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A Study On A New Generalization of δ-Supplemented Modules

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Research Article

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Abstract
For any ring S and an S-module W, a submodule G of W is termed *co*$_\delta$*-coatomic* if the quotient module W/G is δ-coatomic. In this study, we introduce the term ($\oplus$-)*co*$_\delta$*-coatomically* δ*-supplemented module*, or shortly ($\oplus$-)*co*$_\delta$-δ*-supplemented module* to describe a module W where each co$_\delta$-coatomic submodule has a δ-supplement (that is a direct summand) in W. Furthermore, a module W is identified as *co*$_\delta$*-coatomically* δ*-semiperfect*, or shortly *co*$_\delta$-δ*-semiperfect*, provided each δ-coatomic quotient module of W has a projective δ-cover. It has been proved that over a δ-semiperfect ring S, the module ${}_SS$ is $\oplus_\delta$-co-coatomically supplemented if and only if ${}_SS$ is co$_\delta$-δ-semiperfect if and only if ${}_SS$ is $\oplus$-co$_\delta$-δ-supplemented.

Keywords: co$_\delta$-coatomic submodule, co$_\delta$-coatomically δ-supplemented module, $\oplus$-co$_\delta$-coatomically δ-supplemented module, co$_\delta$-coatomically δ-semiperfect module

δ-Tümlenmiş Modüllerin Yeni Bir Genelleştirilişi Üzerine Bir Çalışma

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Öz
Herhangi bir S halkası ve bir W S-modülü için, W modülünün bir G alt modülü, eğer W/G bölüm modülü δ-eşatom ise *eş*$_\delta$*-eşatom* olarak adlandırılır. Bu çalışmada, ($\oplus$-)*eş*$_\delta$*-eşatom* δ*-tümlenmiş modül*, veya kısaca ($\oplus$-)*eş*$_\delta$-δ*-tümlenmiş modül* terimini her eş$_\delta$-eşatom alt modülü (direkt toplam terimi olan) bir δ-tümleyene sahip olan bir W modülünü belirtmek için tanıtıyoruz. Ayrıca, W modülü, eğer her bir δ-eşatom bölüm modülü, projektif bir δ-örtüye sahipse *eş*$_\delta$*-eşatom* δ*-yarı mükemmel* veya kısaca *eş*$_\delta$-δ*-yarı mükemmel* olarak tanımlanır. Bir δ-yarı mükemmel S halkası üzerinde, ${}_SS$ modülünün $\oplus_\delta$-eş-eşatom tümlenmiş olmasının ${}_SS$ modülünün eş$_\delta$-δ-yarı mükemmel olmasına ve ${}_SS$ modülünün $\oplus$-eş$_\delta$-δ-tümlenmiş olmasına denk olduğu kanıtlanmıştır.

Anahtar Kelimeler: eş$_\delta$-eşatom alt modül, eş$_\delta$-eşatom δ-tümlenmiş modül, $\oplus$-eş$_\delta$-eşatom δ-tümlenmiş modül, eş$_\delta$-eşatom δ-yarı mükemmel modül

Introduction

All along the current manuscript, we regard that whole rings are associative having identity element and whole modules are unital left S-modules. Let S be a ring of such nature and W be a module falling into this category. The impressions $G \leq W$ and $G \leq_{\oplus} W$ signify that G functions as a submodule of W and G functions as a direct summand of W, respectively. Referring to a submodule G of W as *small* in W, denoted as $G \ll W$, implies that $W \neq G + L$ for each proper submodule L of W. The symbol $Rad(W)$ represents the intersection of whole maximal submodules of W or, equivalently, the sum of whole small submodules of W. Dually, a submodule G of W is classified as *essential* in W, denoted by $G \trianglelefteq W$, provided $G \cap H \neq 0$ where $0 \neq H \leq W$. A module W is termed *singular* if $W \cong W'/G$ for some module W' and $G \trianglelefteq W'$. The notion of a *projective cover* for a module W refers to a pair comprising a module P and a homomorphism $h : P \rightarrow W$, where P is projective, and h is an epimorphism with $Ker(h) \ll P$ (refer to [1]). A module W is termed *coatomic* if each proper submodule is included in a maximal submodule of W (see [2]). Examples of coatomic modules encompass finitely generated, semisimple, and local modules. Initially introduced by Alizade and Güngör in [3] the concept of co-coatomic submodules is articulated as follows. A submodule G of W is denoted as *co-coatomic* when the quotient module W/G is coatomic.
As a particular instance derived from the concept of coatomic modules, the notion of δ-coatomic modules is described in [4]. A module W is characterized as *δ-coatomic* provided, each submodule which is different from W of W is encompassed within a maximal submodule H of W where W/H is singular (refer to [4, Lemma 2.1]). In the context of [4], a ring S is denoted as *left (right) δ-coatomic* if the left (right) S-module ${}_SS$ (S_S) is δ-coatomic.
The subsequent proposition is a consequence of [4, Proposition 2.5], and we will invoke it consistently in the course of this paper.

Proposition 1. Suppose that $0 \rightarrow G \rightarrow W \rightarrow L \rightarrow 0$ is an exact sequence consisting of modules.

1. When W is δ-coatomic, L is δ-coatomic.

2. When G and L are δ-coatomic, W is δ-coatomic.

In particular, for any δ-coatomic module W, $G \leq_{\oplus} W$ is δ-coatomic.

