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Mersenne Matrix Operator and Its Application in $p-$Summable Sequence Space

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Abstract
In this study, it is introduced the regular Mersenne matrix operator which is obtained by using Mersenne numbers and examined sequence spaces described as the domain of this matrix in the space of p-summable sequences for $1 \leq p \leq \infty$. After that, it investigated some properties and inclusion relations, established the Schauder basis, and stated $\alpha-$, $\beta-$, and $\gamma-$duals of the aforementioned spaces. Additionally, it is characterized by the matrix classes from newly described spaces to classical sequence spaces. Finally, we studied the compactness of matrix operators on related sequence spaces.

Keywords: Compact operators, Duals, Matrix transformations, Mersenne numbers, Sequence spaces
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1. Introduction

Mersenne numbers, named after the French theologian, philosopher, mathematician, music theorist and priest Marin Mersenne, who is known as the father of acoustics, in the first half of the 17th century, have an important place in number theory and computer science. rth Mersenne number m_r is stated by $m_r = 2^r - 1$ with $r \in \mathbb{N}$ and $\mathbb{N} = \{1, 2, 3, ...\}$ and this is called as the Binet formula of the Mersenne sequence.

The Mersenne numbers m_r can be described by the recurrence relations

$$m_{r+2} = 3m_{r+1} - 2m_r \quad \text{and} \quad \sum_{s=1}^{r} m_s = 2m_r - r.$$

The first 10 terms of the Mersenne sequence are as follows:

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023 \ldots .$$

There are prime and non-prime Mersenne numbers, and studies on Mersenne primes have held an important place in the fields of number theory and computer science until today. It is known that if m_r is prime, then r must be a prime, but the its reverse is not true.

Now, we may give basic information about sequence spaces and summability theory. ω represents all real or complex sequence's space and each $\Gamma \subset \omega$ named as sequence space. The spaces ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$ express the set of all bounded, convergent, null and convergent p-absolutely summable sequences' well known spaces, respectively. The spaces mentioned above are Banach spaces with $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{r\in\mathbb{N}} |u_r|$ and $\|u\|_{\ell_p} = (\sum_r |u_r|^p)^{\frac{1}{p}}$, where $\sum_r |u_r| = \sum_{r=1}^{\infty} |u_r|$. Moreover, every finite sequences' space is represented by Ω and by cs, cs_0 and bs, we mean the spaces of all convergent, null and bounded series, respectively.

Banach spaces in which all coordinate functionals t_s described with $t_s(u) = u_s$ are continuous are called BK-spaces. Additionally, metric vector spaces in which all coordinate functionals are continuous are called FK-spaces.

Let $e^{(1)} = (1,0,0,\ldots)$, $e^{(2)} = (0,1,0,\ldots)$, $e^{(3)} = (0,0,1,0,\ldots)$,.... If each $u = (u_r) \in \Gamma \subset \omega$ can be expressed uniquely as $u = \sum_r u_r e^r$, in that case, it is said that the BK-space Γ holds the AK-property. The spaces ℓ_p $(1 \le p < \infty)$ and c_0 hold AK-property however the spaces c and ℓ_∞ do not hold.

For an infinite matrix $B = (b_{rs})$ with real entries, B_r represent the rth row for each $r \in \mathbb{N}$. The B-transform of $u = (u_s) \in \omega$ is described by $(Bu)_r = \sum_s b_{rs} u_s$ provided that the series is convergent for each $r \in \mathbb{N}$. If $Bu \in \Psi$, in that case it is said that B is a matrix transformation from Γ to Ψ for all $u \in \Gamma$. The class of every matrices transform Γ to Ψ is represented by $(\Gamma : \Psi)$. Matrix domain of B in Γ is described as

$$\Gamma_B = \{u \in \omega : Bu \in \Gamma\}. \tag{1.1}$$

If Γ and Ψ are two sequence spaces, then the multiplier set $D(\Gamma : \Psi)$ is described as

$$D(\Gamma : \Psi) = \left\{ x = (x_r) \in \omega : xu = (x_r u_r) \in \Psi \quad \text{for all} \quad (u_r) \in \Gamma \right\}.$$

In that case, α-, β- and γ-duals of Γ are described as $\Gamma^\alpha = D(\Gamma : \ell_1)$, $\Gamma^\beta = D(\Gamma : cs)$ and $\Gamma^\gamma = D(\Gamma : bs)$.

Sequences, their spaces and matrix domains have been seen as interesting topics in mathematics by the authors, and in recent years, many studies have been done in this area. Researchers who want to get more detailed information about summability theory, infinite matrices, sequences and their spaces, matrix domains and other related subjects can benefit from the studies [1]-[10] and textbooks [11]-[13].

Special integer sequences have been used extensively in sequence space studies in recent years. In this context, the first study done is the study with a tag [14] made by Başarır and Kara. After this study, some special integer sequences such as Lucas, Padovan, Pell, Leanardo, Catalan, Bell, Schröder and Motzkin were used to define new sequence spaces in summability theory. Researchers who want to get more detailed information about literature can benefit from the studies [15]-[25].

In parallel with the studies mentioned above, this article aims to construct a novel regular matrix operator μ obtained by the aid of Mersenne sequence and examine sequence spaces described as the domain of μ in ℓ_p $(1 \le p \le \infty)$. It is investigated algebraic and topological properties, established Schauder basis and stated $\alpha-$, $\beta-$ and $\gamma-$duals of the aforementioned spaces and additionally, it is featured the matrix classes from new sequence spaces to the classical sequence spaces. At the end, it is studied the compactness of matrix operators on related sequence spaces.

