive at x_0. From exhaustiveness there exists $\delta > 0$ and $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f_{n_k}(x_0)) < \varepsilon$. Since $n_k \geq n_{k^*} \geq k^*$ so that $n_{k^*+n} \geq k^* + n > n$ for all $k \geq k^*$, then if we choose $m = n_{k^*+n}$ for all $n \in \mathbb{N}$ then for all $x \in B(x_0, \delta)$ we have

$$\rho(f_m(x), f_m(x_0)) = \rho\left(f_{n_{k^*+n}}(x), f_{n_{k^*+n}}(x_0)\right) < \varepsilon.$$

Now, assume that the sequence (f_n) semi-exhaustive at x_0. From here we construct the desired subsequence (n_k) as follows: From the Definition 2.1., there exists $n_1 \geq 1$ such that $\rho(f_{n_1}(x), f_{n_1}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Similarly, there exists $n_2 \geq n_1 + 1$ such that $\rho(f_{n_2}(x), f_{n_2}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. If it continues in this way, there exists $n_k \geq n_{k-1} + 1$ such that $\rho(f_{n_k}(x), f_{n_k}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Consequently, we get a strictly increasing sequence of positive integers (n_k) such that f_{n_k} is exhaustive at x_0.

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Definition 2.4. [22] The sequence (f_n) is called *uniformly exhaustive* on X if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x, y \in X$ that satisfy $d(x, y) < \delta$ implies $\rho(f_n(x), f_n(y)) < \varepsilon$.

Definition 2.5. The sequence (f_n) is called *semi-bounded* on X if there exists $M > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that the sequence $\rho(f_m(x), 0) \leq M$ for all $x \in X$.

Definition 2.6. The sequence (f_n) is called *semi-boundedly exhaustive* at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that

i. $\rho(f_m(x), f(x_0)_m) < \varepsilon$ for all $x \in B(x_0, \delta)$
ii. $\rho(f_m(x_0), 0) < M$

where $M > 0$ is a constant independent from ε and n.

Definition 2.7. The sequence (f_n) is called *almost uniformly bounded* on X if there exists $n_0 \in \mathbb{N}$ and $M > 0$ such that $\rho(f_n(x), 0) \leq M$ for all $n \geq n_0$ and all $x \in X$.

Remark 2.8. It is clear that the uniform boundedness of a sequence implies almost uniformly boundedness. The inverse of this assertion is not true. For example, for $f_n: (1, \infty) \to \mathbb{R}$, $f_n(x) = x^{2n-n^2}$, the sequence (f_n) is not uniformly bounded, but almost uniformly bounded.

Definition 2.8. The sequence (f_n) is called *locally almost uniformly bounded* on X, if for all $x \in X$, there exists $\delta > 0$ such that the sequence (f_n) is almost uniformly bounded on $B_d(x, \delta)$.

Proposition 2.9. If the sequence (f_n) is exhaustive at x_0 and $(f_n(x_0))$ is bounded then (f_n) is almost uniformly bounded in a neighborhood at x_0.

Proof.
By boundedness of the sequence $(f_n(x_0))$, there exists a number $M > 0$ such that $\rho(f_n(x_0), 0) \leq M$ for all $n \in \mathbb{N}$. From exhaustiveness of the sequence (f_n) at x_0, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x \in B_d(x_0, \delta)$ we have $\rho(f_n(x), f_n(x_0)) < 1$. Since $\rho(f_n(x), 0) \leq 1 + \rho(f_n(x_0), 0) \leq 1 + M$ for all $n \geq n_0$ and all $x \in B_d(x_0, \delta)$, we get the desired result.

■

Corollary 2.10. Let the sequence (f_n) is exhaustive and pointwise bounded on X then (f_n) is locally almost uniformly bounded on X.

Proposition 2.11. If $(f_n) \overset{a.u.}{\to} f$ at x_0, then $(f_{n_k}) \overset{a.u.}{\to} f$ at x_0, for any strictly increasing sequence of positive integers (n_k).