Consider submodules G and H of a module W. The term *supplement* is attributed to H in relation to G within W if H is minimal while satisfying $W = G + H$. It is a well-established fact that H serves as a supplement of G in W if and only if $W = G + H$ and $G \cap H \ll H$. A module W is said to be a *supplemented module* provided each submodule of W possesses a supplement in W. Examples of supplemented modules include semisimple modules and hollow modules (refer to [1, Section 41]). In [3], W is termed *co-coatomically supplemented module* when each co-coatomic submodule of W possesses a supplement in W.
In [5], the authors put forth the concept that W is denoted as *$\oplus$-co-coatomically supplemented module* provided any co-coatomic submodule possesses a supplement G with $G \leq_{\oplus} W$. Furthermore, in the same paper, W is characterized as *co-coatomically semiperfect module* provided any coatomic quotient module of W admits a projective cover.

Zhou described the concept of δ-small submodules, a generalization of small submodules that play a pivotal role in the framework of supplemented modules, as presented in [6]. For a submodule $G \leq W$, G is designated as δ-small in W and is denoted by $G \ll_\delta W$ if $W \neq G + H$ holds for each proper submodule H of W where W/H is singular. In accordance with the notation in [6, Lemma 1.5], $\delta(W)$ signifies the sum of whole δ-small submodules of W.
We compile the fundamental features of submodules which are δ-small in the subsequent lemma, sourced from [6, Lemma 1.2 and 1.3].

Lemma 1. Suppose that W is a module.

1. For any submodule G of W, $G \ll_\delta W$ if and only if whenever $W = X + G$ there is a semisimple projective submodule G' of G with $X \oplus G' = W$.
2. When $G \ll_\delta W$ and $h : W \to L$ is a homomorphism, then $h(G) \ll_\delta L$. In particular, $G \ll_\delta W \leq L$, then $G \ll_\delta L$.
3. When $G_1 \ll_\delta H_1 \leq W$ and $G_2 \ll_\delta H_2 \leq W$, $G_1 + G_2 \ll_\delta H_1 + H_2$.
4. When $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$, $\delta(W) = \bigoplus_{\lambda\in\Lambda} \delta(W_\lambda)$.
5. When $G \leq H \leq W$, $G \ll_\delta W$ and $H \leq_\oplus W$, $G \ll_\delta H$.

In [7], a module W is defined as *δ-supplemented* provided any submodule G of W possesses a δ-supplement H in W, characterized by $W = G + H$ and $G \cap H \ll_\delta H$. For more comprehensive details and characterizations of δ-supplemented modules, additional information can be found in [8] and [7].
In [9], a module W is referred to as *co-coatomically δ-supplemented* (*$\oplus_\delta$-co-coatomically supplemented*) provided any co-coatomic submodule of W possesses a δ-supplement G with $G \leq_\oplus W$. In the same paper, a module W is identified as *co-coatomically δ-semiperfect* provided any co-coatomic quotient module of W possesses a projective δ-cover. The authors thoroughly examined the structure of these modules in the same paper and provided new characterizations of rings based on these modules.
In this note, we introduce a special case of co-coatomic submodules. Let G be a submodule of a module W. We designate G as a *co$_\delta$-coatomic submodule* of W if the quotient module W/G is δ-coatomic. It is evident that δ-coatomic modules are coatomic. Consequently, each submodule which is co$_\delta$-coatomic of a module is also co-coatomic. According to [4, Corollary 2.9], since each semisimple singular module is δ-coatomic, it follows that each submodule of semisimple singular modules is co$_\delta$-coatomic.
In the other part of this paper, we delve into the introduction and examination of the concept of co$_\delta$-coatomically δ-supplemented modules, shortly co$_\delta$-δ-supplemented modules, and $\oplus$-co$_\delta$-coatomically δ-supplemented modules, shortly $\oplus$-co$_\delta$-δ-supplemented modules, as notions stronger than the previously defined co-coatomically δ-supplemented modules and $\oplus_\delta$-co-coatomically supplemented modules. We commence by presenting an example of a module that is co$_\delta$-δ-supplemented but not δ-supplemented. It is demonstrated that the class of co$_\delta$-δ-supplemented modules is closed under quotient modules and finite sums. We establish that a ring S is left δ-coatomic if and only if each simple S-module is singular, and this equivalence extends to the statement that each coatomic S-module is δ-coatomic. Consequently, it is deduced that over left δ-coatomic rings, co-coatomically δ-supplemented modules and co$_\delta$-δ-supplemented modules coincide. Additionally, an example is provided of a ring S over which

each co$_\delta$-δ-supplemented S-module is co-coatomically δ-supplemented. It is established that the quotient module of a $\oplus$-co$_\delta$-δ-supplemented module by a fully invariant submodule is also $\oplus$-co$_\delta$-δ-supplemented. Consequently, it is proven that for a $\oplus$-co$_\delta$-δ-supplemented module W, the quotient module $W/\delta(W)$ is $\oplus$-co$_\delta$-δ-supplemented. Moreover, it is shown that the class of $\oplus$-co$_\delta$-δ-supplemented modules is closed under finite direct sums.

