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Araştırma Makalesi / Research Article

The novel kumaraswamy extended garima distribution, statistical properties and its application

Ayşe Metin Karakaş

Bitlis Eren University
Department of Statistics
Bitlis, Turkey
akarakas@beu.edu.tr
ORCID:0000-0003-3552-0105

Murat Karakaş

Bitlis Eren University
Department of Mathematics
Bitlis, Turkey
mkarakas@beu.edu.tr
ORCID:0000-0002-5174-0282

Mine Doğan

Firat University
Department of Statistics
Elazığ, Turkey
mine.dogan@firat.edu.tr
ORCID:0000-0002-2745-9909

ABSTRACT

This essay aims to present a novel Kumaraswamy Extended Garima distribution family. We obtain a cumulative distribution function, the failure rate, the risk rate, the inverse risk function, the odd function, the cumulative risk function, the moment r-th, the characteristic function of the moment generating function, the moments, the mean and the variance, Lorenz and Bonferroni curves, order statistics, MLE, mean time between failures (MTBF), Renyi and Tsallis entropies. The MLE technique estimates the parameters of the new Kumaraswamy Extended Garima distribution. The parameters of the MLE technique are derived using a nonlinear system of equations and the Symmetric Information Matrix. Furthermore, the consequences are analogous to others because they are probability distributions. According to the findings, the proposed distribution fits these data sets better than existing probability distributions.

Key words: Garima distribution, Kumaraswamy distribution, Cumulative distribution function, Characteristic function, Symmetric information matrix.

ÖZ

Yeni Kumaraswamy genişletilmiş Garima dağılımı istatistiksel özellikleri ve uygulaması

Bu makale yeni bir Kumaraswamy Genişletilmiş Garima dağılım ailesini sunmayı amaçlamaktadır. Kümülatif bir dağılım fonksiyonu, başarısızlık oranı, risk oranı, ters risk fonksiyonu, tek fonksiyon, kümülatif risk fonksiyonu, moment r-th, moment üreten fonksiyonun karakteristik fonksiyonu, momentler, ortalama ve varyans, Lorenz ve Bonferroni eğrileri, sıra istatistikleri, MLE, arızalar arasındaki ortalama süre (MTBF), Renyi ve Tsallis entropileri gibi özellikleri elde ettik. MLE tekniği, yeni Kumaraswamy Extended Garima dağılımının parametrelerini tahmin eder. MLE tekniğinin parametreleri, doğrusal olmayan bir denklem sistemi ve Simetrik Bilgi Matrisi kullanılarak türetilir. Ayrıca sonuçlar olasılık dağılımları olduğundan diğerlerine benzer. Bulgulara göre, bu veri setlerine önerilen dağılım, mevcut olasılık dağılımlarından daha iyi uymaktadır.

Anahtar Kelimeler: *Garima dağılımı, Kumaraswamy dağılımı, Kümülatif dağılım fonksiyonu, Karakteristik fonksiyon, Simetrik bilgi matrisi.*

1. Introduction

Kumaraswamy [15] developed the Kumaraswamy probability distribution primarily for hydrological applications. Garg [11], Jones [14], Mitnik [18], and Nadarajah [21] investigated the Kumaraswamy distribution in theoretical terms. Mitnik [19] demonstrated that Kumaraswamy variables exhibit closeness under exponentiation and linear transformation, and he also introduced some of the distribution's limiting distributions and an analytical expression for the mean absolute deviation around the median as a distribution parameter function. Again, the author provided some boundaries for this dispersion measure and the variance in this investigation. Tahir et al. [27] presented a new Kumaraswamy generalized (G) distribution family via a novel generator that could be a replacement for the Kumaraswamy-G family. Cordeiro and de Castro [7] defined a new family of generalized distributions to extend the normal, Weibull, gamma, Gumbel, and inverse Gaussian distributions, among others, and discussed some special distributions in the new family, such as the Kw-normal, Kw-Weibull, Kw-gamma, Kw-Gumbel, and Kw-inverse Gaussian distributions. Carrasco et al. [5] demonstrated the log-Kumaraswamy MW regression model for censored data analysis. Asiribo et al. [2] defined the Lomax-Kumaraswamy distribution with four parameters and offered various statistical features. Wang et al. [11] investigated the estimation of points and intervals for the Kumaraswamy distribution. As a specific model, Gomes et al. [12] introduced the Kumaraswamy - Kumaraswamy (KW-KW) distribution. Kumaraswamy class generalized (KW-G) distributions. El-Sherpieny and Ahmed [13] introduced the Kumaraswamy GR (KwGR) distribution for analyzing lifespan data, as well as a linear log KwGR regression model for analyzing data with real support in order to expand some of the current regression models. Tahir et al. [26] developed a new extension of the Kumaraswamy distribution by using the Weibull link function to add a shape and a scale parameter to the Kumaraswamy distribution. Yang [29] proposed a generalized inverse Weibull distribution that incorporates both the proportional inverse hazard and the Kumaraswamy generalized inverse Weibull distributions. Carrasco et al. [6] used a definite probability integral transform to propose and test a new five-parameter continuous distribution over a unit interval. Dey et al. [8] explored from a different perspective several estimating methods of uncertain parameters of the two-parameter Kumaraswamy distribution and they handled the estimate of the Kumaraswamy distribution's unknown parameters using simple random sampling (SRS). They ordered cluster sampling (RSS), as well as maximum probability estimation and Bayesian estimation approaches. They created a new distribution known as the generalized inverted Kumaraswamy (GIKum). The authors' main purpose in Iqbal et al. [13] is to improve a common structure for the inverse Kumaraswamy (IKum) distribution that is more flexible than the IKum distribution and all its related and submodels. Salman [25] analyzes various strategies for calculating scale and form parameters. Mohiuddin et al. [20] created the Transmuted Garima Distribution and investigated its features and applications. We propose a new model of Kumaraswamy Extended Garima distribution in our work. Several aspects of this novel model are explored and computed, including reliability functions, apparent assertions of the moments, mean deviations, Lorenz and Bonferroni curves, and Renyi and Shannon entropies. The maximum likelihood method was used to estimate the model parameters. We use this method to obtain parameter estimates and then conduct a simulation exercise to determine the performance of the maximum likelihood estimators. Furthermore, we use four real-world data sets to demonstrate the use and significance of the new family of distributions. Finally, we show that our new distribution model outperforms the well-known distributions.