Proof. Let strictly increasing sequence of positive integers (n_k) and $x_0 \in \mathbb{R}$ are given. From almost convergency of (f_n) to f at x_0 there exists $\delta > 0$ and $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_n(x), f(x)) < \varepsilon$ for all $\varepsilon > 0$. For given $\varepsilon > 0$ if we choose $\delta^* = \delta$ and $k^* = \min\{k : n_k \geq n^*\}$ then for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$.

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Definition 2.12. [20] The sequence (f_n) is called *semi-uniformly convergent* to a function f at x_0 if

i. $f_n(x_0) \to f(x_0)$

ii. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that for all $x \in B(x_0, \delta)$ implies $\rho(f_m(x), f(x)) < \varepsilon$.

The notation "$f_n \overset{semi-un.}{\to} f$ at x_0" will be used for semi-uniformly convergence of the sequence (f_n) to f at x_0.

Remark 2.13. Obviously, if a sequence of functions converges almost uniformly to a function at a specific point, then it can be inferred that the sequence converges semi-uniformly to the same function at the same point. Nevertheless, it should be noted that the converse statement does not hold true. For instance, considering the sequence provided in Example 2.2. even though it converges semi-uniformly to the function $f = 0$ at the point $x = 0$, it is not characterized by almost uniform convergence.

Proposition 2.14. Let $x_0 \in X$. The sequence (f_n) semi-uniformly converges to f at x_0 iff

i. $f_n(x_0) \to f(x_0)$

ii. There exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is almost uniformly convergent to f at x_0.

Proof. Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ are given. Assume that $(f_n(x_0))$ converges to $f(x_0)$ and there exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is almost uniformly convergent to f at x_0. From almost uniform convergency there exists $\delta > 0$ and $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$. Since $n_k \geq n_{k^*} \geq k^*$ so that $n_{k^*+n} \geq k^* + n > n$ for all $k \geq k^*$, then if we choose $m = n_{k^*+n}$ for all $n \in \mathbb{N}$ then for all $x \in B(x_0, \delta)$ we have $\rho(f_m(x), f(x)) = \rho\left(f_{n_{k^*+n}}(x), f(x)\right) < \varepsilon$.

Now, assume that the sequence (f_n) semi-uniformly converges to f at x_0. From here we construct the desired subsequence (n_k) as follows: From the second condition of Definition 2.12. there exists $n_1 \geq 1$ such that $\rho(f_{n_1}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Similarly, there exists $n_2 \geq n_1 + 1$ such that $\rho(f_{n_2}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. If it continues in this way, there exists $n_k \geq n_{k-1} + 1$ such that $\rho(f_{n_k}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Consequently, we get a strictly increasing sequence of positive integers (n_k) such that $f_{n_k} \overset{semi-un.}{\to} f$.

∎

With the proposition mentioned earlier serving as motivation, a specific type of semi-exhaustiveness can be defined by incorporating the concept of natural density. However, before delving into this definition, let's review the definition of natural density. For $A \subseteq \mathbb{N}$, we denote the natural density of A by

$$d(A) = \lim_{n\to\infty} \frac{|\{k\in A : k\leq n\}|}{n}$$

if the limit exists, where $|A|$ denotes of the cardinality of the finite set A. Let $A \subset \mathbb{N}$. Then A is called

- a statistically thin set is $d(A) = 0$
- a statistically thick set is $d(A) \neq 0$
- a statistically dense set if $d(A) = 1$.

It is well known that if $d(A_1) = d(A_2) = 1$ for $A_1, A_2 \subset \mathbb{N}$ then $d(A_1 \cap A_2) = 1$ [8].

Definition 2.15. It is called that the sequence (f_n) is *densely semi-exhaustive* at $x_0 \in X$ if there exists a strictly increasing sequence of positive integers (n_k), with $d(\{n_k\}) = 1$, such that (f_{n_k}) is exhaustive at x_0.

It's clear that if a function sequence is densely semi-exhaustive then it is semi-exhaustive. Reverse implication could not be true. For example, the sequence (f_n) defined by $f_n: (-1,1) \to \mathbb{R}$,

$$f_n(x) = \begin{cases} nx, & n \text{ is prime} \\ x/n, & n \text{ is non-prime} \end{cases}$$

is semi-exhaustive at $x_0 = 0$, however it is not densely semi-exhaustive at $x_0 = 0$.