In the last part of this article, we introduce the definitions of co$_\delta$-coatomically δ-semiperfect modules, shortly co$_\delta$-δ-semiperfect modules, as a generalization of co-coatomically δ-semiperfect modules. It is demonstrated that for a projective module W, W is co$_\delta$-δ-semiperfect if and only if W is $\oplus$-co$_\delta$-δ-supplemented. It is established that the class of co$_\delta$-δ-semiperfect modules is closed under quotient modules and δ-covers. Furthermore, it is shown that the finite direct sum of projective co$_\delta$-δ-semiperfect modules is co$_\delta$-δ-semiperfect if and only if each direct summand is co$_\delta$-δ-semiperfect. Additionally, it is proven that for a δ-semiperfect ring S, $_SS$ is $\oplus_\delta$-co-coatomically supplemented if and only if $_SS$ is co$_\delta$-δ-semiperfect if and only if $_SS$ is $\oplus$-co$_\delta$-δ-supplemented.

Co$_\delta$-Coatomically δ-Supplemented Modules

Definition 1. We term a module W *co$_\delta$-coatomically δ-supplemented, shortly co$_\delta$-δ-supplemented* provided each co$_\delta$-coatomic submodule of W possesses a δ-supplement in W.W is named *$\oplus$-co$_\delta$-coatomically δ-supplemented, shortly $\oplus$-co$_\delta$-δ-supplemented* provided each co$_\delta$-coatomic submodule of W has a δ-supplement G with $G \leq_\oplus W$.

It is evident that co-coatomically δ-supplemented modules are co$_\delta$-δ-supplemented, as co$_\delta$-coatomic submodules are co-coatomic. In the subsequent discussion, we will provide an example of a ring for which co$_\delta$-δ-supplemented modules are co-coatomically δ-supplemented. It is apparent that modules which are δ-supplemented are also co$_\delta$-δ-supplemented. However, the example which will be given next illustrates that a co$_\delta$-δ-supplemented module does not necessarily need to be δ-supplemented.

Example 1. Let's consider the $\mathbb{Z}$-module $\mathbb{Q}$. Given that $\mathbb{Q}$ has only $\mathbb{Q}$ as a co$_\delta$-coatomic submodule, $\mathbb{Q}$ qualifies as a co$_\delta$-δ-supplemented module. However, it's important to note that $\mathbb{Q}$ is not a δ-supplemented module.

Proposition 2. Suppose that W is a δ-coatomic module. In that case, W is co$_\delta$-δ-supplemented if and only if W is δ-supplemented.

Proof. The sufficiency is evident. To prove the necessity, let G be any submodule of W. Since W is δ-coatomic, then W/G is δ-coatomic by Proposition 1. Thus G has a δ-supplement in W, by the assumption. Hence W is δ-supplemented.

Corollary 1. Suppose that W is a semisimple singular module. Then W is co$_\delta$-δ-supplemented if and only if W is δ-supplemented.

Proof. Note that by [4, Corollary 2.9], W is δ-coatomic. Thus the result is derived from Proposition 2.

Corollary 2. For a module W, assume that $\delta(W) \ll_\delta W$ and W satisfies either

1. $W/\delta(W)$ is semisimple, or

2. For each submodule G of W, there exists a submodule H of W such that $W = G + H$ and $G \cap H \ll_\delta W$.

Then W being co$_\delta$-δ-supplemented is equivalent to being δ-supplemented.

Proof. W is δ-coatomic module by [4, Theorem 2.2]. Thus the result is supported by Proposition 2.

Proposition 3. Co$_\delta$-δ-supplemented modules exhibit transfer properties through their quotient modules.

Proof. Suppose that W is a co$_\delta$-δ-supplemented module and $G \leq W$. Then any co$_\delta$-coatomic submodule of W/G is a submodule of the form L/G where L is co$_\delta$-coatomic submodule of W. By the hypothesis, L has a δ-supplement in W, say X. According to [8, Proposition 2.3], we achieve that
$(X + G)/G$ is a δ-supplement of L/G in W/G.

Corollary 3. The property of being co$_\delta$-δ-supplemented module is transferred by direct summands.

Proposition 4. Suppose that W is a co$_\delta$-δ-supplemented module. In that case, each co$_\delta$-coatomic submodule of the module $W/\delta(W)$ is a direct summand.

Proof. Suppose that $G/\delta(W)$ is a co$_\delta$-coatomic submodule of $W/\delta(W)$. Then G is co$_\delta$-coatomic submodule of W. By the hypothesis, there exists a submodule H of W such that $W = G + H$ and $G \cap H \ll_\delta H$. Note that $G \cap H \leq \delta(W)$. Thus
$W/\delta(W) = G/\delta(W) + (H + \delta(W))/\delta(W)$,
$(G/\delta(W)) \cap ((H + \delta(W))/\delta(W)) = ((G \cap H) + \delta(W))/\delta(W) = 0$.
Hence $W/\delta(W) = (G/\delta(W)) \oplus (H + \delta(W))/\delta(W)$.

Corollary 4. Suppose that W is a co$_\delta$-δ-supplemented module. In that case, $W/\delta(W)$ is $\oplus$-co$_\delta$-δ-supplemented.

Next, we aim demonstrating that the sum of a finite number of co$_\delta$-δ-supplemented modules becomes a co$_\delta$-δ-supplemented module. To begin with, we establish the following lemma as a preliminary step.

Lemma 2. Suppose that W is a module and $G, H \leq W$. Assume that G is co$_\delta$-coatomic, H is co$_\delta$-δ-supplemented and $G + H$ possesses a δ-supplement in W. Then G possesses a δ-supplement in W.

