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Characterization of a New Type of Topological Sequence Spaces and Some Properties

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Abstract: Examination of spaces in the field of functional analysis, especially revealing their topological and algebraic structures, is very important in terms of forming a basis for studies in the field of pure mathematics and applied sciences. In this context, topology, which was widely used only in the field of geometry at the beginning, gave a solid foundation to the fields in which it was used by causing methodological changes in all branches of mathematics over time. Frechet-Coordinate space (FK space) is a concept that has a functional role in fields such as topological sequence spaces and summability. Topological vector spaces are described as linear spaces defined by a topology that provides continuous vector space operations. If this vector space has a complete metric space structure, it is called Frechet space, and if it has a topology with continuous coordinate functions, it is called Frechet-Coordinate (FK) space. The theory of FK spaces has gained more importance in recent years and has found applications in various fields thanks to the efforts of many researchers. If the topology of an FK space can be derived from the norm, this space is called as a BK space. In this study, $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$, and $bs^\lambda(\Delta)$ difference sequence spaces are defined, and it is revealed that these spaces are BK spaces. In addition, considering the topological properties of these spaces, some spaces that are isomorphic and their duals have been determined.

Yeni Tip Topolojik Dizi Uzaylarının Karakterizasyonu ve Bazı Özellikleri

Anahtar Kelimeler
BK uzayları,
Fark dizi uzayları,
Frechet-koordinat uzayları,
Schauder bazı,
Topolojik dizi uzayları.

Öz: Fonksiyonel analiz alanında uzayların incelenmesi, özellikle topolojik ve cebirsel yapılarının ortaya konulması, pür matematik ve uygulamalı bilimler alanındaki çalışmalara temel oluşturması açısından oldukça önemlidir. Bu bağlamda başlangıçta sadece geometri alanında yaygın olarak kullanılan topoloji, zamanla matematiğin tüm dallarında metodolojik değişikliklere neden olarak kullanıldığı alanlara sağlam bir temel kazandırmıştır. Frechet-Koordinat uzayı (FK uzayı), topolojik dizi uzayları ve toplanabilirlik gibi alanlarda işlevsel rolü olan bir kavramdır. Topolojik vektör uzayları, sürekli vektör uzayı işlemlerini sağlayan bir topoloji tarafından tanımlanan lineer uzaylar olarak tanımlanır. Bu vektör uzayı tam bir metrik uzay yapısına sahipse Frechet uzayı, sürekli koordinat fonksiyonlarına sahip bir topolojiye sahipse Frechet-Koordinat (FK) uzayı olarak adlandırılır. FK uzayları teorisi, son yıllarda daha da önem kazanmış ve birçok araştırmacının çabaları sayesinde çeşitli alanlarda uygulama alanı bulmuştur. Bir FK uzayının topolojisi normdan türetilebiliyorsa, bu uzaya BK uzayı denir. Bu çalışmada ise $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$ ve $bs^\lambda(\Delta)$ fark dizi uzayları tanımlanmıştır ve bu uzayların BK uzayları olduğu sonucuna ulaşılmıştır. Ayrıca bu uzayların topolojik özellikleri dikkate alınarak bu uzaylara izomorf olan bazı uzaylar ve bu uzayların dualleri belirlenmiştir.

1. INTRODUCTION AND PRELIMINARIES

The main motivation point in the studies conducted in functional analysis and topology is to obtain the expansions and generalizations of spaces, to reveal their various properties and finally to form a new space. Researchers working in this field have used various methods to serve this purpose. By using the domain of an infinite triangular matrices, which is one of these methods, on standard sequence spaces, many new sequence spaces have been created by using Cesaro matrix and Nörlund matrix (see the papers [1], [2]). Sequence spaces are one of the subjects that have been the focus of attention of many researchers due to the topological and algebraic structure they contain. Much researches have been made on the basis of the properties of these sequence spaces and their contribution to the field. Especially in studies in the field of summability theory, topological sequence spaces and difference sequence spaces have contributed to obtaining functional results.

The concept of difference sequence space has been introduced by Kızmaz in [3] as follows:

Suppose that $X = l_\infty, c, c_0$. Then,

$$X(\Delta) = \{x = (x_k) \in w: \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \in X\}$$

will be called the difference sequence space.