Definition 2.16.
It is called that the sequence (f_n) is *densely semi-uniformly converges* to f at $x_0 \in X$ if there exists a strictly increasing sequence of positive integers (n_k), with $d(\{n_k\}) = 1$, such that (f_{n_k}) almost converges to f at x_0.

It's clear that if a function sequence is densely semi-uniformly convergent then it is semi-uniformly convergent. Reverse implication could not be true. For instance, consider the sequence (f_n) defined by: $f_n: (-1,1) \to \mathbb{R}$,

$$f_n(x) = \begin{cases} \frac{x}{n}, & n \text{ is prime} \\ nx, & n \text{ is not prime} \end{cases}$$

is semi-uniformly convergent at $x_0 = 0$, it is not densely semi-uniformly convergent at $x_0 = 0$.

Let $C(X)$ denote the space of real valued continuous functions and $B(X)$ denote the space of real valued bounded functions on the metric space (X, ρ). We will deal with the positive and linear operators defined on these spaces. The positivity of an L operator defined on these spaces will be understood as the fact that the $L(f)$ function is also positive for every positive function f. Let be $e_k(x) = x^k$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$. For $X = [a, b]$, let us give Korovkin's Theorem to deal with an approximation property of the sequences of positive and linear operators on $C(X)$:

Theorem 2.17. [18] Let (L_n) be a sequence of positive linear operators on $C[a, b]$. If the sequence $L_n(e_k)$ converges uniformly to e_k on $[a, b]$, for $k = 0,1,2$ then the sequence $L_n(f)$ converges uniformly to f on $[a, b]$ for all $f \in C[a, b]$.

In the next section, we deal with Korovkin-type theorems depending upon the kind of convergences such as almost uniform convergence, semi-uniformly convergence and the notion of semi-exhaustiveness.

3. MAIN RESULTS

Let (X,ρ) be a metric space for a bounded set $X \subset \mathbb{R}$ and $C_b(X)$ be the space of real valued, bounded and continuous functions on the metric space (X,ρ). For every $x \in X$ denote by $B(x;\delta)$, the set $\{y \in X: \rho(y,x) < \delta\}$ and by ρ_x the function $\rho_x(y) = \rho(x,y)$, $(y \in X)$. It is clear that $\rho_x \in C_b(X)$. In [2], Altomare's contribution involved extending Korovkin's Theorem to include metric spaces, thus presenting a broader and more encompassing version of the theorem.

Theorem 3.1. [2] Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators on $C(X)$ and assume that for some compact subset K of X the following properties hold true:

- $\lim_{n\to\infty} L_n(e_0) = e_0$ uniformly on K.
- $\lim_{n\to\infty} L_n(\rho_x) = 0$ uniformly on K.

Then for every function $f \in C(K)$, $lim_{n\to\infty} L_n(f) = f$ uniformly on K.
Using similar method in [2], we give the Korovkin-type theorems based on the concept of semi-exhaustiveness, almost uniform convergence and semi-uniformly convergence.

Theorem 3.2. Let (L_n) be a sequence of positive linear operators on $C(X)$ and $x_0 \in X$. If $L_n(e_0) \overset{a.u.}{\to} e_0$ and $L_n(\rho_{x_0}^r)$ is almost uniformly converges to 0 at x_0 for some $r > 0$, then $L_n(f) \overset{a.u.}{\to} f$ at x_0 for all $f \in C_b(X)$.