Proof. Let X be a δ-supplement of $G + H$ in W. Note that
$H/(H \cap (G + X)) \cong (G + H + X)/(G + X) = W/(G + X)$
is δ-coatomic as a quotient module of the δ-coatomic module W/G. Thus there exists a submodule H' of H such that
$H = H' + (H \cap (G + X))$ and $H' \cap (H \cap (G + X)) = H' \cap (G + X) \ll_\delta H'$. Therefore we have $W = G + H + X = G + (H' + H \cap (G + X)) + X = G + H' + X$ and $G \cap (H' + X) \leq H' \cap (G + X) + X \cap (G + H') \leq H' \cap (G + X) + X \cap (G + H) \ll_\delta H' + X$. Hence $H' + X$ is a δ-supplement of G in W.

Theorem 1. A finite sum of co$_\delta$-δ-supplemented modules remains co$_\delta$-δ-supplemented.

Proof. It is sufficient to prove that the sum, say W, of co$_\delta$-δ-supplemented modules W_1 and W_2 is co$_\delta$-δ-supplemented. For this, assume that G is a co$_\delta$-coatomic submodule of W. Then $W = W_1 + W_2 + G$. Note that $(W/G)/((W_2 + G)/G) \cong W/(W_2 + G)$ is δ-coatomic. Since $W_2 + G$ is co$_\delta$-coatomic submodule of W and W_1 is co$_\delta$-δ-supplemented, then $W_2 + G$ has a δ-supplement in W by Lemma 2. By using Lemma 2 once more again, we conclude that G has a δ-supplement in W because W_2 is co$_\delta$-δ-supplemented module and G is a co$_\delta$-coatomic submodule of W. Hence W is co$_\delta$-δ-supplemented.
Consider modules W and L. L is designated as *finitely W-generated* in case there exists an epimorphism $f : W^{(\Lambda)} \to L$ where Λ is a finite set.
The subsequent corollary naturally follows from Proposition 3 and Theorem 1.

Corollary 5. In case W is co$_\delta$-δ-supplemented module, then any finitely W-generated module is a co$_\delta$-δ-supplemented module.

Theorem 2. Let G be a co$_\delta$-δ-supplemented submodule of a module W such that W/G has no maximal submodule. Then W is a co$_\delta$-δ-supplemented module.

Proof. Let L be a co$_\delta$-coatomic submodule of W. Then $W/(G + L)$ is δ-coatomic as a quotient module of the δ-coatomic module W/L . Thus $W/(G+L)$ is coatomic. Since W/G has no maximal submodule, $W/(G + L)$ also has no maximal submodule. Thus $W = G + L$. By Lemma 2, L has a δ-supplement in W. Hence W is co$_\delta$-δ-supplemented module.

Corollary 6. Suppose that W is a module and $W/Soc(W)$ has no maximal submodule. Then W is co$_\delta$-δ-supplemented module.

As a reminder from [10], a module W is termed *δ-local* in case $\delta(W) \ll_\delta W$ and the submodule $\delta(W)$ is maximal in W.

Proposition 5. Let W be a co$_\delta$-δ-supplemented module. If W contains a maximal submodule H with singular W/H, then W contains a δ-local or projective semisimple submodule.

Proof. Assume that W is a co$_\delta$-δ-supplemented module and H is a maximal submodule of W such that W/H is singular. Then H is co$_\delta$-coatomic submodule of W. By the assumption, there exists a submodule X of W such that $W = H + X$ and $H \cap X \ll_\delta X$. Therefore, X is δ-local or semisimple projective by [11, Lemma 2.22].

Lemma 3. Suppose that W is a module and H is a maximal submodule that contains whole semisimple singular submodules of W. If W/H is singular and X is a δ-supplement of H in W, then X is δ-local.

Proof. By the hypothesis, $W = H + X$ and $H \cap X \ll_\delta X$. Assume that $H \cap X$ is not an essential submodule of X. Then there exists a submodule X' of X such that $(H \cap X) \cap X' = 0$, and so $X = (H \cap X) \oplus X'$. Thus $X' \cong X/(H \cap X) \cong W/H$ is simple singular, by the assumption. Also we get $W = H+X = H+(H\cap X)\oplus X' = H+X'$. It leads to the conclusion that H does not contain X'. This contradicts with the hypothesis. Hence $H \cap X$ is an essential submodule of X, and so $\delta(X) \leq H \cap X$. Hence $\delta(X) = H \cap X$.

Proposition 6. The expressions below are equivalent for a δ-coatomic module W:

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1. W is co$_\delta$-δ-supplemented.

2. Each maximal submodule H of W with singular W/H has a δ-supplement in W.

3. $W = W_1 + W_2 + ... + W_n$ where each W_λ is either simple or δ-local.

Proof. When W is semisimple singular, then these stated equivalences above are evidently valid. So, we posit that W is not semisimple singular.
$(1) \Longrightarrow (2)$ It is evident.
$(2) \Longrightarrow (3)$ Let X be the sum of whole submodules which are δ-supplement of maximal submodules H of W with singular W/H and $Soc(W) \leq H$. Then by Lemma 3, X is a sum of δ-local submodules of W. Now assume that $W \neq Soc(W) + X$. Since W is δ-coatomic module, then there is a maximal submodule G of W such that $Soc(W) + X \leq G$ and W/G is singular. By the hypothesis, G possesses a δ-supplement Y in W. Therefore, $Y \leq X \leq G$ and $G = W$, but this is a contradiction. Hence, $W = Soc(W) + X$ and the result holds.
$(3) \Longrightarrow (1)$ It is supported by [10, Lemma 3.3] that δ-local modules are δ-supplemented, and so co$_\delta$-δ-supplemented. Moreover, it is evident that simple modules are co$_\delta$-δ-supplemented. Hence W is co$_\delta$-δ-supplemented as a finite sum of co$_\delta$-δ-supplemented modules by Theorem 1.