2. Mersenne Matrix Operator and Mersenne Sequence Spaces

It is described the Mersenne matrix operator generated with the help of the Mersenne numbers and it is observed that this aforementioned matrix is regular. After that, we introduced the normed spaces $\ell_p(\mu)$ and $\ell_\infty(\mu)$ and shown that these are complete and linearly isomorphic to ℓ_p and ℓ_∞, respectively, for $1 \le p < \infty$. Then, it is shown that except for the case $p = 2$, $\ell_p(\mu)$ is not a Hilbert space, it is established Schauder basis and to determine the location of the newly defined spaces among the other spaces, it is given the inclusion relations at the end.

Now, we construct the Mersenne matrix operator $\mu = (\mu_{rs})$ with the help of Mersenne numbers as follows:

$$\mu_{rs} := \begin{cases} \dfrac{m_s}{2m_r - r} &, \quad \text{if } 1 \le s \le r, \\ 0 &, \quad \text{if } s > r, \end{cases}$$

for all $r, s \in \mathbb{N}$. The Mersenne matrix μ can be expressed more clearly in the following form:

$$\mu := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{11} & \frac{3}{11} & \frac{7}{11} & 0 & 0 & \cdots \\ \frac{1}{26} & \frac{3}{26} & \frac{7}{26} & \frac{15}{26} & 0 & \cdots \\ \frac{1}{57} & \frac{3}{57} & \frac{7}{57} & \frac{15}{57} & \frac{31}{57} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

From its definition, we can understand that μ is a triangle. Moreover, μ-transform of a sequence $u = (u_s)$ is stated as

$$v_r := (\mu u)_r = \frac{1}{2m_r - r}\sum_{s=1}^{r} m_s u_s \quad (r \in \mathbb{N}). \tag{2.1}$$

It is known that, an infinite matrix is named as regular if it maps any convergent sequence into a convergent sequence with the same limit.

Lemma 2.1. *An infinite matrix B is regular if and only if the following conditions hold:*

(i) $\sup_{r\in\mathbb{N}} \sum_s |b_{rs}| < \infty$,

(ii) $\lim_{r\to\infty} \sum_s b_{rs} = 1$,

(iii) $\lim_{r\to\infty} b_{rs} = 0$.

Theorem 2.2. *The Mersenne matrix μ is regular.*

Proof. From the equality

$$\sum_s |\mu_{rs}| = \sum_s \mu_{rs} = \sum_{s=1}^{r} \frac{m_s}{2m_r - r} = 1,$$

it is easily seen that the conditions (**i**) and (**ii**) hold. It is reached the validity of the condition (**iii**) from the equality

$$\begin{aligned} \lim_{r\to\infty} \mu_{rs} = \lim_{r\to\infty} \frac{m_s}{2m_r - r} &= m_s. \lim_{r\to\infty} \frac{1}{2m_r - r} \\ &= m_s. \lim_{r\to\infty} \frac{1}{2^{r+1} - r - 2} = 0. \end{aligned}$$

□

Now, let us introduce the sets $\ell_p(\mu)$ and $\ell_\infty(\mu)$ of all Mersenne p-absolutely convergent and Mersenne bounded sequences by

$$\ell_p(\mu) = \left\{ u = (u_s) \in \omega : \sum_{r=1}^{\infty} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(\mu) = \left\{ u = (u_s) \in \omega : \sup_{r\in\mathbb{N}} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right| < \infty \right\}.$$

In that case, the sets $\ell_p(\mu)$ can be rewritten as $\ell_p(\mu) = (\ell_p)_\mu$ for $1 \le p \le \infty$ with the notation (1.1). If $\Gamma \subset \omega$ is normed, in that case $\Gamma(\mu)$ is called as a Mersenne sequence space.

Unless otherwise stated in the following parts of the study, $1 \le p < \infty$ will be assumed.

Wilansky [26] proved that, if B is triangle and Γ is BK-space, in that case the domain Γ_B is BK-space too, with $\|u\|_{\Gamma_B} = \|Bu\|_\Gamma$. Therefore, we are ready to give the theorem without proof regarding the BK-spaceness of the sets we just defined.

Theorem 2.3. *$\ell_p(\mu)$ and $\ell_\infty(\mu)$ are BK-spaces with*

$$\|u\|_{\ell_p(\mu)} = \left(\sum_{r=1}^{\infty} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|^p \right)^{\frac{1}{p}}$$

and

$$\|u\|_{\ell_\infty(\mu)} = \sup_{r\in\mathbb{N}} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|,$$

respectively.

Theorem 2.4. *$\ell_p(\mu)$ and $\ell_\infty(\mu)$ are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively.*

Proof. Since, it can be shown similarly for the other spaces, the theorem will be proven only for the spaces $\ell_\infty(\mu)$ and ℓ_∞.

For the proof, it must be shown that there is a norm-preserving bijection between the aforementioned spaces. The linearity of the function described for this purpose as $\mathscr{A}:\ell_\infty(\mu)\to\ell_\infty$, $\mathscr{A}(u)=\mu u$ can be seen immediately. Besides this, from the proposition $\mathscr{A}(u)=0\Rightarrow u=0$, $\mathscr{A}$ is decided to be an injection.