2. Materials and Methods

Definition 2.1. The Kumaraswamy Distribution

Kumaraswamy presented a two-parameter distribution on (0,1). Let 'Kw' represent the abbreviated name of this distribution. Its cdf is as follows:

$$G(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}, x \in (0,1) \tag{1}$$

as well as the probability density function

$$g(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, x \in (0,1) \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ denote shape parameters. Kumaraswamy [15] proposed the Kumaraswamy-G (Kw-G) distribution with the following pdf "f(x)" and cdf "F(x)" for any baseline cumulative function G(x).

$$f(x) = \alpha\beta g(x)G^{\alpha-1}(x)\left((1-G(x)^{\alpha})\right)^{\beta-1} \tag{3}$$

and

$$F(x) = 1 - (1 - G(x)^{\alpha})^{\beta} \tag{4}$$

g(x) = dG(x)/dx, etc. The parameters of the Kw-G and G distributions are identical. If X is a random variable with a density function (3), it may be represented as X Kw G(a, b) [11].

Definition 2.2. Garima Distribution

Assume X is a random variable with the Garima distribution and parameter. As a result, probability density is represented by;

$$g(x;\theta) = \frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}, x > 0, \theta > 0. \tag{5}$$

The pertinent cumulative distribution function (c.d.f.) is as follows:

$$G(x;\theta) = 1 - \left[1 + \frac{\theta x}{\theta+2}\right]e^{-\theta x}, x > 0, \theta > 0. \tag{6}$$

3. The Novel Kumaraswamy Extended Garima Distribution

Definition 3.1. Let X $G(x,\theta)$ represent the cdf of the Garima distribution provided by (6). The novel three-parameter cdf Kumaraswamy Substituting (6) into equation (4) yields the Extended Garima (KwEG) distribution.

$$F(x) = 1 - \left(1 - \left\{1 - \left[1 + \frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}. \tag{7}$$

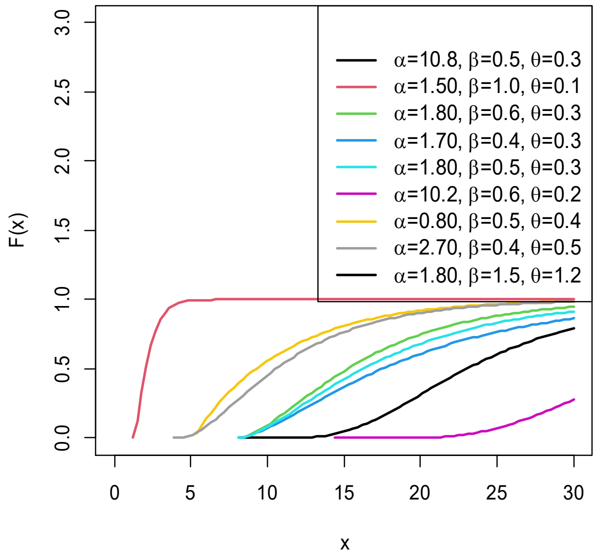

Figure 1. The cdf's of Kw Extended Garima distributions

The probability density function corresponding to f(x) is given by

$$f(x)=\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}. \quad (8)$$

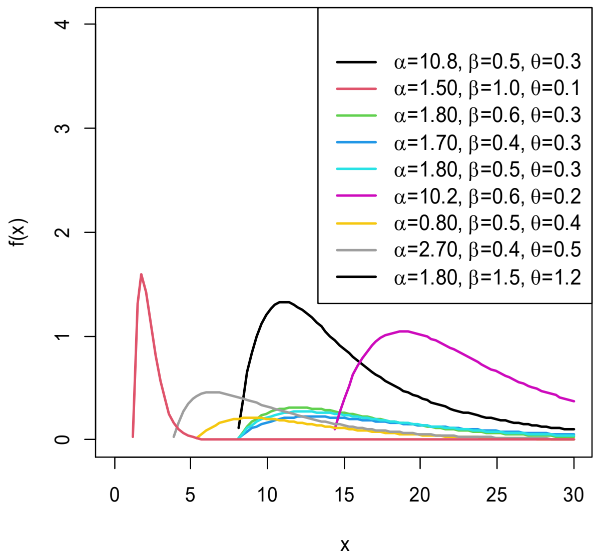

Figure 2. The pdf's of Kw- Extended Garima distributions

Lemma 3.1. When a=1 and b=1, the Kw- Extended Garima distribution in Equation (8) is reduced to the Garima distribution in Equation (5) with parameter.

Lemma 3.2. When b=1, the Kw- Extended Garima distribution is reduced to the generalized exponentiated Garima distribution, with parameters shape a and scale θ .

Theorem.3.1. If X follows the probability density function (8) for all $\alpha,\beta,\theta > 0$, then X's quantile function is

$$Q_u = -\frac{2}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left(-(\theta+2)e^{-(\theta+2)}\left(1-\left(1-(1-u)^{1/\beta}\right)\right)^{1/\alpha} \right).$$

W_{-1} denotes the negative branch of the Lambert W function in this case.