Corollary 7. Suppose that W is a δ-coatomic module. In case W is δ-supplemented, $W = W_1 + W_2 + ... + W_n$ where each W_λ is either simple or δ-local.

Proposition 7. Suppose that S is a ring. The expressions below are equivalent:

1. S is a left δ-coatomic ring.

2. Each simple S-module is singular.

3. Each coatomic S-module is δ-coatomic.

Proof. $(1) \Longrightarrow (2)$ It is supported by [4, Theorem 2.18].
$(2) \Longrightarrow (3)$ Let W be a coatomic module. Then each submodule which is proper of W is included in a maximal submodule G. By the assumption, W/G is singular. Thus we conclude that each submodule which is proper of W is included in a maximal submodule G where W/G is singular. This means that W is δ-coatomic.
$(3) \Longrightarrow (1)$ By (3), the module $_SS$ is δ-coatomic. Hence S is a left δ-coatomic ring.

Corollary 8. Suppose that S is a δ-coatomic ring and W is an S-module. In that case, the expressions below are equivalent:

1. W is co-coatomically δ-supplemented.

2. W is co$_\delta$-δ-supplemented.

Proof. $(1) \Longrightarrow (2)$ It is evident.
$(2) \Longrightarrow (1)$ Suppose that G is a co-coatomic submodule of W, that is, W/G is coatomic. Then W/G is δ-coatomic by Proposition 7, as S is a left δ-coatomic ring, by the assumption. Thus $G \leq W$ is co$_\delta$-coatomic. As W is co$_\delta$-δ-supplemented, G possesses a δ-supplement in W, as desired.

Heed the information derived from [6] that, concerning the modules P and W, an epimorphism $h : P \to W$ is denominated a *δ-cover* of W under the condition that $Ker(h) \ll_\delta P$. A δ-cover $h : P \to W$ is labeled as a *projective δ-cover* provided that P is a projective module. A ring S is characterized as *δ-semiperfect* in case each simple S-module possesses a projective δ-cover (refer to [6]).

Corollary 9. Suppose that S is a δ-semiperfect ring and W is an S-module. In that case, the expressions below are equivalent:

1. W is co-coatomically δ-supplemented.
2. W is co$_\delta$-δ-supplemented.

Proof. Since δ-semiperfect rings are left δ-coatomic by [4, Proposition 2.15], then the result is supported by Corollary 8.

Corollary 10. The expressions below are equivalent for a δ-semiperfect ring S:

1. $_SS$ is ($\oplus_\delta$-co-coatomically supplemented) co-coatomically δ-supplemented.
2. $_SS$ is ($\oplus$-co$_\delta$-δ-supplemented) co$_\delta$-δ-supplemented.

Presently, we provide an instance of a ring where each co$_\delta$-δ-supplemented S-module is simultaneously co-coatomically δ-supplemented.

Example 2. Suppose that S is an incomplete discrete valuation ring and Q denotes its field of fractions. Let p be the maximal ideal of S. Then p is essential in S. Hence the simple S-module S/p is singular. Hence each simple S-module is singular. So S has no simple projective S-modules. Also as stated in the [11, Example 2.2], δ-small submodules are small over the ring S. Therefore over this ring, modules which are co-coatomically δ-supplemented are co-coatomically supplemented, and also co$_\delta$-δ-supplemented modules are co-coatomically δ-supplemented by Proposition 7. Moreover, $Q \oplus Q$ is co$_\delta$-δ-supplemented as it is supplemented by [12, Theorem 2.2].

Consider a module W. A submodule G within W is denoted as *fully invariant* if $\gamma(G) \leq G$ for any endomorphism γ of W. In case each submodule of W is fully invariant, W is termed *duo* (refer to [13]).

Proposition 8. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module and G is a fully invariant submodule of W. In that case, the quotient module W/G is $\oplus$-co$_\delta$-δ-supplemented.

Proof. Assume that L/G is a co$_\delta$-coatomic submodule of W/G. Then $(W/G)/(L/G) \cong (W/L)$ is δ-coatomic, and hence L is a co$_\delta$-coatomic submodule of W. By the assumption, there exists a decomposition $W = X \oplus X'$ of W such that $W = L + X$ and $L \cap X \ll_\delta X$. This leads to the conclusion that $(G + X)/G$ is a δ-supplement of L/G in W/G by [8, Proposition 2.3]. Therefore, we conclude that $G = (G \cap X) \oplus (G \cap X')$, since G is fully invariant by [13, Lemma 2.1]. This decomposition implies that $W/G = ((G + X)/G) \oplus ((G + X')/G)$. Hence W/G is $\oplus$-co$_\delta$-δ-supplemented.

Retrieve from [6, Lemma 1.5(2)] the fact that the submodule $\delta(W)$ of a module W is fully invariant.

Corollary 11. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module. In that case, $W/\delta(W)$ is $\oplus$-co$_\delta$-δ-supplemented.

Corollary 12. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented duo module. In that case each quotient module of W is $\oplus$-co$_\delta$-δ-supplemented.