In [5], this sequence spaces have been extended by Et as following:

$$X(\Delta^2) = \{x = (x_k) \in w: \Delta^2 x = (\Delta^2 x_k) = (x_k - x_{k+1}) \in X\}.$$

By a similar methodology, the authors have given modification of these spaces for the integer m as:

$$X(\Delta^m) = \{x = (x_k) \in w: \Delta^m x \in X\}$$

where $\Delta^0 x = (x_k \Delta^m x = (\Delta^m x_k - \Delta^m x_{k+1})$ and $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

On all of these efforts, another motivated generalization has been established by Et and Esi (see [7]) as follows:

Suppose that $v = (v_k)$ is a sequence for complex numbers.

$$X(\Delta_v^m) = \{x = (x_k) \in w: \Delta_v^m x \in X\}$$

where $m, k \in \mathbb{N}$

$$\Delta_v^0 x = v_k x_k \Delta_v^m x = (\Delta_v^m x_k - \Delta_v^m x_{k+1})$$

and

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$

Sequence spaces are an important concept in mathematical analysis and play a methodologically functional key role in the work of many researchers. Although it was known as a branch of topology used only in geometry in the beginning, it has become a structure that contributes to all pure and applied sciences in time. The theory of FK spaces is a structure used in sequence spaces, summability, and matrix transformations as a topological subject. Topological vector spaces are linear spaces with a topology that enables continuous vector space operations. If this vector space has a complete metric space structure, it is called Frechet space, and if it also has a topology with continuous coordinate functions, it is called Frechet-Coordinate space (FK space).

By ω, we mean the vector space containing all real- or complex-valued sequences that are topologized through coordinatewise convergence. Any vector subspace of ω is said to be a sequence space. A sequence space X with a locally convex topology τ is referred to as a K-space if the inclusion mapping $(X,\tau) \to \omega$ is continuous when ω has the topology of coordinatewise convergence. Additionally, if τ is complete and metrizable, (X,τ) is referred to be an FK-space. A BK-space is an FK-space with a normable topology. For further results on these concepts, see the papers [4-14].

2. MATRIX TRANSFORMATIONS

In this section, we will present some lemmas related to the matrix transformations by introducing matrix transformations in the sequence spaces.

Definition 1 *([13]) Suppose that $A = (a_{nk})$ is a infinite matrix with real or complex terms and $x = (x_k)$ is a sequence. For $n \in N$, the following sequences are convergent*

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

then, the sequence $((Ax)_n)$ is called the transformation sequence obtained by the matrix A of the sequence (x_k).

In the sequel of the paper, every negative index term such as λ_{-1} and x_{-1} will be assumed to be equal to zero.

Lemma 2.1 *([16]) $A = (a_{nk}) \in (cs_0: l_1)$ if and only if the following condition holds:*

$$\sup_{N,K \in F} \left|\sum_{n \in N} \sum_{k \in K} \left(a_{nk} - a_{n,k+1}\right)\right| < \infty. \quad (2.1)$$

Lemma 2.2 *([16]) $A = (a_{nk}) \in (cs: l_1)$ if and only if the following condition holds:*

$$\sup_{N,K \in F} \left|\sum_{n \in N} \sum_{k \in K} \left(a_{nk} - a_{n,k-1}\right)\right| < \infty. \quad (2.2)$$

Lemma 2.3 *([16]) $A = (a_{nk}) \in (bs: l_1)$ if and only if the following condition holds: $\forall k \in \mathbb{N}$*

$$\lim_k a_{nk} = 0. \quad (2.3)$$

Lemma 2.4 *([16]) $A = (a_{nk}) \in (cs_0: c)$ if and only if the condition*

$$\sup_n \sum_k \left|a_{nk} - a_{n,k+1}\right| < \infty \quad (2.4)$$

will be held and for $\forall k \in \mathbb{N}$, the following limit will be existed

$$\lim_k \left(a_{nk} - a_{n,k+1}\right). \quad (2.5)$$

Lemma 2.5 *([16]) $A = (a_{nk}) \in (cs: c)$ if and only if the following condition holds: $\forall k \in \mathbb{N}$*

$$\lim_k a_{nk} \text{ is exist.} \quad (2.6)$$

Lemma 2.6 *([16]) $A = (a_{nk}) \in (bs: c)$ if and only if the following condition holds: $\forall k \in \mathbb{N}$*

$$\sum_k \left|a_{nk} - a_{n,k-1}\right| \text{ is convergent.} \quad (2.7)$$

Lemma 2.7 *([16]) $A = (a_{nk}) \in (cs_0: l_\infty)$ if and only if the condition that is given in (2.4) is satisfied.*