Proof. If F_x is a continuous and strictly increasing function, then the quantile function Q_u of X is defined as follows:

$$Q(u) = F^{-1}(u), u \in (0,1) \tag{9}$$

We have an equation to solve here, derived from (7) and (9).

$$u = 1 - \left(1 - \left\{1 - \left[1 + \frac{\theta Q(u)}{\theta + 2}\right] e^{-\theta Q(u)}\right\}^{\alpha}\right)^{\beta}.$$

We can deduce from this equation,

$$\left(\theta + 2 + \theta Q(u)\right) e^{-\theta Q(u)} = (\theta + 2)\left(1 - \left(1 - (1-u)^{1/\beta}\right)^{1/\alpha}\right) \tag{10}$$

and

$$-\left(\theta + 2 + \theta Q(u)\right) e^{-(\theta Q(u)+2+\theta)} = e^{-(\theta+2)}(\theta + 2)\left(1 - \left(1 - (1-u)^{1/\beta}\right)^{1/\alpha}\right) \tag{11}$$

The equation's solution is then

$$\left(\theta + 2 + \theta Q(u)\right) = W_{-1}\left(e^{-(\theta+2)}(\theta + 2)\left(1 - \left(1 - (1-u)^{1/\beta}\right)^{1/\alpha}\right)\right) \tag{12}$$

From (12), we obtained

$$Q_u = -\frac{2}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left(-(\theta+2)e^{-(\theta+2)}\left(1-\left(1-(1-u)^{1/\beta}\right)\right)^{1/\alpha} \right). \quad (13)$$

Collary 3.1. Setting u to 0.5 in equality (13), we get the median (M) of X as;

$$M = -\frac{2}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left(-(\theta+2)e^{-(\theta+2)}\left(1-\left(1-(0.5)^{1/\beta}\right)\right)^{1/\alpha} \right).$$

However, by setting u to 0.25 and 0.75 inequality (9), the 25th and 75th percentiles for the random variable X are obtained as;

$$Q_1 = -\frac{2}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left(-(\theta+2)e^{-(\theta+2)}\left(1-\left(1-(0.75)^{1/\beta}\right)\right)^{1/\alpha} \right).$$

$$Q_3 = -\frac{2}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left(-(\theta+2)e^{-(\theta+2)}\left(1-\left(1-(0.25)^{1/\beta}\right)\right)^{1/\alpha} \right).$$

The quantile function yields the Bowley's skewness as

$$S_k = \frac{Q_{0.75} - 2Q_{0.50} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}.$$

The kurtosis of the Moor is written as

$$M_k = \frac{Q_{0.875} - Q_{0.025} - Q_{0..375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

We can easily generate X by treating u as a uniform random variable in the range (0,1).

Theorem.3.2. The r-th moment $E(X^r)$ of the Kw- Extended Garima distributed random variable X is given by Theorem.3.2.

$$E\left(X^r\right) = \alpha\beta \frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{aj+i} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty} (-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}$$
$$\theta\left(\theta\left(1+i+\alpha j\right)\right)^{-2-i-\alpha j-r}\left(\left(2+\theta\right)\left(1+i+\alpha j\right)+r\right)\Gamma\left[1+i+\alpha j+r\right].$$

Proof. $\mu_r^{'} = E(X^r) = \int_0^\infty x^r f(x)dx$

$$= \int_0^\infty x^r \left\{ \alpha\beta \frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}\right\}dx$$

and

$$= \alpha\beta \frac{\theta}{\theta+2}\int_0^\infty x^r \left\{(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}\right\}dx. \quad (14)$$

Using binomial expansions of

$$\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)\right)^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}\left(e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)\right)^{i}(-1)^{\alpha-1-i}$$

$$\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1} = \sum_{j=0}^{\infty}\binom{\beta-1}{j}\left(\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{j}(-1)^{\alpha-1-j},$$

Then, equation (14) becomes,

$$= \alpha\beta\frac{\theta}{\theta+2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{a-1-i}(-1)^{\beta-1-j}\binom{\alpha-1}{i}\binom{\beta-1}{j}\int_0^\infty x^r\left\{(1+\theta+\theta x)e^{-\theta x}\left(\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{i}\left(\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{j}\right\}dx. \quad (15)$$

if the following equation is written in equation (15),

$$\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha j} = \sum_{k=0}^{\infty}\binom{\alpha j}{k}\left(\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)^{aj}(-1)^{aj-k},$$

Then, there's

$$\alpha\beta\frac{\theta}{\theta+2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\int_0^\infty x^r(1+\theta+\theta x)e^{-\theta x(i+\alpha j+1)}\left(1+\frac{\theta x}{\theta+2}\right)^{i+\alpha j}dx.$$

As a result, we have

$$\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{b-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{b-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}\int_0^{\infty}x^r(1+\theta+\theta x)e^{-\theta x(i+\alpha j+1)}x^{aj+i}dx. \tag{16}$$

We can deduce from equation (16), that

$$\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{aj+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}\theta\big(\theta(1+i+\alpha j)\big)^{-2-i-\alpha j-r}$$
$$\big((2+\theta)(1+i+\alpha j)+r\big)\Gamma\left[1+i+\alpha j+r\right]. \tag{17}$$

All of the moments exist because the series in (17) is convergent.

Theorem.3.3. . Assume X has the Kw- Extended Garima distribution. Then, say $M_X(t)$, the moment generating function of X.

$$\phi(e^{mt})=\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\frac{t^s}{s!}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}$$
$$\theta\big(\theta(1+i+\alpha j)\big)^{-2-i-\alpha j-ms}\big((2+\theta)(1+i+\alpha j)+ms\big)\Gamma\left[1+i+\alpha j+ms\right].$$

Proof.

$$M_X(t)=E(e^{tx})=\int_0^{\infty}e^{tx}f(x)dx. \tag{18}$$

When we apply Taylor's series to equation (18), we get

$$M_X(t)=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\ldots\right)f(x)dx. \tag{19}$$

From equality (19), we can write

$$M_X(t)=\int_0^{\infty}\sum_{s=0}^{\infty}\frac{t^s}{s!}x^s f(x)dx. \tag{20}$$

From equality (20), we have

$$M_X(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!}\mu_s'$$

$$= \alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \frac{t^s}{s!}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}$$
$$\theta\left(\theta\left(1+i+\alpha j\right)\right)^{-2-i-\alpha j-s}\left(\left(2+\theta\right)\left(1+i+\alpha j\right)+s\right)\Gamma\left[1+i+\alpha j+s\right].$$

The characteristic function of X is then calculated as follows:

$$\phi(e^{mt}) = \alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \frac{t^s}{s!}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}$$
$$\theta\left(\theta\left(1+i+\alpha j\right)\right)^{-2-i-\alpha j-ms}\left(\left(2+\theta\right)\left(1+i+\alpha j\right)+ms\right)\Gamma\left[1+i+\alpha j+ms\right].$$

(21)