By taking into account the sequences $\nu=(\nu_s)\in\ell_\infty$ and $u=(u_s)\in\omega$ whose terms are

$$u_s=\sum_{i=s-1}^{s}(-1)^{s-i}\frac{2m_i-i}{m_s}\nu_i$$

with $u_1=\nu_1$ for all $s\geq 2$, we reach the surjectivity of $\mathscr{A}$ from the expression

$$\begin{aligned}(\mu u)_r &= \frac{1}{2m_r-r}\sum_{s=1}^{r}m_s u_s\\ &= \frac{1}{2m_r-r}\sum_{s=1}^{r}m_s\sum_{i=s-1}^{s}(-1)^{s-i}\frac{2m_i-i}{m_s}\nu_i\\ &= \nu_r.\end{aligned}$$

Additionally, since the relation $\|u\|_{\ell_\infty(\mu)}=\|\mu u\|_{\ell_\infty}$ holds, then $\mathscr{A}$ keeps the norm. □

Theorem 2.5. *Except for the case $p=2$, $\ell_p(\mu)$ is not a Hilbert space.*

Proof. If we consider that $x=(1,1,-\frac{4}{7},0,0,\ldots)$ and $y=(1,-\frac{5}{3},\frac{4}{7},0,0,\ldots)$, in that case it is obtain that $\mu x=(1,1,0,0,\ldots)$ and $\mu y=(1,-1,0,0,\ldots)$ and

$$\|x+y\|^2_{\ell_p(\mu)}+\|x-y\|^2_{\ell_p(\mu)}=8\neq 2^{2+\frac{2}{p}}=2\left(\|x\|^2_{\ell_p(\mu)}+\|y\|^2_{\ell_p(\mu)}\right).$$

Hence, the norm associated with the space $\ell_p(\mu)$ for $p\neq 2$ doesn't hold the parallelogram equality, which is desired result. □

Consider the normed sequence space $(\Gamma,\|.\|)$ and $(\eta_r)\in\Gamma$. In that case, (η_r) is Schauder basis for Γ if for any $u\in\Gamma$, there is a unique scalars' sequence (σ_r) as

$$\left\|u-\sum_{s=1}^{r}\sigma_s\eta_s\right\|\longrightarrow 0$$

as $r\to\infty$ and it is written as $u=\sum_s\sigma_s\eta_s$.

Now, it will be given the result that determines the Schauder basis of $\ell_p(\mu)$. It is concluded that the inverse image of the basis $(e^{(r)})_{r\in\mathbb{N}}$ of ℓ_p composes the basis of $\ell_p(\mu)$ because the function $\mathscr{A}:\ell_p(\mu)\to\ell_p$ described above is an isomorphism. In this way, we can present the following theorem about the Schauder basis without proof.

Theorem 2.6. *Let us consider the sequences $\sigma_s=(\mu u)_s$ and $\eta^{(s)}=\left(\eta_r^{(s)}\right)\in\ell_p(\mu)$ described as*

$$\eta_r^{(s)}:=\begin{cases}(-1)^{r-s}\dfrac{2m_s-s}{m_r} & , \quad \text{if } r-1\leq s\leq r,\\ 0 & , \quad \text{otherwise}.\end{cases}$$

In that case; the set $\eta^{(s)}$ is a basis for the space $\ell_p(\mu)$ and the unique representation of any $u\in\ell_p(\mu)$ is stated as $u=\sum_s\sigma_s\eta^{(s)}$ for $1\leq p<\infty$.

Theorem 2.7. *The inclusion $\ell_p(\mu)\subset\ell_{\tilde{p}}(\mu)$ strictly holds for $1\leq p<\tilde{p}<\infty$.*

Proof. Consider the sequence $u=(u_s)\in\ell_p(\mu)$ such that $\mu u\in\ell_p$. Furthermore, it is known that $\ell_p\subset\ell_{\tilde{p}}$ for $1\leq p<\tilde{p}<\infty$ and thus $\mu u\in\ell_{\tilde{p}}$. Consequently, we can write $u=(u_s)\in\ell_{\tilde{p}}(\mu)$.

The strictness of inclusion can be easily seen when $\tilde{\nu}=\mu\tilde{u}\in\ell_{\tilde{p}}\setminus\ell_p$ is taken. □

Theorem 2.8. *The inclusion $\ell_\infty\subset\ell_\infty(\mu)$ holds.*

Proof. By taking a sequence $u=(u_s)\in\ell_\infty$, from the inequality

$$\begin{aligned}\|u\|_{\ell_\infty(\mu)} &= \sup_{r\in\mathbb{N}}\left|\frac{1}{2m_r-r}\sum_{s=1}^{r}m_su_s\right| \\ &\leq \|u\|_\infty\sup_{r\in\mathbb{N}}\left|\frac{1}{2m_r-r}\sum_{s=1}^{r}m_s\right| \\ &= \|u\|_\infty<\infty,\end{aligned}$$

it is reached that $u\in\ell_\infty(\mu)$, which is desired result. □

Theorem 2.9. *The inclusion $\ell_p\subset\ell_p(\mu)$ holds.*

Proof. By taking a sequence $u=(u_s)\in\ell_p$ for $1<p<\infty$, from the inequality

$$\begin{aligned}\sum_{r=1}^{\infty}|(\mu u)_r|^p &\leq \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|\right)^p \\ &\leq \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|^p\right)\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}\right)^{p-1} \\ &= \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|^p\right) \\ &= \sum_{s=1}^{\infty}|u_s|^p\left(\sum_{r=s}^{\infty}\frac{m_s}{2m_r-r}\right),\end{aligned}$$

we reach that $\|u\|_{\ell_p(\mu)}^p\leq N.\|u\|_{\ell_p}^p$ for $N=\sup_{s\in\mathbb{N}}\left\{\sum_{r=s}^{\infty}\frac{m_s}{2m_r-r}\right\}$. This implies that $u\in\ell_p(\mu)$ and $\ell_p\subset\ell_p(\mu)$. It can be shown that $\ell_1\subset\ell_1(\mu)$ similarly. □

3. Dual Spaces

It will be calculated duals of the spaces $\ell_p(\mu)$ in the current part. Since, the following results related the duals can be seen similar to the case $1<p\leq\infty$, the proofs of results involving the case $p=1$ will be omitted. In the rest of the paper, unless otherwise stated, $q=\frac{p}{p-1}$ will be assumed and $\mathscr{F}$ will represented the family of all finite subsets of $\mathbb{N}$.