Lemma 2.8 *([16]) $A = (a_{nk}) \in (cs_0: l_\infty)$ if and only if*

$$\sup_n \sum_k \left|a_{nk} - a_{n,k-1}\right| < \infty. \quad (2.8)$$

Lemma 2.9 *([16]) $A = (a_{nk}) \in (bs: l_\infty)$ if and only if the conditions of (2.3) and (2.4) are satisfied.*

Lemma 2.10 *([16]) $A = (a_{nk}) \in \left(cs_0: l_p\right)$ if and only if*

$$\sup_k \sum_n \left|\sum_{k \in K} \left(a_{nk} - a_{n,k+1}\right)\right|^p < \infty \quad (2.9)$$

for $1 < p < \infty$.

Lemma 2.11 *([16]) $A=(a_{nk})\in(cs:l_p)$ if and only if*

$$\sup_k \sum_n \left|\sum_{k\in K}\left(a_{nk}-a_{n,k-1}\right)\right|^p<\infty \qquad (2.10)$$

for $1<p<\infty$.

Lemma 2.12 *$A=(a_{nk})\in(bs:l_p)$ if and only if the conditions of (2.3) and (2.9) are satisfied.*

3. DIFFERENCE SEQUENCE SPACES

In this section, we will define the sequence spaces $cs_0^\lambda(\Delta), cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$, then we will show that these spaces are BK-spaces. In addition, we will calculate the Schauder bases of $cs_0^\lambda(\Delta), cs^\lambda(\Delta)$ spaces and obtain the duals of $cs_0^\lambda(\Delta), cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$.

Definition 2 *([3]) Suppose that $\Delta x=(a_k-a_{k+1})$ for any sequence $x\in w$. Assume that X is any sequence space, the difference sequence spaces can be defined as:*

$$\Delta x=\{x=(x_k)\in w:\Delta x\in X\}.$$

In [17], Mursaleen ve Noman have defined the spaces c_0^λ, c^λ and l_∞^λ by using the domain of $\wedge=(\lambda_{nk})$ for $\forall n,k\in N$

$$\lambda_{nk}=\begin{cases}\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}, & 0\leqslant k\leqslant n,\\ 0, & k>n,\end{cases} \qquad (3.1)$$

on c_0, c and l_∞ where $\lambda=(\lambda_k)$ is an increasing sequence with the following assumptions:

$$0<\lambda_0<\lambda_1<\lambda_2\ldots,\text{ve}\lim_{k\to\infty}\lambda_k=\infty$$

Then, in [4], the authors established the following sequence spaces as follows:

$$cs^\lambda=\left\{x=(x_k)\in w:\lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k \text{ exist}\right\},$$

$$cs_0^\lambda=\left\{x=(x_k)\in w:\lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k=0\right\},$$

$$bs^\lambda=\left\{x=(x_k)\in w:\sup_m\left|\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k\right|<\infty\right\}$$

by using the domain of $\wedge=(\lambda_{nk})$ on cs, cs_0 and bs.

Now, it is time to define the difference sequence spaces of $cs_0^\lambda(\Delta), cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ by using the sequence spaces cs_0^λ, cs^λ and bs^λ with matrix transformations:

$$cs_0^\lambda(\Delta)=\left\{x=(x_k):\lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(x_k-x_{k-1})=0\right\},$$

$$cs^\lambda(\Delta)=\left\{x=(x_k):\lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(x_k-x_{k-1})\text{ exist}\right\},$$

$$bs^\lambda(\Delta)=\left\{x=(x_k):\sup_m\left|\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(x_k-x_{k-1})\right|<\infty\right\}.$$

Let us define a new matrix as for $n,k\in N$:

$$\bar{\lambda}_{nk}=\begin{cases}\frac{(\lambda_k-\lambda_{k-1})-(\lambda_{k+1}-\lambda_k)}{\lambda_n}, & k<n,\\ \frac{\lambda_n-\lambda_{n-1}}{\lambda_n}, & k=n,\\ 0, & k>n\end{cases}$$

that is obtained by multiplying $\wedge=(\lambda_{nk})$ and difference matrix.