Theorem.3.4. Assume X has the Kw- Extended Garima distribution. Renyi Entropy of X is thus given by

$$= \frac{1}{1-\gamma}\log\left(\begin{array}{l} \alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty} (-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t} \\ \frac{1}{\Gamma[-\gamma]}(1+\theta)^{\gamma}\left(\frac{\theta}{1+\theta}\right)^{-1-i-\alpha j}\Gamma\left[1+i+\alpha j\right]\Gamma\left[-1-i-\alpha j-\gamma\right] \\ Hypergeometric1F1\left[1+i+\alpha j, 2+i+\alpha j+\gamma, \alpha\left(1+\theta\right)\left(j+\gamma\right)\right] \\ +\left(\frac{\theta}{1+\theta}\right)^{\gamma}\left(\alpha\theta\left(j+\gamma\right)\right)^{-1-i-\alpha j-\gamma}\Gamma\left[-\gamma\right]\Gamma\left[1+i+\alpha j+\gamma\right] \\ Hypergeometric1F1\left[-\gamma, -i-\alpha j-\gamma, \alpha\left(1+\theta\right)\left(j+\gamma\right)\right] \end{array}\right).$$

The hypergeometric series, which covers many other special functions as specific or limiting cases, is used to show the hypergeometric1F1 function here.

Proof.

$$e(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^{\infty} f^{\gamma}(x)dx\right)$$

where $\gamma > 0$ and $\gamma \neq 1$. As a result, we have

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\int_0^\infty\left\{\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{a-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{a}\right)^{\beta-1}\right\}^{\gamma}dx\right)$$

$$=\frac{1}{1-\gamma}\log\left(\int_0^\infty\left\{(\alpha\beta)^{\gamma}\left(\frac{\theta}{\theta+2}\right)^{\gamma}(1+\theta+\theta x)^{\gamma}e^{-\gamma cx}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{(\alpha-1)\gamma}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{(\beta-1)\gamma}\right\}dx\right).$$

(22)

If the expressions below are written in equality (22),

$$\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)\right)^{\gamma(\alpha-1)}=\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)}{i}\left(e^{-\theta x}\left(1+\frac{\theta x}{\theta+2}\right)\right)^{i}(-1)^{\gamma(\alpha-1)-i}$$

,

$$\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\gamma(b-1)}=\sum_{j=0}^{\infty}\binom{\gamma(\beta-1)}{j}\left(\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{j}(-1)^{\gamma(\beta-1)-j}$$

$$\left(\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{j}=\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha j}\sum_{k=0}^{\infty}\binom{\alpha j}{k}\left(\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)^{\alpha j}(-1)^{\alpha j-k}$$

We then write, correspondingly,

$$=\frac{1}{1-\gamma}\log\left(\begin{array}{l}(\alpha\beta)^{\gamma}\left(\frac{\theta}{\theta+2}\right)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=o}^{\infty}(-1)^{\gamma(\alpha-1)-i}(-1)^{\gamma(\beta-1)-j}(-1)^{\alpha j-k}\binom{\gamma(\alpha-1)}{i}\binom{\gamma(\beta-1)}{j}\binom{\alpha j}{k}\\ \int_0^\infty\left\{(1+\theta+\theta x)^{\gamma}e^{-\gamma\theta x}\left(\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{(a-1)\gamma}\left(\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{aj}\right\}dx\end{array}\right)$$

$$=\frac{1}{1-\gamma}\log\left(\begin{array}{l}\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}\\ \int_0^\infty(1+\theta+\theta x)^{\gamma}e^{-\gamma\theta x-\theta x(\alpha-1)\gamma-\theta x\alpha j}x^{\alpha j+i}dx\end{array}\right).$$

When we integrate the above equation, we get the following equation. This concludes the evidence.

$$=\frac{1}{1-\gamma}\log\left(\begin{array}{l}\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}\\ \frac{1}{\Gamma[-\gamma]}(1+\theta)^{\gamma}\left(\frac{\theta}{1+\theta}\right)^{-1-i-\alpha j}\Gamma[1+i+\alpha j]\Gamma[-1-i-\alpha j-\gamma]\\ Hypergeometric1F1[1+i+\alpha j,2+i+\alpha j+\gamma,\alpha(1+\theta)(j+\gamma)]\\ +\left(\frac{\theta}{1+\theta}\right)^{\gamma}(\alpha\theta(j+\gamma))^{-1-i-\alpha j-\gamma}\Gamma[-\gamma]\Gamma[1+i+\alpha j+\gamma]\\ Hypergeometric1F1[-\gamma,-i-\alpha j-\gamma,\alpha(1+\theta)(j+\gamma)]\end{array}\right).$$

Theorem.3.5. . Suppose X has the Kw- Extended Garima distribution. Tsallis Entropy of X is thus given by

$$=\frac{1}{1-\lambda}\left(\begin{array}{l}\alpha\beta\frac{\theta}{\theta+2}\left(\frac{\theta}{\theta+2}\right)^{\alpha j+i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{\alpha-1-i}(-1)^{\beta-1-j}(-1)^{\alpha j-k}\binom{\alpha-1}{i}\binom{\beta-1}{j}\binom{\alpha j}{k}\binom{\alpha j+i}{t}\\ \frac{1}{\Gamma[-\lambda]}(1+\theta)^{\lambda}\left(\frac{\theta}{1+\theta}\right)^{-1-i-\alpha j}\Gamma[1+i+\alpha j]\Gamma[-1-i-\alpha j-\lambda]\\ Hypergeometric1F1[1+i+\alpha j,2+i+\alpha j+\lambda,\alpha(1+\theta)(j+\lambda)]\\ +\left(\frac{\theta}{1+\theta}\right)^{\lambda}(\alpha\theta(j+\lambda))^{-1-i-\alpha j-\lambda}\Gamma[-\lambda]\Gamma[1+i+\alpha j+\lambda]\\ Hypergeometric1F1[-\lambda,-i-\alpha j-\lambda,\alpha(1+\theta)(j+\lambda)]\end{array}\right)$$

The hypergeometric series, which covers many other special functions as specific or limiting cases, is used to present the hypergeometric1F1 function here.