For the determination of duals, it may be given the following lemmas collected from the study [27] to characterize some classical matrix classes:

Lemma 3.1. *For $1<p\leq\infty$, $B=(b_{rs})\in(\ell_p:\ell_1)$ if and only if*

$$\sup_{E\in\mathscr{F}}\sum_{s=1}^{\infty}\left|\sum_{r\in E}b_{rs}\right|^q<\infty.$$

Lemma 3.2. *For $1<p<\infty$, $B=(b_{rs})\in(\ell_p:c)$ if and only if*

$$\lim_{r\to\infty}b_{rs}\ \textit{exists for all}\ s\in\mathbb{N}, \tag{3.1}$$

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|b_{rs}|^q<\infty. \tag{3.2}$$

Lemma 3.3. *$B=(b_{rs})\in(\ell_\infty:c)$ if and only if the conditions (3.1),*

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|b_{rs}|<\infty,$$
$$\lim_{r\to\infty}\sum_{s=1}^{\infty}\left|b_{rs}-\lim_{r\to\infty}b_{rs}\right|=0$$

hold.

Lemma 3.4. *$B=(b_{rs})\in(\ell_p:\ell_\infty)$ if and only if (3.2) holds for $1<p\leq\infty$.*

Theorem 3.5. *Let us consider the set ϖ_1 and the infinite matrix $G=(g_{rs})$ described by*

$$\varpi_1=\left\{\tau=(\tau_s)\in\omega:\sup_{E\in\mathscr{F}}\sum_{s=1}^{\infty}\left|\sum_{r\in E}g_{rs}\right|^q<\infty\right\}$$

and

$$g_{rs}:=\begin{cases}(-1)^{r-s}\dfrac{2m_s-s}{m_r}\tau_r & , \quad \text{if } r-1\leq s\leq r,\\[2ex] 0 & , \quad \text{otherwise}.\end{cases}$$

In that case; $[\ell_p(\mu)]^\alpha=\varpi_1$ for $1<p\leq\infty$.

Proof. By using the equality (2.1), we obtain that

$$\begin{aligned}\tau_r u_r &= \tau_r\left(\sum_{s=r-1}^{r}(-1)^{r-s}\frac{2m_s-s}{m_r}v_s\right)\\ &= \sum_{s=r-1}^{r}\left((-1)^{r-s}\frac{2m_s-s}{m_r}\tau_r\right)v_s=(Gv)_r\end{aligned}\tag{3.3}$$

for all $r\in\mathbb{N}$. Hence, it is obtained by the relation (3.3) that $\tau u=(\tau_r u_r)\in\ell_1$ when $u\in\ell_p(\mu)$ if and only if $Gv\in\ell_1$ when $v\in\ell_p$. In that case, it is reached the biconditional statement $\tau\in[\ell_p(\mu)]^\alpha$ if and only if $G\in(\ell_p:\ell_1)$. By taking into consideration the condition of Lemma 3.1 with together $G=(g_{rs})$ in place of $B=(b_{rs})$, it is seen that $[\ell_p(\mu)]^\alpha=\varpi_1$ for $1<p\leq\infty$, which is desired result. □

Theorem 3.6. *Let us consider the sets $\varpi_2^{(q)}$, ϖ_3 and ϖ_4 by*

$$\begin{aligned}\varpi_2^{(q)} &= \left\{\tau=(\tau_s)\in\omega:\sum_{s=1}^{\infty}\left|(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)\right|^q<\infty\right\},\\ \varpi_3 &= \left\{\tau=(\tau_s)\in\omega:\sup_{r\in\mathbb{N}}\left|\frac{2m_r-r}{m_r}\tau_r\right|<\infty\right\},\\ \varpi_4 &= \left\{\tau=(\tau_s)\in\omega:\lim_{r\to\infty}\frac{2m_r-r}{m_r}\tau_r=0\right\}.\end{aligned}$$

In that case; $[\ell_p(\mu)]^\beta=\varpi_2^{(q)}\cap\varpi_3$ for $1<p<\infty$ and $[\ell_\infty(\mu)]^\beta=\varpi_2^{(1)}\cap\varpi_4$.