Here, we can consider these new difference sequence spaces as the domain of $\bar{\wedge}=(\bar{\lambda}_{nk})$ on the sequence spaces, namely: $cs_0^\lambda(\Delta)=(cs_0)_{\bar{\wedge}}$, $cs^\lambda(\Delta)=(cs)_{\bar{\wedge}}$, $bs^\lambda(\Delta)=(bs)_{\bar{\wedge}}$.

$$(\bar{\wedge}x)_n=\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(x_k-x_{k-1}). \qquad (3.2)$$

Theorem 3.1 *The spaces $cs_0^\lambda(\Delta), cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ are linear spaces. Besides, the spaces are BK −spaces with the following norm:*

$$\|x\|_{bs^\lambda(\Delta)}=\|\bar{\wedge}x\|_{bs}=\sup_m\left|\sum_{n=0}^{m}(\bar{\wedge}x)_n\right|.$$

Proof. Firstly, we will prove that $cs_0^\lambda(\Delta)$ is a linear space. For $x,y\in cs_0^\lambda(\Delta)$ and α,β scalars, we will show that $\alpha x+\beta y\in cs_0^\lambda(\Delta)$. Let us consider

$$cs_0^\lambda(\Delta)=\{x=(x_k)\in w:\bar{\wedge}x\in cs_0\}$$

and $\bar{\wedge}x$, $\bar{\wedge}y\in cs_0$. For all $m\in N$, we can write

$$\begin{aligned}\sum_{n=0}^{m}\bar{\wedge}_n(\alpha x+\beta y)&=\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(\alpha x_k+\beta y_k-\alpha x_{k-1}-\beta y_{k-1})\\&=\alpha\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(x_k-x_{k-1})\\&\quad+\beta\sum_{n=0}^{m}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(y_k-y_{k-1})\\&=\alpha\sum_{n=0}^{m}\bar{\wedge}_n(x)+\beta\sum_{n=0}^{m}\bar{\wedge}_n(y).\end{aligned}$$

If we set $m\to\infty$ in the last step, we have

$$\lim_{m\to\infty}\sum_{n=0}^{m}\bar{\wedge}_n(\alpha x+\beta y)=0.$$

Then, we obtain $\bar{\wedge}(\alpha x+\beta y)\in cs_0$, namely $\alpha x+\beta y\in cs_0^\lambda(\Delta)$. This implies that $cs_0^\lambda(\Delta)$ is alinear space. By a similar argument, one can show that $cs^\lambda(\Delta)$ is a linear space. We omit the details. It is clear to show that $bs^\lambda(\Delta)$ is a linear space as following:

$$\sup_m\left|\sum_{n=0}^{m}\bar{\wedge}_n(\alpha x+\beta y)\right|$$

$$= \sup_m \left| \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(\alpha x_k + \beta y_k - \alpha x_{k-1} - \beta y_{k-1}) \right|$$

$$\leqslant |\alpha| \sup_m \left| \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) \right| + |\beta| \sup_m \left| \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(y_k - x_{yk-1}) \right|$$

$$= |\alpha| \sup_m \left| \sum_{n=0}^{m} \bar{\wedge}_n (x) \right| + |\beta| \sup_m \left| \sum_{n=0}^{m} \bar{\wedge}_n (y) \right|$$

for $x, y \in bs^{\lambda}(\Delta)$ and α, β scalars. Then, from the definition of $bs^{\lambda}(\Delta)$, we can write $\bar{\wedge} x, \bar{\wedge} y \in bs$. Therefore, for $\bar{\wedge} x, \bar{\wedge} y \in bs$, we provide

$$\sup_m \left| \sum_{n=0}^{m} \bar{\wedge}_n (\alpha x + \beta y) \right| < \infty.$$

This completes the proof. Also, one can say that these spaces are BK −spaces.

Theorem 3.2 *The sequence spaces* $cs_0^{\lambda}(\Delta), cs^{\lambda}(\Delta)$ *and* $bs^{\lambda}(\Delta)$ *are isometrically isomorphic to the sequence spaces* cs_0, cs *and* bs*, respectively, namely* $cs_0^{\lambda}(\Delta) \cong cs_0$, $cs^{\lambda}(\Delta) \cong cs$ *and* $bs^{\lambda}(\Delta) \cong bs$.