Proof. Tsallis Entropy for the Kw-Extended Garima Distribution;

$$S_{\lambda}=\frac{1}{1-\lambda}\left(1-\int_{0}^{\infty}f^{\lambda}(x)dx\right)$$

$$=\frac{1}{1-\lambda}\left(1-\int_{0}^{\infty}\left\{\begin{array}{l}\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\\ \left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}\end{array}\right\}^{\lambda}dx\right).$$

The Tsallis entropy is obtained by integrating the above equation. This brings the proof to a close.

4. Reliability Analysis

The reliability or survival function for the Kw-Extended Garima distribution is stated as (7) in R(t), and Figure 3 depicts the reliability function of the Kw-Extended Garima distribution for different parameter values:

$$R(t)=\left(1-\left\{1-\left[1+\frac{\theta t}{\theta+2}\right]e^{-\theta t}\right\}^{\alpha}\right)^{\beta}. \tag{23}$$

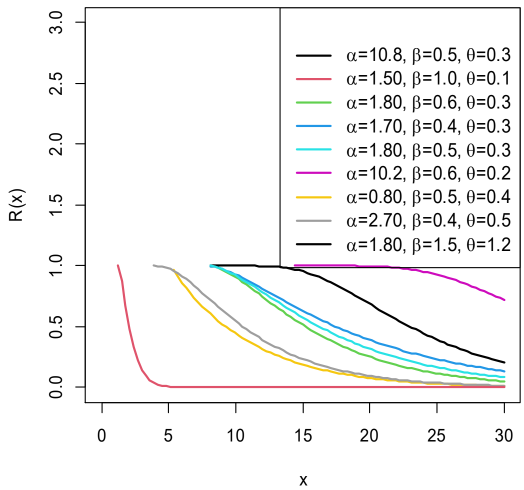

Figure 3. The reliability function of Kw- Extended Garima distributions

The hazard rate function, defined h as the event at time t conditional on survival until time t, is the other function. Assume that an item has survived for time t and state the likelihood,

$$h(t)=\lim_{dt\to 0}\frac{P\left(t\le T\le t+dt\right)}{dtR(t)}=\frac{f(t)}{R(t)}=-\frac{R'(t)}{R(t)}. \tag{24}$$

The hazard rate for the Kw- Extended Garima distribution can be calculated by combining (8) and (23) and defining it as follows;

$$h(t)=\frac{\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta t)e^{-\theta t}\left(1-\left[1+\frac{\theta t}{\theta+2}\right]e^{-\theta t}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta t}{\theta+2}\right]e^{-\theta t}\right\}^{\alpha}\right)^{\beta-1}}{\left(1-\left\{1-\left[1+\frac{\theta t}{\theta+2}\right]e^{-\theta t}\right\}^{\alpha}\right)^{\beta}}. \tag{25}$$

Figure 4. Depicts the hazard rate function of the Kw-Extended Garima distribution for various parameter values.

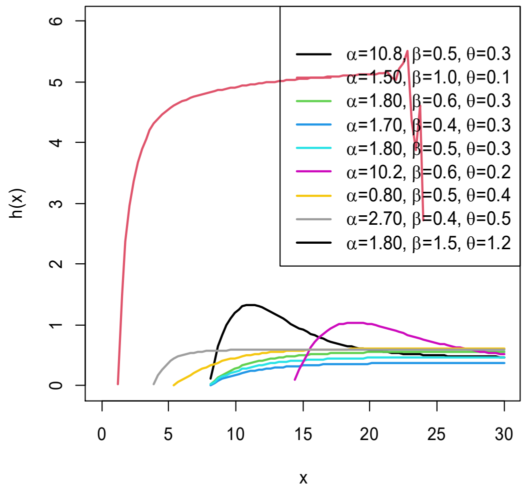

Figure 4. The hazard rate function of Kw- Extended Garima distributions

The reversed hazard rate is defined as follows;

$$r(x)=\frac{f(x)}{F(x)}. \tag{26}$$

Setting (7) and (8) in (26) yields the reversed hazard rate for the Kw- Extended Garima distribution, which has the following format. Figure 5 depicts the reverse hazard rate function of the Kw- Extended Garima distribution for different parameter values.

$$r(x)=\frac{\alpha\beta\dfrac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\dfrac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\dfrac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}}{1-\left(1-\left\{1-\left[1+\dfrac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}}.$$

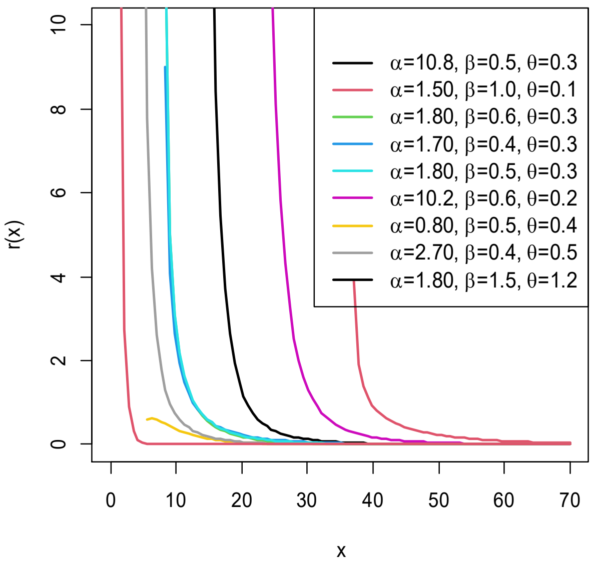

Figure 5. The reversed hazard rate function of Kw- Extended Garima distributions

The following format describes the odds function;

$$O(x)=\frac{F(x)}{R(x)}. \tag{27}$$

Setting (8) and (10) in (27) yields the odds function for the Kw- Extended Garima distribution, which has a following statement.

$$O(x)=\frac{1-\left(1-\left\{1-\left[1+\dfrac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}}{\left(1-\left\{1-\left[1+\dfrac{\theta t}{\theta+2}\right]e^{-\theta t}\right\}^{\alpha}\right)^{\beta}}. \tag{28}$$

Figure 6 depicts the odds function of the Kw- Extended Garima distribution for various parameter values.