Proof. Let us choose two sequences $\tau=(\tau_s)\in\omega$ and $u\in\ell_p(\mu)$ such that $v\in\ell_p$ with the relation (2.1). Then, we reach that

$$\begin{aligned}\psi_r=\sum_{s=1}^{r}\tau_s u_s &= \sum_{s=1}^{r}\tau_s\left(\sum_{i=s-1}^{s}(-1)^{s-i}\frac{2m_i-i}{m_s}v_i\right)\\ &= \sum_{s=1}^{r-1}(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)v_s+\frac{2m_r-r}{m_r}\tau_r v_r\\ &= (Ov)_r\end{aligned}\tag{3.4}$$

where the matrix $O=(o_{rs})$ is described as

$$o_{rs}:=\begin{cases}(2m_s-s)\left(\dfrac{\tau_s}{m_s}-\dfrac{\tau_{s+1}}{m_{s+1}}\right) & , \quad 1\leq s<r,\\ \dfrac{2m_r-r}{m_r}\tau_r & , \quad s=r,\\ 0 & , \quad \text{otherwise.}\end{cases} \tag{3.5}$$

It can be checked that

$$\lim_{r\to\infty} o_{rs}=(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right). \tag{3.6}$$

In that case, from the relation (3.4), it is infered that $\tau u\in cs$ whenever $u=(u_s)\in\ell_p(\mu)$ if and only if $\psi=(\psi_r)\in c$ when $v\in\ell_p$. Thus, $\tau\in[\ell_p(\mu)]^\beta$ if and only if $O\in(\ell_p:c)$ for $1<p<\infty$. Hence, in view of (3.4), (3.6) and the conditions of Lemma 3.2, it is reached that

$$\sum_{s=1}^{\infty}\left|(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)\right|^q<\infty \quad \text{and} \quad \sup_{r\in\mathbb{N}}\left|\frac{2m_r-r}{m_r}\tau_r\right|<\infty$$

which is desired result.

It can be shown similarly for the case $p=\infty$ by the aid of Lemma 3.3 and the relations (3.4) and (3.6). □

Theorem 3.7. *For* $1<p\leq\infty$, $[\ell_p(\mu)]^\gamma=\varpi_2^{(q)}\cap\varpi_4$.

Proof. It can be obtained with similar approach in the proof of the Theorem 3.6 by considering with together the Lemma 3.4 with the matrix $O=(o_{rs})$ described by (3.5). □

4. Matrix Transformations

Current part aims to present the matrix classes $(\ell_p(\mu):\Psi)$, where $\Psi\in(\ell_\infty,c,c_0)$ and $1\leq p\leq\infty$. For brevity, we take

$$\phi_{rs}=(2m_s-s)\left(\frac{b_{rs}}{m_s}-\frac{b_{r,s+1}}{m_{s+1}}\right) \tag{4.1}$$

in the rest for infinite matrices $\Phi=(\phi_{rs})$ and $B=(b_{rs})$ and $r,s\in\mathbb{N}$.

Consider that u and v with the relation (2.1). In that case, it is reached that

$$\sum_{s=1}^{n}b_{rs}u_s=\sum_{s=1}^{n-1}\phi_{rs}v_s+\frac{2m_n-n}{m_n}b_{rn}v_n. \tag{4.2}$$

Now, it may be given the following conditions to characterize new matrix classes:

$$\left(\frac{2m_s-s}{m_s}b_{rs}\right)_{s=1}^{\infty}\in\ell_\infty \text{ for all } r\in\mathbb{N}, \tag{4.3}$$

$$\sup_{r,s\in\mathbb{N}}|\phi_{rs}|<\infty, \tag{4.4}$$

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|\phi_{rs}|^q<\infty, \tag{4.5}$$

$$\left(\frac{2m_s-s}{m_s}b_{rs}\right)_{s=1}^{\infty}\in c_0 \text{ for all } r\in\mathbb{N}, \tag{4.6}$$

$$\lim_{r\to\infty}\phi_{rs} \text{ exists for all } s\in\mathbb{N}, \tag{4.7}$$

$$\lim_{r\to\infty}\sum_{s=1}^{\infty}|\phi_{rs}-\rho_s|=0 \text{ for all } s\in\mathbb{N} \text{ and } (\rho_s)\in\omega, \tag{4.8}$$

$$\lim_{r\to\infty}|\phi_{rs}|=0 \text{ for all } s\in\mathbb{N}. \tag{4.9}$$

In that case; from the conditions of the matrix classes in [27] with together Theorem 3.6 and the relation (4.2), it may be given the following results:

Theorem 4.1. *The following statements hold:*

(i) $B = (b_{rs}) \in (\ell_1(\mu) : \ell_\infty)$ *if and only if (4.3) and (4.4) hold.*

(ii) $B = (b_{rs}) \in (\ell_1(\mu) : c)$ *if and only if (4.3), (4.4) and (4.7) hold.*

(iii) $B = (b_{rs}) \in (\ell_1(\mu) : c_0)$ *if and only if (4.3), (4.4) and (4.9) hold.*

Theorem 4.2. *For $1 < p < \infty$, the following statements hold:*

(i) $B = (b_{rs}) \in (\ell_p(\mu) : \ell_\infty)$ *if and only if (4.3) and (4.5) hold.*

(ii) $B = (b_{rs}) \in (\ell_p(\mu) : c)$ *if and only if (4.3), (4.5) and (4.7) hold.*

(iii) $B = (b_{rs}) \in (\ell_p(\mu) : c_0)$ *if and only if (4.3), (4.5) and (4.9) hold.*

Theorem 4.3. *The following statements hold:*

(i) $B = (b_{rs}) \in (\ell_\infty(\mu) : \ell_\infty)$ *if and only if (4.5) and (4.6) hold with $q = 1$.*

(ii) $B = (b_{rs}) \in (\ell_\infty(\mu) : c)$ *if and only if (4.5), (4.6), (4.7) and (4.8) hold with $q = 1$.*

(iii) $B = (b_{rs}) \in (\ell_\infty(\mu) : c_0)$ *if and only if (4.6) and (4.8) hold for $\rho_s = 0$ and $s \in \mathbb{N}$.*

5. Compactness by Hausdorff Measure of Non-compactness

This part aims to acquire the necessary and sufficient conditions for an operator to be compact from $\ell_p(\mu)$ to the space Ψ, where $1 \leq p \leq \infty$ and $\Psi \in \{c_0, c, \ell_\infty, \ell_1, cs_0, cs, bs\}$.