Proof. Assume that $X = \{cs, cs_0, bs\}$ and $X^{\lambda}(\Delta) = \{cs^{\lambda}(\Delta), cs_0^{\lambda}(\Delta), bs^{\lambda}(\Delta)\}$. To prove the result, we must show the existence of linear, injective and surjective mapping as:

$$T: X^{\lambda}(\Delta) \to X$$
$$x \to T(x) = \bar{\wedge}(x) = y.$$

For $x = (x_j), u = (u_j) \in X^{\lambda}(\Delta)$ and α, β scalars, we cam write

$$T(\alpha x + \beta u) = \bar{\wedge}(\alpha x + \beta u) = \alpha \bar{\wedge}(x) + \beta \bar{\wedge}(u) = \alpha T(x) + \beta T(u).$$

Then, T is linear.

Let we prove that T is injective. We must prove that if $Tx = \theta$, then $x = \theta$. If we assume that $Tx = \theta$, then we have

$$k = 0, x_0 = 0,$$
$$k = 1, x_1 = 0,$$
$$\vdots$$
$$k = n,\ x_n = 0.$$

This implies that $x = \theta$.

Let us consider $y = (y_k) \in X$ and the sequence $x = (x_k(\Delta))$ is defined as:

$$x_k(\Delta) := \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i;\ (k \in N). \quad (3.3)$$

Then, for $\forall k \in N$, we have

$$x_k(\Delta) - x_{k-1}(\Delta) = \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} y_i.$$

By using (3.2), for $\forall n \in \mathbb{N}$, we get

$$(\bar{\wedge} x)_n = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{i=k-1}^{k} (-1)^{k-i} \lambda_i y_i = y_n$$

Therefore, we obtain $\bar{\wedge} x = y$. Since $y = (y_k) \in X$ we provide $\bar{\wedge} x \in X$. T is surjective. Finally, by using

$$\|Tx\|_{bs} = \|y(\lambda)\|_{bs} = \|\bar{\wedge} x\|_{bs} = \|x\|_{bs^{\lambda}(\Delta)}$$

we conclude that T is a linear, bijective and surjective mapping.

The Schauder basis of $cs_0^{\lambda}(\Delta)$ ve $cs^{\lambda}(\Delta)$ will be presented in the following result.

Remark 1 *Suppose that* $\alpha_k(\lambda) = (\bar{\wedge} x)_k$ *for* $\forall k \in N$. *Let us define the sequence* $b^{(k)}(\lambda) = \left\{b_n^{(k)}(\lambda)\right\}_{n=0}^{\infty}$ *as*

$$b_n^{(k)}(\lambda) = \begin{cases} 0, & n < k, \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}, & n = k, \\ \dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \dfrac{\lambda_k}{\lambda_{k+1} - \lambda}, & n > k, \end{cases}$$

for $k \in N$. In this case, the sequence $\left\{b^{(k)}(\lambda)\right\}_{k=0}^{\infty}$ is the Schauder basis of the spaces $cs_0^{\lambda}(\Delta)$ and $cs^{\lambda}(\Delta)$. Thus, $\forall x \in cs_0^{\lambda}(\Delta)$ or the sequence $cs^{\lambda}(\Delta)$ has a unique form as follows:

$$x = \sum_k a_k(\lambda) b^{(k)}(\lambda).$$

4. THE INCLUSION RELATIONS

In this section, we will present some inclusion relations of the spaces $cs_0^{\lambda}(\Delta), cs^{\lambda}(\Delta)$ and $bs^{\lambda}(\Delta)$.

Theorem 4.1 *The inclusion relations hold as*

$$cs_0^{\lambda}(\Delta) \subset cs^{\lambda}(\Delta) \subset bs^{\lambda}(\Delta).$$

Proof. It is obvious that $cs_0^{\lambda}(\Delta) \subset cs^{\lambda}(\Delta) \subset bs^{\lambda}(\Delta)$. To prove the sharpness of these inclusion relations, let us consider the following sequence:

$$x_k = \sum_{i=0}^{k} \frac{\frac{1}{(i+2)(i+3)} \lambda_i - \frac{1}{(i+1)(i+2)} \lambda_{i-1}}{\lambda_i - \lambda_{i-1}}, \qquad (\forall k \in \mathbb{N}).$$

For $n \in \mathbb{N}$, we can write

$$(\bar{\Lambda} x)_n = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \left(\frac{1}{(k+2)(k+3)} \lambda_k - \frac{1}{(k+1)(k+2)} \lambda_{k-1} \right) = \frac{1}{(n+2)(n+3)}.$$

Then, for $m \in \mathbb{N}$, we have

$$\sum_{n=0}^{m} (\bar{\Lambda} x)_n = \frac{1}{2} - \frac{1}{m+3}.$$

If we set $m \to \infty$, we get

$$\sum_{n=0}^{\infty} (\bar{\Lambda} x)_n = \frac{1}{2}.$$

This shows that $\bar{\Lambda} x \in cs$, cs_0. We conclude that $cs_0^{\lambda}(\Delta) \subset cs^{\lambda}(\Delta)$ is sharp.