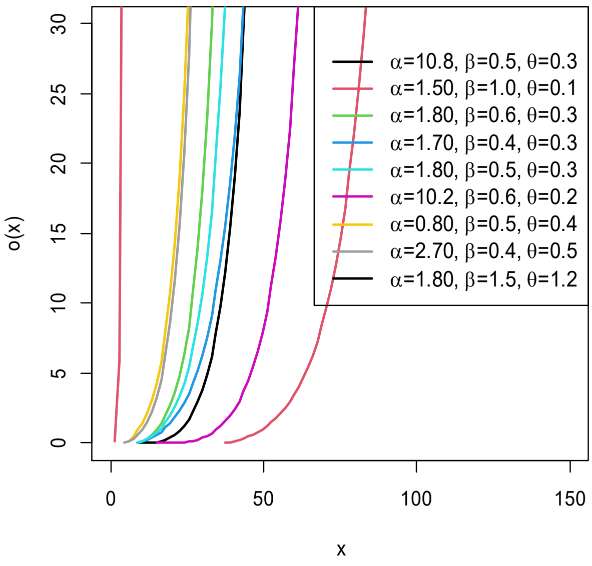

Figure 6. The odds function The Kw- Extended Garima distributions

5. Order Statistics

The jth order statistics of the Kw- Extended Garima distribution's pdf and cdf are as follows:

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-i)!}\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j} f(x)$$

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}$$
$$\left(1-\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}\right)^{j-1}\left(1-\left(1-\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}\right)\right)^{n-j}.$$

$$F_{j:n}(x) = \sum_{r=j}^{n}\left(F(x)\right)^{r}\left(1-F(x)\right)^{n-r}$$

$$=\binom{n}{j}\left(1-\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)\right)^{i}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{n-i}$$

For the largest order statistic, the pdf is

$$f_{X_{(n)}}(x) = n\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}$$
$$\left(1-\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}\right)^{n-1},$$

and here is the pdf for the smallest order statistic:

$$f_{X_{(1)}}(x) = n\alpha\beta\frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta-1}$$
$$\left(1-\left(1-\left(1-\left\{1-\left[1+\frac{\theta x}{\theta+2}\right]e^{-\theta x}\right\}^{\alpha}\right)^{\beta}\right)\right)^{n-1}.$$

6. Maximum Likelihood Estimation (MLE)

This function of Kw- Extended Garima is described in the following format:

$$\phi(X) = \alpha^n\beta^n\left(\frac{\theta}{\theta+2}\right)^n e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(1+\theta+\theta x_i)\left(1-\left[1+\frac{\theta x_i}{\theta+2}\right]e^{-\theta x_i}\right)^{\alpha-1}\left(1-\left\{1-\left[1+\frac{\theta x_i}{\theta+2}\right]e^{-\theta x_i}\right\}^{\alpha}\right)^{\beta-1}. \tag{29}$$

The function is as follows:

$$l_n = \log(\phi) = n\log\alpha + n\log\beta + n\log\theta - n\log(\theta+2) - \theta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(1+\theta+\theta x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-\left[1+\frac{\theta x_i}{\theta+2}\right]e^{-\theta x_i}\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\log\left(1-\left\{1-\left[1+\frac{\theta x_i}{\theta+2}\right]e^{-\theta x_i}\right\}^{\alpha}\right). \tag{30}$$

Now setting,

$$\frac{\partial l_n}{\partial\alpha} = 0, \frac{\partial l_n}{\partial\beta} = 0, \frac{\partial l_n}{\partial\theta} = 0.$$

We have obtained,

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]+\left(-1+\beta\right)\sum_{i=1}^{n}-\frac{\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}=0$$

$$\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}\right]=0$$

$$\frac{n}{\theta}-\frac{n}{2+\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{1+x_i}{1+\theta+\theta x_i}+\left(-1+\alpha\right)\sum_{i=1}^{n}\frac{-e^{-\theta x_i}\left(-\frac{\theta x_i}{\left(2+\theta\right)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)}$$

$$+\left(-1+\beta\right)\sum_{i=1}^{n}-\frac{\alpha\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{\left(2+\theta\right)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}=0$$

Estimates are derived by partially differentiating the equation (30) with respect to α,β and θ , but the EM method is the best strategy for estimating the parameters. By solving this nonlinear system of equations, the MLE $\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ of $\left(\alpha,\beta,\theta\right)$ is obtained. To numerically optimize the sample likelihood function given in (29), it is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton approach. The MLE $\hat{\gamma}=\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ can be approximated as tri-variate normal with mean $\hat{\eta}$ and variance-covariance matrix equal to the inverse of the expected information matrix. $I^{-1}(\gamma)$ denotes the variance-covariance matrix of $\hat{\gamma}$. The members of the three-dimensional matrix $I(\gamma)$ can be estimated using $\sqrt{n}\left(\hat{\gamma}-\gamma\right)\rightarrow N_3\left(0,nI^{-1}(\gamma)\right)$.

Where, $I^{-1}(\gamma)$ is the variance-covariance matrix of $\hat{\gamma}$ .The elements of the 3 × 3 matrix $I(\gamma)$ can be estimated by

$$I^{-1}=-E\begin{bmatrix}\frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\partial\beta} & \frac{\partial^2 \log L}{\partial\alpha\partial\theta}\\ \frac{\partial^2 \log L}{\partial\beta\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} & \frac{\partial^2 \log L}{\partial\beta\partial\theta}\\ \frac{\partial^2 \log L}{\partial\theta\partial\alpha} & \frac{\partial^2 \log L}{\partial\theta\partial\beta} & \frac{\partial^2 \log L}{\partial\theta^2}\end{bmatrix}=\begin{bmatrix} I_{11}^{-1} & I_{12}^{-1} & I_{13}^{-1}\\ I_{21}^{-1} & I_{22}^{-1} & I_{23}^{-1}\\ I_{31}^{-1} & I_{32}^{-1} & I_{33}^{-1}\end{bmatrix}$$