For a normed space Γ, $\mathscr{D}_\Gamma$ represents the unit sphere in Γ. It is used the notation

$$\|u\|_\Gamma^\diamond = \sup_{x \in \mathscr{D}_\Gamma} \left| \sum_s u_s x_s \right|$$

for a BK-space $\Gamma \supset \Omega$ and $u = (u_s) \in \omega$, where Ω represents all finite sequences's space and it is assumed that the series above is exists and then it is reached that $u \in \Gamma^\beta$.

Lemma 5.1. *[28] The following statements hold:*

(i) $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ *and* $\|u\|_\Gamma^\diamond = \|u\|_{\ell_1}$ *for all $u \in \ell_1$ and $\Gamma \in \{\ell_\infty, c, c_0\}$.*

(ii) $\ell_1^\beta = \ell_\infty$ *and* $\|u\|_{\ell_1}^\diamond = \|u\|_{\ell_\infty}$ *for all $u \in \ell_\infty$.*

(iii) $\ell_p^\beta = \ell_q$ *and* $\|u\|_{\ell_p}^\diamond = \|u\|_{\ell_q}$ *for all $u \in \ell_q$.*

The set $\mathfrak{B}(\Gamma : \Psi)$ represents all bounded (continuous) linear operators' set from Γ to Ψ.

Lemma 5.2. *[28] Let Γ and Ψ are the BK-spaces. In that case, for all $B \in (\Gamma : \Psi)$, there exists a linear operator $\mathscr{L}_B \in \mathfrak{B}(\Gamma : \Psi)$ as $\mathscr{L}_B(u) = Bu$ for every $u \in \Gamma$.*

Lemma 5.3. *[28] Consider that $\Gamma \supset \Omega$ is a BK-space. If $B \in (\Gamma : \Psi)$, in that case $\|\mathscr{L}_B\| = \|B\|_{(\Gamma:\Psi)} = \sup_{r\in\mathbb{N}} \|B_r\|_\Gamma^\diamond < \infty$.*

Let us consider a metric space Γ and $A \subset \Gamma$ is bounded. The Hausdorff measure of non-compactness of A is represented with $\chi(A)$ and it is described by

$$\chi(A) = \inf\left\{\varepsilon > 0 : A \subset \cup_{j=1}^r A(u_j, n_j), u_j \in \Gamma, n_j < \varepsilon, r \in \mathbb{N}\right\},$$

where $A(u_j, n_j)$ is the open ball centred at u_j and radius n_j for each $j = 1, 2, ..., r$. Researchers who want to get more detailed information about Hausdorff measure of non-compactness can benefit from [28] and its references.

Theorem 5.4. *[29] Let $A \subset \ell_p$ is bounded and the operator $\lambda_n : \ell_p \longrightarrow \ell_p$ described as $\lambda_n(u) = (u_1, u_2, u_3, ..., u_n, 0, 0, ...)$ for every $u = (u_s) \in \ell_p$, $1 \leq p < \infty$ and each $n \in \mathbb{N}$. In that case, for the identity operator I on ℓ_p, it is reached that*

$$\chi(A) = \lim_{n\to\infty} \left(\sup_{u \in A} \|(I - \lambda_n)(u)\|_{\ell_p} \right).$$

For the Banach spaces Γ and Ψ, a linear operator $\mathscr{L}:\Gamma\to\Psi$ is named as compact operator if domain of $\mathscr{L}$ is whole of Γ and $\mathscr{L}(A)$ is totally bounded set in Ψ for all $u=(u_s)\in\ell_\infty\cap\Gamma$. Equivalently, $\mathscr{L}$ is compact if $(\mathscr{L}(u))$ has a convergent subsequence in Ψ for all $u=(u_s)\in\ell_\infty\cap\Gamma$.

Let $\|\mathscr{L}\|\chi$ represents Hausdorff measure of non-compactness of $\mathscr{L}$ and it is described by $\|\mathscr{L}\|\chi=\chi(\mathscr{L}(\mathscr{D}_\Gamma))$. The notions Hausdorff measure of non-compactness and compact operators have a distinct relationship of "$\mathscr{L}$ is compact if and only if $\|\mathscr{L}\|\chi=0$".

Readers can use the studies [30, 31, 32, 33, 34, 35] for sequence space studies where Hausdorff measure of non-compactness is used to determine compact operators between BK-spaces.

Lemma 5.5. *[30] Let $\Gamma\supset\Omega$ is BK-space. In that case:*

(i) *If $B\in(\Gamma:c_0)$, then $\|\mathscr{L}_B\|_\chi=\limsup_r\|B_r\|_\Gamma^\diamond$ and $\mathscr{L}_B$ is compact if and only if $\lim_r\|B_r\|_\Gamma^\diamond=0$.*

(ii) *If Γ has AK property or $\Gamma=\ell_\infty$ and $B\in(\Gamma:c)$, then*

$$\frac{1}{2}\limsup_r\|B_r-\kappa\|_\Gamma^\diamond\le\|\mathscr{L}_B\|_\chi\le\limsup_r\|B_r-\kappa\|_\Gamma^\diamond$$

and $\mathscr{L}_B$ is compact if

$$\lim_r\|B_r-\kappa\|_\Gamma^\diamond=0$$

where $\kappa=(\kappa_s)$ and $\kappa_s=\lim_r b_{rs}$.