Proposition 9. Each co$_\delta$-coatomic fully invariant direct summand of a $\oplus$-co$_\delta$-δ-supplemented module is $\oplus$-co$_\delta$-δ-supplemented.

Proof. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module and $G \leq_\oplus W$ is co$_\delta$-coatomic fully invariant. Let X be a co$_\delta$-coatomic submodule of G. By the assumption, there exists a δ-coatomic submodule G' of W such that $W = G \oplus G'$. Thus we have that $W/X = ((G \oplus G')/X) \oplus G' \cong G/X \oplus G'$ is δ-coatomic since it is a direct sum of two δ-coatomic modules by [4, Proposition 2.6]. Therefore, by the assumption there is a submodule T which is δ-supplement of X in W such that $W = T \oplus T'$. By the modular law, we get that $G = G \cap (X + T) = X + (G \cap T)$. Note here that $G = (G \cap T) \oplus (G \cap T')$ by [13, Lemma 2.1], since G is a fully invariant submodule of W. So, $G \cap T \leq_\oplus G$. Since $X \cap (G \cap T) = X \cap T \ll_\delta T$, then $X \cap (G \cap T) \ll_\delta G \cap T$ by Lemma 1-(5). Hence G is $\oplus$-co$_\delta$-δ-supplemented.

Proposition 10. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module and G is a submodule of W. Assume that for each $X \leq_\oplus W$, $(G + X)/G \leq_\oplus W/G$. Then W/G is a $\oplus$-co$_\delta$-δ-supplemented.

Proof. Assume that L/G is a co$_\delta$-coatomic submodule of W/G. Then L is a δ-coatomic submodule of W. By the assumption, there exists a decomposition $W = X \oplus X'$ of W such that $W = L + X$ and $L \cap X \ll_\delta X$. Thus we get that $W/G = (L/G) + ((G + X)/G)$. By the hypothesis, $(G + X)/G \leq_\oplus W/G$. On the other hand, since $L \cap X \ll_\delta X$, $(L \cap (G + X))/G = (G + (L \cap X))/G = \pi(L \cap X) \ll_\delta \pi(X) = (G + X)/G$ where $\pi : W \to W/G$ is a canonical projection by Lemma 1-(2). This completes the proof.

As per the definition in [14], a module W is designated to possess the *Summand Sum Property (SSP)* in case the sum $W_1 + W_2 \leq_\oplus W$ where $W_1, W_2 \leq_\oplus W$.

Proposition 11. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module. In case W possesses (SSP), each $G \leq_\oplus W$ is $\oplus$-co$_\delta$-δ-supplemented.

Proof. Assume that $W_1 \leq_\oplus W$ such that $W = W_1 \oplus W_2$. Regarding any $X \leq_\oplus W$, we achieve that $W = (X + W_1) \oplus K$ for some submodule K of W, as W has (SSP) by the assumption. Note that $W/W_2 = ((X + W_1)/W_2) \oplus ((K + W_2)/W_2)$. Here according to Proposition 10, W/W_2 is $\oplus$-co$_\delta$-δ-supplemented module.

Referencing [15], it can be recalled that a module W is termed *distributive* if, for submodules G, H, L of W, the equality $G \cap (H + L) = (G \cap H) + (G \cap L)$ or $G + (H \cap L) = (G + H) \cap (G + L)$ is satisfied.

Proposition 12. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented module and G is a submodule of W. In case W is distributive, W/G is $\oplus$-co$_\delta$-δ-supplemented.

Proof. The claim can be proved by using Proposition 10. For this, take $K \leq_\oplus W$. Then we have the decomposition $W = K \oplus K'$ for some $K' \leq W$. Thus $W/G = ((G + K)/G) + ((G + K')/G)$. By the assumption, since W is distributive, then $G = G + (K \cap K') = (G + K) \cap (G + K')$. We infer from here that $W/G = ((G + K)/G) \oplus ((G + K')/G)$. Thus W/G is $\oplus$-co$_\delta$-δ-supplemented module by Proposition 10.

Refer to [16] for the reminder that a module W is termed an (D_3)-*module* in case the intersection of direct summands, the sum of which yields W, is also a direct summand of W.

Proposition 13. Suppose that W is a $\oplus$-co$_\delta$-δ-supplemented (D_3)-module. In that case, each co$_\delta$-coatomic $K \leq_\oplus W$ is $\oplus$-co$_\delta$-δ-supplemented.

Proof. Assume that $K \leq_\oplus W$ is co$_\delta$-coatomic and X is a co$_\delta$-coatomic submodule of K. Then there exists a δ-coatomic submodule K' of M such that $W = K \oplus K'$. Therefore, we deduce that $W/X \cong (K/X) \oplus K'$ is δ-coatomic as a direct sum of two δ-coatomic modules by [4, Proposition 2.6]. Since W is $\oplus$-co$_\delta$-δ-supplemented, then there exists a δ-supplement T of X in W with $T \leq_\oplus W$. It gives the result that $K = K \cap (T + X) = (K \cap T) + X$. Since W is a (D_3)-module, $K \cap T \leq_\oplus W$, and thus $K \cap T \leq_\oplus K$. Also, $X \cap (K \cap T) = X \cap T \ll_\delta K \cap T$ by Lemma 1-(5). Hence K is $\oplus$-co$_\delta$-δ-supplemented.

In the sequel, we provide a helpful lemma to establish that any finite direct sum of modules that are $\oplus$-co$_\delta$-δ-supplemented remains $\oplus$-co$_\delta$-δ-supplemented.