The Hessian matrix entries that correspond to the elements in Equation (29),

$$\frac{\partial^2 \log L}{\partial\alpha^2}=-\frac{n}{\alpha^2}+\left(-1+\beta\right)$$

$$\sum_{i=1}^{n}\left(-\frac{\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]^2\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{2\alpha}}{\left(1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}\right)^2}-\frac{\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]^2\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}\right)$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}-\frac{\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\theta} = \sum_{i=1}^{n} \frac{-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)} + (-1+\beta)$$

$$\sum_{i=1}^{n}\left(\begin{array}{l} -\dfrac{\alpha\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+2\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)}{\left(1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}\right)^2} \\ -\dfrac{\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}} \\ -\dfrac{\alpha\log\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right]\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}} \end{array}\right)$$

$$\frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\theta} = \sum_{i=1}^{n} -\frac{\alpha\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}+\frac{n}{(2+\theta)^2}+\sum_{i=1}^{n} -\frac{(1+x_i)^2}{(1+\theta+\theta x_i)^2}+(-1+\alpha)A-(1-\beta)\sum_{i=1}^{n} B+C+D$$

$$A = \sum_{i=1}^{n}\left(-\frac{\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^2}{\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^2}+\frac{-e^{-\theta x_i}\left(\frac{2\theta x_i}{(2+\theta)^3}-\frac{2x_i}{(2+\theta)^2}\right)+2e^{-\theta x_i}x_i\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)-e^{-\theta x_i}x_i^2\left(1+\frac{\theta x_i}{2+\theta}\right)}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)}\right)$$

$$B = \frac{\alpha^2\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-2+2\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^2}{\left(1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}\right)^2},$$

$$C = \frac{(-1+\alpha)\alpha\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-2+\alpha}\left(-e^{-\theta x_i}\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)+e^{-\theta x_i}x_i\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^2}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}},$$

$$D = \frac{\alpha\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{-1+\alpha}(-e^{-\theta x_i}\left(\frac{2\theta x_i}{(2+\theta)^3}-\frac{2x_i}{(2+\theta)^2}\right)++2e^{-\theta x_i}x_i\left(-\frac{\theta x_i}{(2+\theta)^2}+\frac{x_i}{2+\theta}\right)-e^{-\theta x_i}x_i^2\left(1+\frac{\theta x_i}{2+\theta}\right))}{1-\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{2+\theta}\right)\right)^{\alpha}}$$

The approximate two-sided confidence intervals for α, β, θ are as follows:

$$\widehat{\alpha} \pm z_{\alpha/2}\left(I_{11}^{-1}(\widehat{\gamma})\right)^{1/2} \quad \widehat{\beta} \pm z_{\alpha/2}\left(I_{22}^{-1}(\widehat{\gamma})\right)^{1/2} \text{ and } \widehat{\theta} \pm z_{\alpha/2}\left(I_{22}^{-1}(\widehat{\gamma})\right)^{1/2}$$

where z_{α} denotes the upper α th quantile of the standard normal distribution.

7. Application

7.1. Simulation Studies

We ran a simulation to explore the flexibility and competency of the Kumaraswamy Extended Garima distribution class. Eghwerido et al. [9], Team, R.C [28] used R for computing. The simulation was investigated further below;

- The quantile function of the Kumaraswamy Extended Garima distribution was used to create the data, as shown in equation (9)
- The sample sizes of 5, 10, 20, 30, 50, 80 and 100, for, $\alpha = 0.5$, $\beta = 2$, $\theta = 0.5$,The MSE of the parameters $\gamma = (\alpha, \beta, \theta)$ .The simulation study investigated the mean estimates, variance, MSE (Mean Square Error).
- $$MSE = \sum_{i=1}^{1000} \frac{(\hat{\gamma} - \gamma)^2}{1000}.$$

Table 1. Monte Carlo simulation study for the Kumaraswamy Garima Distrubution

	Parameter	Variance	MSE
5	$\alpha = 0.5$	0.098978	0.1572
	$\beta = 2$	0.097515	0.1516
	$\theta = 0.5$	0.099191	0.1338
10	$\alpha = 0.5$	0.8425	0.1226
	$\beta = 2$	0.089813	0.1377
	$\theta = 0.5$	0.080249	0.1096
20	$\alpha = 0.5$	0.089228	0.0753
	$\beta = 2$	0.076534	0.0832
	$\theta = 0.5$	0.072762	0.0804
30	$\alpha = 0.5$	0.080662	0.0716

	$\beta = 2$	0.090617	0.0685
	$\theta = 0.5$	0.084565	0.0738
50	$\alpha = 0.5$	0.093203	0.0700
	$\beta = 2$	0.085992	0.0694
	$\theta = 0.5$	0.084664	0.0704
80	$\alpha = 0.5$	0.07661	0.0692
	$\beta = 2$	0.08511	0.0681
	$\theta = 0.5$	0.070994	0.0677
100	$\alpha = 0.5$	0.074335	0.0652
	$\beta = 2$	0.070458	0.0586
	$\theta = 0.5$	0.070579	0.0590

In our simulation analysis, the experiment from Table 1 was repeated 1000 times, and the MSE values fell as the sample size increased. This demonstrates that the parameter estimate is consistent with the asymptotic theory or big sample theory.

7.2. Real Data Analysis

We employed four real lifespan data sets in Kumaraswamy's Extended Garima distribution and compared the model to exponentiated Garima (Rather and Subramanian [24]), Garima, Exponential distribution, Generalized Exponential, and Exponentiated Weibull. To compare the Kumaraswamy Extended Garima, exponentiated Garima distribution with Garima, Exponential distribution, Generalized Exponential, and Exponentiated Weibull distributions, these distributions are provided in the following order:

(i) Exponential distribution

$$f(x;\theta) = \theta e^{-\theta x}, x \geq 0, \theta > 0$$

(ii) Garima distribution

$$f(x;\theta) = \frac{\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}, x > 0, \theta > 0.$$

(iii) Exponentiated Garima

$$\mathrm{f}(x;\theta,\alpha) = \frac{\alpha\theta}{\theta+2}(1+\theta+\theta x)e^{-\theta x}\left(1-\left(1+\frac{\theta x}{\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}, x > 0, \alpha, \theta > 0.$$