Proof. Let $f \in C_b(X)$ and $x_0 \in X$. By the continuity of f at x_0, there exists $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

holds for all $t \in X$ that satisfies $\rho(x_0,t) < \delta$. On the other hand, in the case $\rho(x_0,t) \geq \delta$, we have

$$|f(t) - f(x_0)| \leq 2\sup_{x\in X}|f(x)| \leq \frac{2M}{\delta}\rho(x_0,t)$$

where $M := \sup_{x\in X}|f(x)|$. Let $r > 0$. From the discussion above, the inequality pertaining to the set X can be written as follows:

$$|f(t) - f(x_0)| \leq \varepsilon e_0 + \frac{2M}{\delta^r}\rho_{x_0}^r.$$

By almost uniformly convergence of $(L_n(e_0))$ at x_0, there exists $\delta_1 > 0$ and $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ and for all $x \in B(x_0,\delta_1)$ we have

$$|L_n(e_0;x) - e_0(x)| < 1.$$

Also, by almost uniformly convergence of $\left(L_n(\rho_{x_0}^r)\right)$ at x_0, there exists $\delta_2 > 0$ and $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ and for all $x \in B(x_0,\delta_2)$ we have

$$L_n(\rho_{x_0}^r;x) < \frac{\varepsilon\delta^2}{6M}.$$

By utilizing the well-known properties of positive and linear operators, we can establish the following:

$$\begin{aligned}|L_n(f;x) - f(x)| &\leq L_n(|f - f(x)|;x)\\ &\leq \varepsilon L_n(e_0;x) + \frac{2M}{\delta^r}L_n(\rho_{x_0}^r;x)\\ &\leq \frac{\varepsilon}{3}|L_n(e_0;x) - e_0(x)| + \frac{2M}{\delta^r}L_n(\rho_{x_0}^r;x) + \frac{\varepsilon}{3}e_0(x)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for all $n \geq n_0$ and for all $x \in B(x_0,\delta)$ where $n_0 = \max\{n_1,n_2\}$ and $\delta_0 = \min\{\delta_1,\delta_2,\delta\}$. Consequently, we obtain the almost uniform convergence of the sequence $(L_n(f))$ to f at x_0.

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Corollary 3.3. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $L_n(e_0) \overset{a.u.}{\to} e_0$ and $L_n(\rho_{x_0}^r)$ is almost uniformly converges to 0 on X for some $r > 0$, then $L_n(f) \overset{a.u.}{\to} f$ on X for all $f \in C_b(X)$.

Example 3.4. For $X = (0,2)$, consider the operators L_n on $C_b(X)$

$$L_n(f;x) = \begin{cases} f(1) + nf(x), & x \leq 1/2^n \\ f(x), & x > 1/2^n. \end{cases}$$

It is evident that the operators L_n possess both linearity and positivity. While the sequence (L_n) does not meet the conditions of Theorem 3.1, it does satisfy the conditions outlined in Theorem 3.2.

Remark 3.5. Korovkin's Theorem is not true for the concept of semi-uniformly convergence. However, as we can see in the next theorem, it can be written for densely semi-uniformly convergence. An example is given after the next theorem.

Theorem 3.6. Let (L_n) be a sequence of positive linear operators on $C_b(X)$ and $x_0 \in X$. If the sequence $(L_n(e_0))$ densely semi-uniformly convergent to e_0 and the sequence $L_n(\rho^r_{x_0})$ densely semi-uniformly convergent to 0, for some $r > 0$, at x_0, then $L_n(f)$ densely semi-uniformly convergent to f at x_0 for all $f \in C_b(X)$.

Proof. Let $f \in C_b(X)$ and $\varepsilon > 0$ be given. Since the sequence $(L_n(e_0))$ has densely semi-uniformly convergent to e_0 at x_0, then there exists a strictly increasing sequence of positive integers $(n_k^{(1)})$, with $d(\{n_k^{(1)}\}) = 1$, such that $(L_{n_k^{(1)}}(e_0))$ is almost uniformly convergent to e_0 at x_0. Similarly, since the sequence $(L_n(\rho^r_{x_0}))$ has densely semi-uniformly convergent to 0 at x_0 then there exists a strictly increasing sequence of positive integers $(n_k^{(2)})$, with $d(\{n_k^{(2)}\}) = 1$, such that $(L_{n_k^{(2)}}(\rho^r_{x_0}))$ is almost uniformly convergent to 0 at x_0. Because of the densely semi-uniformly convergence implies the semi-uniformly convergence , if we take the strictly increasing sequence of positive integers 0 in the set $\{n_k^{(1)}\} \cap \{n_k^{(2)}\}$ which has natural density 1, we obtain that $L_{n_k}(e_0) \overset{a.u}{\rightarrow} e_0$ and $L_{n_k}(\rho^r_x)$ almost uniformly converges to 0 at x_0 by using Proposition 2.11. Now, the desired result follows from Theorem 3.2.