Now, to show the sharpness of the inclusion $cs^{\lambda}(\Delta) \subset bs^{\lambda}(\Delta)$, we can write

$$y_k = \sum_{i=0}^{k} (-1)^i \left(\frac{\lambda_i + \lambda_{i-1}}{\lambda_i - \lambda_{i-1}} \right), \qquad (\forall k \in \mathbb{N}).$$

For all $n \in \mathbb{N}$, we have

$$\sum_{n=0}^{m} (\bar{\Lambda}y)_n = \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{k=0}^{n} (-1)^k (\lambda_k + \lambda_{k-1})$$
$$= \sum_{n=0}^{m} (-1)^n.$$

Then, $\bar{\Lambda}y \in bs, cs$. This implies that $cs^{\lambda}(\Delta) \subset bs^{\lambda}(\Delta)$ is sharp.

Theorem 4.2 *The inclusion $cs^{\lambda}(\Delta) \subset c_0^{\lambda}(\Delta)$ is sharp.*

Proof. Since, we know that when $x \in cs^{\lambda}(\Delta)$, $\bar{\Lambda}x \in cs$ therefore $\bar{\Lambda}x \in c_0$. The inclusion $cs^{\lambda}(\Delta) \subset c_0^{\lambda}(\Delta)$ is valid.

To demonsrate the sharpness of the inclusion relation, we define the sequence as

$$x_k = \sum_{i=0}^{k} \frac{1}{i+1}; \qquad (k \in \mathbb{N}).$$

Therefore, we have

$$\Delta x = (x_k - x_{k-1}) = \left(\frac{1}{k+1}\right) \in c_0$$

and $\Delta x \in c_0^{\lambda}$. This implies that $x \in c_0^{\lambda}(\Delta)$.

For all $n \in \mathbb{N}$, we get

$$(\bar{\Lambda}x)_n = \frac{1}{\lambda_n} \sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{k+1}$$
$$\geqslant \frac{1}{\lambda_n (n+1)} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})$$
$$= \frac{1}{n+1}.$$

Then, $\bar{\Lambda}x \notin cs$ and so $x \notin cs^{\lambda}(\Delta)$. Since x belongs to $c_0^{\lambda}(\Delta)$ spaces but not to $cs^{\lambda}(\Delta)$, we can write $cs^{\lambda}(\Delta) \subset c_0^{\lambda}(\Delta)$.

5. DUAL SPACES

In this section, we will determine the $\alpha-, \beta-$ and $\gamma-$duals of the sequence spaces $cs_0^{\lambda}(\Delta), cs^{\lambda}(\Delta)$ and $bs^{\lambda}(\Delta)$.

Theorem 5.1 *For $n, k \in \mathbb{N}$ the matrix $B^{\lambda} = (b_{nk}^{\lambda})$ can be defined as*

$$b_{nk}^{\lambda} = \begin{cases} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k}\right) a_n, & k < n, \\ \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n, & k = n, \\ 0, & k > n. \end{cases}$$

Then, we have $\{cs_0^{\lambda}(\Delta)\}^{\alpha} = \{bs^{\lambda}(\Delta)\}^{\alpha} = f_1^{\lambda}$ and $\{cs^{\lambda}(\Delta)\}^{\alpha} = f_2^{\lambda}$ where

$$f_1^{\lambda} = \left\{a = (a_n) \in w : \sup_{N, K \in \mathcal{F}} \left|\sum_{n \in N} \sum_{k \in K} (b_{nk}^{\lambda} - b_{n,k+1}^{\lambda})\right| < \infty\right\} \tag{5.1}$$

and

$$f_2^{\lambda} = \left\{a = (a_n) \in w : \sup_{N, K \in \mathcal{F}} \left|\sum_{n \in N} \sum_{k \in K} (b_{nk}^{\lambda} - b_{n,k-1}^{\lambda})\right| < \infty\right\}. \tag{5.2}$$