(iv) Generalized Exponential

$$\mathrm{f}(x;\theta,\alpha) = \alpha\theta e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}, x > 0, \alpha, \theta > 0.$$

(v) Exponentiated Weibull

$$\mathrm{f}(x;\theta,\alpha,\beta) = \frac{\alpha\theta}{\beta^{\alpha}}x^{\alpha-1}e^{-(x/\beta)^{\alpha}}\left(1-e^{-(x/\beta)^{\alpha}}\right)^{\theta-1},$$

$x > 0, \alpha, \theta > 0, \beta > 0.$

We take into account metrics such as the Bayesian Information Criterion (BIC), the Akaike Information Criterion (AIC), the Hannan Quinn (HIC), and -2 logL. The model utilized is best if the values of AIC, BIC, HIC, AICC, and - 2 log L are as low as possible. The following formulas can be used to calculate AIC, BIC, HIC, AICC, and -2 logL;

$$AIC = 2k - 2\log L$$

$$BIC = k\log n - 2\log L$$

$$HIC = 2\mathrm{kln}(\mathrm{In}(n)) - 2\log L$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Data set 1: The data set exhibits the tensile strength, measured in GPa, of n=69 carbon fibers tested under tension at gauge lengths of 20mm, as reported by (see M. Bader and Priest [3], Almanjahie et al. [1], Mead et al. [17]). The information is as follows:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145,3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027 ,4.225, 4.395, 5.020.

Data set 2: To define the novel findings presented in this paper, we applied the kw extended garima distribution to the data set utilized by Nichols and Padgett [22], which included 100 research on the fracture stress of carbon fibers (in Gba).

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95,2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22,1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57,1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Data Set 3. Bjerkedalen [4] investigated and informed on the survival periods in days of 72 guinea pigs infected with virulent tubercle bacilli.

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08,1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6,1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45,2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data Set 4. The data shown below represent the failure times of 84 aircraft windshields as reported by Cordeiro, G. M., and Castro, M. A [7]. The data sets are listed below.

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 2. MLEs and the AIC, BIC and Logl statistics.

Data Set	**Distribution**	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	**-2 logL**	**AIC**	**BIC**	**HIC**	**AICC**
1	Kw Extended Garima	0.39	7.66	0.1148	61.205	67.205	66.721	69.864	67.574
	Exponentiated Garima		-	2.1527	112.92	116.92	115.94	118.04	117.10
	Garima	-	-	0.4784	256.32	258.32	258.15	259.20	258.38
	Exponential	-	-	0.3266	266.89	268.89	268.73	269.77	268.95
	Generalized Exponantial	218.23	-	1.9458	113.03	115.03	116.71	118.80	115.23
	Exponantiated Weibull	0.81	1.50	31.802	116.62	118.62	117,66	125.28	118.99
2	Kw Extended Garima	0.13	2.90	0.0363	111.87	117.87	117.87	121.03	118.12
	Exponentiated Garima	6.68	-	1.2109	240.49	244.49	244.49	246.60	245.05

	Garima	-	-	0.5485	379.41	381.41	381.41	382.47	381.46
	Exponenti al	-	-	0.3814	392.74	394.74	394.74	395.79	394.78
	Generalize d Exponanti al	7.78	-	1.0131	292.36	294.34	296.36	298.47	294.46
	Exponanti ated Weibull	0.37	2.40	1.3198	286.91	288.91	292.91	296.07	289.16
3	Kw Extended Garima	0.08	1.90	0.1781	103.74	109.74	107.46	112.46	110.09
	Exponenti ated Garima	3.22	-	1.3626	188.13	192.13	191.85	193.94	192.3
	Garima	-	-	0.78350	220.05	222.05	221.90	221.50	222.1
	Exponenti al	-	-	0.56554	226.07	228.07	227.93	228.98	228.1
	Generalize d Exponanti al	3.62	-	1.12713	188.47	190.47	192.18	194.28	190.6
	Exponanti ated Weibull	0.88	1.16	2.6523	192.1	194.1	197.73	218.32	194.5

4	Kw Extended Garima	6	2.43	0.5	92.25	98.25	98.026	101.18	98.55
	Exponentiated Garima	3.22	-	0.9508	278.6	282.6	282.54	284.65	282.8
	Garima	-	-	0.5585	319.6	321.6	321.57	322.62	321.7
	Exponential			0.3902	329.9	331.9	331.89	332.95	332.0
	Generalized Exponantial	3.59	-	0.7594	282.7	284.7	286.64	288.7	284.9
	Exponantiated Weibull	0.25	5.82	0.28168	261.58	263.58	267.358	270.51	263.88

Table 2 clearly shows that the Kumaraswamy Garima distribution has values of AIC, BIC, HIC, AICC, and -2 logL when compared to exponentiated Garima distribution, Garima, Exponential distributions, Generalized Exponential, and Exponentiated Weibull. As a result, the Kumaraswamy Garima distribution fits better than the exponentiated Garima, Garima, Exponential, Generalized Exponential, and Exponentiated Weibull distributions.

8. Conclusion

In our research, we developed a novel distribution known as the Kw- Extended Garima, where Garima is a single-parameter life distribution. We've also gotten the probability and cumulative distribution functions for this new distribution. The reliability function, hazard rate, failure rate, inverse hazard function, odd function, cumulative hazard function, r-th moment, moment generating function, characteristic function, moments, mean and variance, Bonferroni and Lorenz curves, order statistics, MLE, and mean time between failures (MTBF) were then obtained based on these functions. We plotted these functions in the R program and obtained some results in the Mathematica program. We use maximum likelihood to forecast the parameters of the innovative model. In addition, we create the information matrix. With the help of simulation analysis, our distribution supported the findings of the asymptotic theory. We discovered that our unique model outperformed previous models when applied to four real-world data sets. The Kumaraswamy Garima distribution has been shown to be superior to the other distributions.

Data Availability

There is data supporting the findings of this study, where it was taken from is clearly stated in the text, it can be sent again by the relevant author if desired.

Conflict of Interest

The article's authors declare that there is no conflict of interest between them.

Author's Contributions

The contribution of the authors is equal.

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