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Example 3.7. Let the linear positive operators L_n on $C[0,1]$ defined by

$$L_n(f;x) = \begin{cases} f(\frac{1}{2}), & x = \frac{1}{2} \\ \int_0^1 f(t)K_n(t,x)\,dt, & x \neq \frac{1}{2} \end{cases}$$

where $K_n(t,x) = (m+1)x^m + \frac{1}{n}\left|x - \frac{1}{2}\right|$ with $n \equiv m (\text{mod} 3)$ for $n \in \mathbb{N}$. It's obvious that $L_n(e_i) \overset{\text{semi}-u}{\rightarrow} e_i$ at $\frac{1}{2}$ for $i = 0,1,2$ but $L_n(f)$ does not semi-uniformly converge to f at $\frac{1}{2}$ for $f(x) = x^3$.

In the next theorem, let $X \subset \mathbb{R}$ be any set, bounded or unbounded.

Theorem 3.8. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-exhaustive and bounded at $x_0 \in X$, then $(L_n(f))$ is semi-exhaustive at x_0 for all $f \in C(X)$.

Proof.
Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. By semi-exhaustiveness of $(L_n(e_0))$ at x_0, there exists $\delta_0 > 0$ and for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $x \in X$ satisfying $\rho(x, x_0) < \delta_0$, we have

$$|L_m(e_0;x) - L_m(e_0;x_0)| < \frac{\varepsilon}{3(|f(x_0)|+1)} := A_1(\varepsilon).$$

By boundedness of the sequence $(L_n(e_0;x_0))$, there exists $M > 0$ such that $L_n(e_0;x_0) \leq M$. By the continuity of f at x_0, there exists $\delta_1 > 0$ such that for all $x \in X$ that satisfies $\rho(x, x_0) < \delta_1$, we get

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3(A_1(\varepsilon)+M)} := A_2(\varepsilon).$$

From properties of positive linear operators, we have

$$L_n(|f - f(x_0)|;x) < A_2(\varepsilon)|L_n(e_0;x) - L_n(e_0;x_0)| + A_2(\varepsilon)|L_n(e_0;x_0)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1\}$ and for all $n \in \mathbb{N}$, $m^* = m$ $(m^* > n)$ then for all $x \in B(x_0; \delta)$, we have

$$\begin{aligned}|L_m(f;x) - L_m(f;x_0)| &\leq |L_m(f;x) - L_m(f(x_0);x)| + |L_m(f(x_0);x) - L_m(f(x_0);x_0)| \\ &\quad +|L_m(f(x_0);x_0) - L_m(f;x_0)| \\ &\leq L_m(|f - f(x_0)|;x) + |f(x_0)||L_m(e_0;x) - L_m(e_0;x_0)| \\ &\quad +L_m(|f - f(x_0)|;x_0) \\ &\leq 2A_2(\varepsilon)(A_1(\varepsilon) + M) + |f(x_0)|A_1(\varepsilon) \\ &< \varepsilon.\end{aligned}$$

Hence $(L_m(f))$ is semi-exhaustive at x_0. ■

Theorem 3.8 can also be expressed as follows:

Theorem 3.9. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-boundedly exhaustive at $x_0 \in X$, then $(L_n(f))$ is semi-exhaustive at x_0 for all $f \in C(X)$.

Theorem 3.10. Let (L_n) be positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-exhaustive and pointwise bounded on X then $(L_n(f))$ is semi-exhaustive on X for all $f \in C(X)$.

Proof.

Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. From Proposition 2.3. there exists an increasing sequence of positive integers (n_k) such that (L_{n_k}) is exhaustive at x_0. Then by exhaustiveness of $\left(L_{n_k}(e_0)\right)$ at x_0, there exists $\delta_0 > 0$ and $k_0 \in \mathbb{N}$ such that for all $x \in X$ and for all $k \geq k_0$ that satisfy $\rho(x, x_0) < \delta_0$, we have

$$\rho\left(L_{n_k}(e_0;x), L_{n_k}(e_0;x_0)\right) < \frac{\varepsilon}{3(|f(x_0)|+1)} = A_1(\varepsilon).$$

Exhaustiveness and pointwise boundedness of $(L_n(e_0))$ on X implies locally almost uniformly boundedness from Corollary 2.10. Consequently, there is a positive real number that exists $M > 0$, $\delta_1 > 0$ and $k_1 \in \mathbb{N}$ such that for all $x \in X$ that satisfy $\rho(x, x_0) < \delta_1$ and for all $k \geq k_1$, we have $|L_{n_k}(e_0;x)| \leq M$. By the continuity of f at x_0, there exists $\delta_2 > 0$ such that for all $x \in X$ that satisfies $\rho(x, x_0) < \delta_2$, we have

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3M}.$$

From properties of positive linear operators, we have

$$L_{n_k}(|f - f(x_0)|;x) < \frac{\varepsilon}{3M}|L_{n_k}(e_0;x)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and for all $n \in \mathbb{N}$, $m = n_k (m > n_k > n)$ then for all $x \in B(x_0, \delta)$, we have

$$\begin{aligned}|L_m(f;x) - L_m(f;x_0)| &\leq |L_m(f;x) - L_m(f(x_0);x)| + |L_m(f(x_0);x) - L_m(f(x_0);x_0)| \\ &\quad + |L_m(f(x_0);x_0) - L_m(f;x_0)| \\ &\leq L_m(|f - f(x_0)|;x) + |f(x_0)||L_m(e_0;x) - L_m(e_0;x_0)| \\ &\quad + L_m(|f - f(x_0)|;x_0) \\ &\leq 2\frac{\varepsilon}{3M}|L_m(e_0;x)| + |f(x_0)|A_1(\varepsilon) \\ &< \varepsilon.\end{aligned}$$

Hence $(L_n(f))$ is semi-exhaustive at x_0. Thus $(L_n(f))$ is semi-exhaustive on X.

■

Example 3.11. Scrutinize the linear positive operators L_n on $C[-1,1]$ defined by

$$L_n(f;x) = \begin{cases} f(x)/n, & x \leq 0 \text{ and n is prime} \\ f(x)/2n, & x > 0 \text{ and n is prime} \\ f(0), & \text{n is not prime.} \end{cases}$$

It is clear that $(L_n(e_0))$ is semi-exhaustive at $x = 0$ and bounded on $[-1,1]$, so for every $f \in C[-1,1]$, $(L_n(f))$ is semi-exhaustive at $x = 0$. Indeed, for every $\varepsilon > 0$, we choose $\delta < 1/2$ and for every $n \in \mathbb{N}$

we choose m to be the least prime integer that greater than n, then $|L_m(f;x) - L_m(f;0)| < \varepsilon$ hold for all $x \in B(0,\delta)$.

Example 3.12. Consider the linear positive operators L_n on $C(0,1)$ defined by

$$L_n(f;x) = \begin{cases} f(x) + nf(x_0), & n \text{ is odd} \\ nf(x), & n \text{ is even} \end{cases}$$

and $x_0 \in (0,1)$ be fixed. For a function $f \in C(0,1)$ with $f(x_0) \neq 0$. the sequence $(L_n(f))$ does not converge uniformly on $(0,1)$, but it is semi-exhaustive on $(0,1)$.

Remark 3.13. The condition about boundedness cannot remove from Theorem 3.8.

Example 3.14. Consider the linear positive operators L_n on $C[0,1]$ defined by $L_n(f;x;) = n^2 f(x)$. It's clear that $(L_n(e_0))$ is not bounded. Although $(L_n(e_0))$ is semi-exhaustive, the sequence $(L_n(f))$ is not semi-exhaustive on $[0,1]$ for every $f \in C[0,1]$ which is not constant.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Alper Erdem: Formal analysis, Conceptualization, Investigation, Writing - original draft.
Tuncay Tunç: Supervision, Investigation, Conceptualization.

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