Lemma 4. Let W be a module and G, H be submodules of W such that G is co$_\delta$-δ-supplemented, H is co$_\delta$-coatomic and $G+H$ possesses a δ-supplement T in W. Then $G\cap(H+T)$ possesses a δ-supplement T' in G and $T + T'$ is a δ-supplement of H in W.

Proof. Since T is a δ-supplement of $G+H$ in W, we have that $W = (G+H)+T$ and $(G+H)\cap T \ll_\delta T$. Moreover,
$G/(G\cap(H+T)) \cong (G+H+T)/(H+T) = W/(H+T) \cong (W/H)/((H+T)/H)$ is δ-coatomic as it is quotient module of the δ-coatomic module W/H. Thus there exists a δ-supplement X of $G\cap(H+T)$ in G, by the hypothesis. So, $G = (G\cap(H+T)) + X$ and $(G\cap(H+T))\cap X = (H+T)\cap X \ll_\delta X$. Therefore, we have $W = (G+H)+T = (G\cap(H+T)+X)+H)+T = H+T+X$ and
$H\cap(T+X) \leq T\cap(H+X) + X\cap(T+H) \leq T\cap(H+G) + X\cap(T+H) \ll_\delta T+X$ by Lemma 1-(3). Hence $T+X$ is a δ-supplement of H in W where X is a δ-supplement of $G\cap(H+T)$ in G.

Theorem 3. Any finite direct sum of $\oplus$-co$_\delta$-δ-supplemented modules is $\oplus$-co$_\delta$-δ-supplemented.

Proof. Assume that $W_1, W_2, ..., W_n$ is a finite collection of $\oplus$-co$_\delta$-δ-supplemented modules and $W = \bigoplus_{\lambda=1}^{n} W_\lambda$. To show that W is $\oplus$-co$_\delta$-δ-supplemented, we will prove the claim in case $n = 2$ and the proof is completed by induction. Let W be the direct sum of $\oplus$-co$_\delta$-δ-supplemented modules W_1, W_2 and G be a co$_\delta$-coatomic submodule of W. Then note that $W = W_1 + W_2 + G$ and 0 is a δ-supplement of $W_1 + W_2 + G$ in W. Since
$W_2/(W_2 \cap (W_1 + G)) \cong (W_1 + W_2 + G)/(W_1 + G) \cong (W/G)/((W_1 + G)/G)$
is δ-coatomic as it is quotient module of the δ-coatomic module W/G, then $W_2 \cap (W_1 + G) \leq W_2$ is co$_\delta$-coatomic. Thus by the hypothesis, W_2 possesses a $K \leq_\oplus W_2$ that is a δ-supplement of $W_2 \cap (W_1 + G)$ in W_2. Then K is a δ-supplement of W_1+G by Lemma 4. Now it is easy to see that $W_1/W_1\cap(G+K)$ is δ-coatomic, and thus $W_1 \cap (G + K)$ possesses a δ-supplement T in W with $T \leq_\oplus W_1$. By using Lemma 4 once again, $K + T$ is a δ-supplement of G in W_1 such that $K \oplus T \leq_\oplus W$. Hence W is $\oplus$-co$_\delta$-δ-supplemented module.

Co$_\delta$-Coatomically δ-Semiperfect Modules

Definition 2. We term a module W *co$_\delta$-coatomically δ-semiperfect, shortly co$_\delta$-δ-semiperfect*, provided each quotient module of W by a co$_\delta$-coatomic submodule, or equivalently each δ-coatomic quotient module of W possesses a projective δ-cover.

Proposition 14. Suppose that W is a projective module. In that case, W is co$_\delta$-δ-semiperfect if and only if W is $\oplus$-co$_\delta$-δ-supplemented.

Proof. ($\Longrightarrow$) Assume that G is a co$_\delta$-coatomic submodule of W. There is a projective δ-cover $h : P \to W/G$, by the assumption. Then according to [6, Lemma 2.4] there are submodules W_1, W_2 of W such that $W = W_1 \oplus W_2$ with $W_1 \leq G$ and $W_2 \cap G \ll_\delta W$. Applying Lemma 1-(5), we conclude that $W_2 \cap G \ll_\delta W_2$. It leads to the conclusion that W_2 is a δ-supplement of G.
($\Longleftarrow$) Assume that G is a co$_\delta$-coatomic submodule of W. As W is $\oplus$-co$_\delta$-δ-supplemented, then W has a decomposition $W = W_1 \oplus W_2$ such that $W = W_1 + G$ and $W_1 \cap G \ll_\delta W_1$. Note that W_1 is projective. Consider the canonical injection $\iota : W_1 \to W$ and the canonical projection $\pi : W \to W/G$. By this way, there exists an epimorphism $\pi\iota : W_1 \to W/G$ with $Ker(\pi\iota) = W_1 \cap G \ll_\delta W_1$.

Theorem 4. Suppose that W is a co$_\delta$-δ-semiperfect module. In that case, each homomorphic image of W is co$_\delta$-δ-semiperfect.

Proof. Assume that $h : W \to L$ is a homomorphism and G is co$_\delta$-coatomic submodule of $h(W)$. Consider the epimorphism $\psi : W \to h(W)/G$ defined by $\psi(w) = h(w)+G$ for whole $w \in W$. As W is co$_\delta$-δ-semiperfect, $h(W)/G \cong W/h^{-1}(G)$ has a projective δ-cover. Hence $h(W)$ is co$_\delta$-δ-semiperfect module.