Proof. Let $a = (a_n) \in w$. Then, by using the relation (3.3) we have

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} a_n y_j$$
$$= \sum_{k=0}^{n} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} y_k - \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} y_{k-1}\right) a_n$$
$$= \sum_{k=0}^{n-1} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k}\right) y_k a_n + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} y_n a_n$$
$$= (B^{\lambda} y). \tag{5.3}$$

Thus, by the equation (5.3) when $x = (x_k) \in cs_0^{\lambda}(\Delta)$, $ax = (a_n x_n) \in \ell_1$ if and only if $y = (y_k) \in cs_0$ with $B^{\lambda} y \in \ell_1$. Namely, $a = (a_n) \in \{cs_0^{\lambda}(\Delta)\}^{\alpha}$ if and only if $B^{\lambda} \in (cs_0 : \ell_1)$. By using Lemma 2.1, with the matrix B^{λ} instead of A, we show that $a = (a_n) \in \{cs_0^{\lambda}(\Delta)\}^{\alpha}$ if and only if

$$\sup_{N, K \in \mathcal{F}} \left|\sum_{n \in N} \sum_{k \in K} (b_{nk}^{\lambda} - b_{n,k+1}^{\lambda})\right| < \infty. \tag{5.4}$$

Indeed for all $n \in \mathbb{N}$, if we have

$$\lim_k b_{nk}^{\lambda} = 0$$

then the condition of Lemma 2.3 holds. This implies that $\{cs_0^{\lambda}(\Delta)\}^{\alpha} = \{bs^{\lambda}(\Delta)\}^{\alpha} = f_1^{\lambda}$.

Similarly, by using the equation (5.3) it is obvious that $a = (a_n) \in \{cs^{\lambda}(\Delta)\}^{\alpha}$ if and only if $B^{\lambda} \in (cs : \ell_1)$. Consequently, if we set the matrix B^{λ} instead of the matrix A in Lemma 2.2, $a = (a_n) \in \{cs^{\lambda}(\Delta)\}^{\alpha}$ if and only if

$$\sup_{N, K \in \mathcal{F}} \left|\sum_{n \in N} \sum_{k \in K} (b_{nk}^{\lambda} - b_{n,k-1}^{\lambda})\right| < \infty. \tag{5.5}$$

Thus, we provide $\{cs^{\lambda}(\Delta)\}^{\alpha} = f_2^{\lambda}$. This completes the proof.

Theorem 5.2 *For all $k \in \mathbb{N}$ and*

$$\bar{a}_k(n) = \lambda_k \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right) \sum_{j=k+1}^{n} a_j\right]$$
$$(k < n),$$

let us define the sets of $f_3^{\lambda}, f_4^{\lambda}, f_5^{\lambda}, f_6^{\lambda}, f_7^{\lambda}$ and f_8^{λ} as follows

$$f_3^{\lambda} = \left\{a = (a_k) \in w : \sup_n \sum_{k=0}^{n-2} |\bar{a}_k(n) - \bar{a}_{k+1}(n)| < \infty\right\},$$
$$f_4^{\lambda} = \left\{a = (a_k) \in w : \sup_k \left|\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\right| < \infty\right\},$$
$$f_5^{\lambda} = \left\{a = (a_k) \in w : \lim_{n \to \infty} \left(\bar{a}_k(n) - \bar{a}_{k+1}(n)\right) \text{ exist } \quad (k \in \mathbb{N})\right\},$$
$$f_6^{\lambda} = \left\{a = (a_k) \in w : \sum_{j=k}^{\infty} a_j \quad \text{exist} \quad (k \in \mathbb{N})\right\},$$
$$f_7^{\lambda} = \left\{a = (a_k) \in w : \sum_{k=0}^{\infty} |\bar{a}_k(n) - \bar{a}_{k+1}(n)| \quad \text{convergent}\right\},$$
$$f_8^{\lambda} = \left\{a = (a_k) \in w : \lim_{k \to \infty} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\right) \quad \text{convergent}\right\}.$$

Then, we have $\{cs_0^{\lambda}(\Delta)\}^{\beta} = f_3^{\lambda} \cap f_4^{\lambda} \cap f_5^{\lambda}$, $\{cs^{\lambda}(\Delta)\}^{\beta} = f_3^{\lambda} \cap f_4^{\lambda} \cap f_6^{\lambda}$ and $\{bs^{\lambda}(\Delta)\}^{\beta} = f_6^{\lambda} \cap f_7^{\lambda} \cap f_8^{\lambda}$.