(iii) *If $B\in(\Gamma:\ell_\infty)$, then $0\le\|\mathscr{L}_B\|_\chi\le\limsup_r\|B_r\|_\Gamma^\diamond$ and $\mathscr{L}_B$ is compact if $\lim_r\|B_r\|_\Gamma^\diamond=0$.*

(iv) *If $B\in(\Gamma:\ell_1)$, then*

$$\lim_j\left(\sup_{E\in\mathscr{F}_j}\left\|\sum_{r\in E}B_r\right\|_\Gamma^\diamond\right)\le\|\mathscr{L}_\mathscr{B}\|_\chi\le 4.\lim_j\left(\sup_{E\in\mathscr{F}_j}\left\|\sum_{r\in E}B_r\right\|_\Gamma^\diamond\right)$$

and $\mathscr{L}_\mathscr{B}$ is compact if and only if $\lim_j\left(\sup_{E\in\mathscr{F}_j}\|\sum_{r\in E}B_r\|_\Gamma^\diamond\right)=0$, where $\mathscr{F}$ represents the family of all finite subsets of $\mathbb{N}$ and $\mathscr{F}_j$ is the subcollection of $\mathscr{F}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than j.

In the sequel of the study, it is used the matrices $\Phi=(\phi_{rs})$ and $B=(b_{rs})$ connected with the relation (4.1).

Lemma 5.6. *Let $\Psi\subset\omega$. If $B\in(\ell_p(\mu):\Psi)$, then $\Phi\in(\ell_p:\Psi)$ and $Bu=\Phi v$ hold for all $u\in\ell_p(\mu)$ and $1\le p\le\infty$.*

Theorem 5.7. *Let $1<p<\infty$. In that case:*

(i) *If $B\in(\ell_p(\mu):c_0)$, then $\|\mathscr{L}_B\|_\chi=\limsup_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}=0$.*

(ii) *If $B\in(\ell_p(\mu):c)$, then*

$$\frac{1}{2}\limsup_r\left(\sum_s|\phi_{rs}-a_s|^q\right)^{\frac{1}{q}}\le\|\mathscr{L}_B\|_\chi\le\limsup_r\left(\sum_s|\phi_{rs}-a_s|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r(\sum_s|\phi_{rs}-a_s|^q)^{\frac{1}{q}}=0$, where $a_s=\lim_r\phi_{rs}$.

(iii) *If $B\in(\ell_p(\mu):\ell_\infty)$, then $0\le\|\mathscr{L}_B\|_\chi\le\limsup_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if $\lim_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}=0$.*

(iv) *If $B\in(\ell_p(\mu):\ell_1)$, then $\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}\le\|\mathscr{L}_B\|_\chi\le 4.\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}$ and $\mathscr{L}_B$ is compact if and only if $\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}=0$, where $\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}=\sup_{E\in\mathscr{F}_j}(\sum_s|\sum_{r\in E}\phi_{rs}|^q)^{\frac{1}{q}}$.*

Proof. **(i)** Let $B \in (\ell_p(\mu) : c_0)$. It is seen that

$$\|B_r\|^{\diamond}_{\ell_p(\mu)} = \|\Phi_r\|^{\diamond}_{\ell_p} = \|\Phi_r\|_{\ell_q} = \left(\sum_s |\phi_{rs}|^q\right)^{\frac{1}{q}}.$$

Thus, in view of Lemma 5.5-(i), it is reached that

$$\|\mathscr{L}_B\|_\chi = \limsup_r \|B_r\|^{\diamond}_{\ell_p(\mu)} = \limsup_r \left(\sum_s |\phi_{rs}|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if $\lim_r \left(\sum_s |\phi_{rs}|^q\right)^{\frac{1}{q}}$.

(ii) Let $B \in (\ell_p(\mu) : c)$. In that case, $\Phi \in (\ell_p : c)$ by Lemma 5.6. From Lemma 5.1-(iii) it is concluded that

$$\|\Phi_r - a\|^{\diamond}_{\ell_p} = \|\Phi_r - a\|_{\ell_q} = \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}}. \tag{5.1}$$

By the aid of the Lemma 5.5-(ii), it is reached that

$$\frac{1}{2}\limsup_r \|\Phi_r - a\|^{\diamond}_{\ell_p} \le \|\mathscr{L}_B\|_\chi \le \limsup_r \|\Phi_r - a\|^{\diamond}_{\ell_p}. \tag{5.2}$$

Then, considering (5.1) and (5.2) together, it is obtained that

$$\frac{1}{2}\limsup_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}} \le \|\mathscr{L}_B\|_\chi \le \limsup_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}}.$$

Moreover, it is seen by Lemma 5.5-(ii) that $\mathscr{L}_B$ is compact if and only if

$$\lim_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}} = 0.$$

(iii) This proof can be made analogous to that of (i) and (ii) considering Lemma 5.5-(iii).