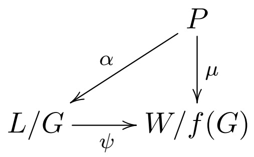
Corollary 13. Each quotient module of a co$_\delta$-δ-semiperfect module is co$_\delta$-δ-semiperfect.

Corollary 14. Each projective quotient module of a co$_\delta$-δ-semiperfect module is $\oplus$-co$_\delta$-δ-supplemented module.

Proof. The proof can be easily seen by Corollary 13 and Proposition 14.

Theorem 5. Each δ-cover of a co$_\delta$-δ-semiperfect module is co$_\delta$-δ-semiperfect.

Proof. Assume that W is co$_\delta$-δ-semiperfect module and $f : L \to W$ is a δ-cover of W. For the δ-coatomic quotient module L/G of L, the homomorphism $\psi : L/G \to W/f(G)$ defined by $\psi(l+G) = f(l) + f(G)$ for all $l \in L$ is an epimorphism. Also, we claim that $Ker(\psi) \ll_\delta L/G$. To prove this, assume that $Ker(\psi) + X/G = L/G$ where $(L/G)/(X/G)$ is singular. Note that $Ker(\psi) = (G + Ker(f))/G$. Thus we have that $L = X + Ker(f)$ and $(L/G)/(X/G) \cong L/X$ is singular. As $Ker(f) \ll_\delta L$, then $X = L$. This leads to the conclusion that $Ker(\psi) \ll_\delta L/G$. Since $W/f(G) = \psi(L/G) \cong (L/G)/(G+Ker(f))/G$ is δ-coatomic, then by assumption $W/f(G)$ possesses a projective δ-cover $\mu : P \to W/f(G)$. As P is projective, there is a homomorphism $\alpha : P \to L/G$ such that the following diagram



is commutative, that is, $\mu = \psi\alpha$. Then $L/G = \alpha(P) + Ker(\psi)$. Since $Ker(\psi) \ll_\delta L/G$, there exists a semisimple projective submodule S of $Ker(\psi)$ such that $L/G = \alpha(P) + S$ by Lemma 1-(1). Now we can define the homomorphism $\beta : P \oplus S \to L/G$ via $\beta(p, s) = \alpha(p) + G$. β is an epimorphism and $Ker(\beta) = Ker(\alpha) \oplus 0$. Since $Ker(\alpha) \leq Ker(\mu) \ll_\delta P$, then $Ker(\alpha) \oplus 0 \ll_\delta P \oplus S$. Consequently, $P \oplus S$ is a δ-cover of L/G.

Corollary 15. Suppose that G is a δ-small submodule of a module W and W/G is co$_\delta$-δ-semiperfect. In that case, W is a co$_\delta$-δ-semiperfect module.

Corollary 16. Suppose that W is and that $f : P \to W$ is a projective δ-cover of W. In that case, the expressions below are equivalent:

1. W is co$_\delta$-δ-semiperfect.
2. P is co$_\delta$-δ-semiperfect.
3. P is $\oplus$-co$_\delta$-δ-supplemented.

Proof. $(1) \Longrightarrow (2)$ It is derived from Theorem 5.
$(2) \Longrightarrow (1)$ It is supported by Corollary 13.
$(2) \Longleftrightarrow (3)$ It is Proposition 14.

Theorem 6. Suppose that W_λ is a collection of projective modules where $\lambda \in \Lambda$ and Λ is a finite index set. In that case, $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$ is co$_\delta$-δ-semiperfect module if and only if W_λ is co$_\delta$-δ-semiperfect for each $\lambda \in \Lambda$.

Proof. $(\Longrightarrow)$ Let $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$ is co$_\delta$-δ-semiperfect module. Then by Corollary 13, $W_\lambda \cong W/(\bigoplus_{\eta\in\Lambda\setminus\{\lambda\}} W_\eta)$ is co$_\delta$-δ-semiperfect for each $\lambda \in \Lambda$.
$(\Longleftarrow)$ Let W_λ be a projective co$_\delta$-δ-semiperfect module for each $\lambda \in \Lambda$. Thus W_λ is $\oplus$-co$_\delta$-δ-supplemented by Proposition 14, for each $\lambda \in \Lambda$. Therefore, W is $\oplus$-co$_\delta$-δ-supplemented by Theorem 3. Applying Proposition 14 once again, we deduce that W is co$_\delta$-δ-semiperfect.

Now the next result is obviously seen by Corollary 13 and Theorem 6.

Corollary 17. Suppose that W is a projective module. In case W is co$_\delta$-δ-semiperfect module, any finitely W-generated module is co$_\delta$-δ-semiperfect.

Finally, we determine over which rings the R-module $_RR$ is equivalent to being $\oplus_\delta$-co-coatomically supplemented, being co$_\delta$-δ-semiperfect and being $\oplus$-co$_\delta$-δ-supplemented.

Theorem 7. The expressions below are equivalent for a δ-semiperfect ring S:

1. ${}_S S$ is $\oplus_\delta$-co-coatomically supplemented.

2. ${}_S S$ is co$_\delta$-δ-semiperfect.

3. ${}_S S$ is $\oplus$-co$_\delta$-δ-supplemented.

Proof. (1) $\iff$ (3) It is Corollary 10.
(2) $\iff$ (3) It is Proposition 14.

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