Proof. Assume that $a = (a_k) \in w$ be a sequence and for all $n, k \in \mathbb{N}$ the matrix $T^\lambda = (t_{nk}^\lambda)$ is given as

$$(t_{nk}^\lambda) = \begin{cases} \bar{a}_k(n) & k < n, \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n & k = n, \\ 0 & k > n. \end{cases}$$

Then, we consider

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] a_k$$

$$= \sum_{k=0}^{n-1} \lambda_k \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{n} a_j \right] y_k + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n y_n$$

$$= \sum_{k=0}^{n-1} \bar{a}_k(n) y_k + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n y_n$$

$$= (T^\lambda y)_n; \qquad (n \in \mathbb{N}). \tag{5.6}$$

From the equation (5.6) when $x = (x_k) \in cs_0^\lambda(\Delta)$, $ax = (a_n x_n) \in cs$ if and only if when $y = (y_k) \in cs_0$, $T^\lambda y \in c$. Namely, $a = (a_n) \in \{cs_0^\lambda(\Delta)\}^\beta$ if and only if $T^\lambda \in (cs_0 : c)$. Thus, from Lemma 2.4, we have

$$\sup_n \sum_{k=0}^{n-2} |\bar{a}_k(n) - \bar{a}_{k+1}(n)| < \infty, \tag{5.7}$$

$$\sup_n \left| \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n \right| < \infty \tag{5.8}$$

and we get

$$\lim_{n\to\infty} \left(\bar{a}_k(n) - \bar{a}_{k+1}(n) \right) \quad \text{(there exist for } k \in \mathbb{N}). \tag{5.9}$$

Then, $\{cs_0^\lambda(\Delta)\}^\beta = f_3^\lambda \cap f_4^\lambda \cap f_5^\lambda$.

Similarly, from the equation of (5.6), $a = (a_n) \in \{cs^\lambda(\Delta)\}^\beta$ if and only if $T^\lambda \in (cs : c)$. Then, we get (5.7) and (5.8) from Lemma 2.5. From the condition (2.6), we have

$$\sum_{j=k}^{\infty} a_j \text{ there exist for all } (k \in \mathbb{N}). \tag{5.10}$$

This implies that $\{cs_0^\lambda(\Delta)\}^\beta = f_3^\lambda \cap f_4^\lambda \cap f_6^\lambda$.

Finally, from the equation (5.6) $a = (a_n) \in \{bs^\lambda(\Delta)\}^\beta$ if and only if $T^\lambda \in (bs : c)$. Then for all $n \in \mathbb{N}$, since

$$\lim_{k\to\infty} t_{nk}^\lambda = 0$$

the condition that is given in Lemma 2.6 holds. Also, by the condition (2.6) we can see that (5.10) is valid. From (2.7), we can write

$$\sum_{k=0}^{\infty} |\bar{a}_k(n) - \bar{a}_{k+1}(n)| \text{ convergent,} \tag{5.11}$$

$$\lim_{k\to\infty} \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n \right) \text{ exist.} \tag{5.12}$$

We conclude that $\{bs^\lambda(\Delta)\}^\beta = f_6^\lambda \cap f_7^\lambda \cap f_8^\lambda$.

6. CONCLUSION

As a result, we defined non-absolute type difference sequence spaces $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ based on the definitions of cs_0^λ, cs^λ and bs^λ sequence spaces defined by Kaya and Furkan in 2015, and the difference sequence space defined by Kızmaz (1981), and show that the difference sequence spaces $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ are BK-spaces. Additionally, it is defined that these spaces are isomorphic to the spaces, cs_0, cs and bs respectively, and their Schauder basis are given. Also, the classes of matrix transformations from the spaces, $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ to the spaces l_∞, c and c_0 are characterized, where $1 \le p \le \infty$. Finally, some inclusion relations are examined and the $\alpha -$, $\beta -$ and $\gamma -$duals of these sequence spaces are calculated. This article provides a significant contribution to the field of functional analysis and topology by establishing that the sequence spaces $cs_0^\lambda(\Delta)$, $cs^\lambda(\Delta)$ and $bs^\lambda(\Delta)$ are BK spaces. The implications of this result extend to applied sciences and topology, offering a new perspective on sequence space theory and its practical applications. This work opens new avenues for studying topological sequence spaces and their properties, with potential applications in diverse fields.

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