(iv) It is reached that

$$\|\sum_{r\in E} B_r\|^{\diamond}_{\ell_p(\mu)} = \|\sum_{r\in E} \Phi_r\|^{\diamond}_{\ell_p} = \|\sum_{r\in E} \Phi_r\|^{\diamond}_{\ell_q} = \left(\sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}}.$$

Let $B \in (\ell_p(\mu) : \ell_1)$, then by Lemma 5.6, $\Phi \in (\ell_p : \ell_1)$ holds. In that case, by taking account the Lemma 5.5-(iv), it is concluded that

$$\lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}} \le \|\mathscr{L}_B\|_\chi \le 4.\lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if

$$\lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}} = 0.$$

□

Lemma 5.8. *[30] Let $\Gamma \supset \Omega$ is BK-space and*

$$\|B\|^{[r]}_{(\Gamma:bs)} = \left\|\sum_{n=1}^{r} B_n\right\|^{\diamond}_{\Gamma}.$$

In that case:

(i) *If $B \in (\Gamma : cs_0)$, then $\|\mathscr{L}_B\|_\chi = \limsup_r \|B\|^{[r]}_{(\Gamma:bs)}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r \|B\|^{[r]}_{(\Gamma:bs)} = 0$.*

(ii) *If Γ has AK and $B \in (\Gamma : cs)$, in that case*

$$\frac{1}{2}\limsup_{r}\left\|\sum_{n=1}^{r} B_n - \xi\right\|_{\Gamma}^{\diamond} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left\|\sum_{n=1}^{r} B_n - \xi\right\|_{\Gamma}^{\diamond}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r \|\sum_{n=1}^{r} B_n - \xi\|_{\Gamma}^{\diamond} = 0$, where $\xi = \xi_s$ with $\xi_s = \lim_{r\to\infty}\sum_{n=1}^{r} b_{ns}$ for each $s \in \mathbb{N}$.

(iii) *If $B \in (\Gamma : bs)$, then $0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \|B\|_{(\Gamma:bs)}^{[r]}$ and $\mathscr{L}_B$ is compact if $\lim_r \|B\|_{(\Gamma:bs)}^{[r]} = 0$.*

Theorem 5.9. *Let $1 < p < \infty$. In that case:*

(i) *If $B \in (\ell_p(\mu) : cs_0)$, then $\|\mathscr{L}_B\|_\chi = \limsup_r (\sum_s |\sum_{n=1}^{r} \phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{rs}|^q)^{\frac{1}{q}} = 0$.*

(ii) *If $B \in (\ell_p(\mu) : cs)$, then*

$$\frac{1}{2}\limsup_{r}\left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns} - \tilde{a}_s\right|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns} - \tilde{a}_s\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s|^q)^{\frac{1}{q}} = 0$, where $\tilde{a} = (\tilde{a}_s)$ and $\tilde{a}_s = \lim_r \sum_{n=1}^{r} \phi_{ns}$.

(iii) *If $B \in (\ell_p(\mu) : bs)$, then*

$$0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns}\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}} = 0$.

Proof. **(i)** It is clear that

$$\left\|\sum_{n=1}^{r} B_n\right\|_{\ell_p(\mu)}^{\diamond} = \left\|\sum_{n=1}^{r} \Phi_n\right\|_{\ell_p}^{\diamond} = \left\|\sum_{n=1}^{r} \phi_{ns}\right\|_{\ell_q}^{\diamond} = \left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns}\right|^q\right)^{\frac{1}{q}}.$$

Hence, by using Lemma 5.8-(i), it is obtained that $\|\mathscr{L}_B\|_\chi = \limsup_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}}$.
(ii) We have

$$\left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond} = \left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_q} = \left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}}. \tag{5.3}$$

If $B \in (\ell_p(\mu) : cs)$, in that case by Lemma 5.6, it is reached that $\Phi \in (\ell_p : cs)$. In that case, by the aid of the Lemma 5.8-(b), it is deduced that

$$\frac{1}{2}\limsup_{r}\left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond},$$

which on using (5.3) gives us

$$\frac{1}{2}\limsup_{r}\left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left(\sum_s\left|\sum_{n=1}^{r}\phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}}$$

and also, $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s|^q)^{\frac{1}{q}} = 0$.
(iii) It can be done similarly to the proof of the first part, considering Lemma 5.8-(iii). □

Theorem 5.10. **(i)** *If $B \in (\ell_\infty(\mu) : c_0)$, in that case $\|\mathscr{L}_B\|_\chi = \limsup_r \sum_s |\phi_{rs}|$ and $\mathscr{L}_B$ is compact if $\lim_r \sum_s |\phi_{rs}| = 0$.*

(ii) *If* $B \in (\ell_\infty(\mu) : c)$*, then*

$$\frac{1}{2}\limsup_r \left(\sum_s |\phi_{rs} - a_s|\right) \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \left(\sum_s |\phi_{rs} - a_s|\right)$$

and $\mathscr{L}_B$ *is compact if and only if* $\lim_r \left(\sum_s |\phi_{rs} - a_s|\right) = 0$.

(iii) *If* $B \in (\ell_\infty(\mu) : \ell_\infty)$*, in that case* $0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \sum_s |\phi_{rs}|$ *and* $\mathscr{L}_B$ *is compact if* $\lim_r \sum_s |\phi_{rs}| = 0$.

(iv) *If* $B \in (\ell_\infty(\mu) : \ell_1)$*, then* $\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} \leq \|\mathscr{L}_B\|_\chi \leq 4.\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)}$ *and* $\mathscr{L}_B$ *is compact if and only if* $\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} = 0$*, where* $\|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} = \sup_{E\in\mathscr{F}_j}\left(\sum_s |\sum_{r\in E}\phi_{rs}|\right)$ *for all* $j \in \mathbb{N}$.

Proof. It can be obtained in a similar way to the proof of Theorem 5.7. So, it is omitted. □

Theorem 5.11. **(i)** *If* $B \in (\ell_1(\mu) : c_0)$*, in that case* $\|\mathscr{L}_B\|_\chi = \limsup_r \left(\sup_s |\phi_{rs}|\right)$ *and* $\mathscr{L}_B$ *is compact if and only if* $\lim_r \left(\sup_s |\phi_{rs}|\right) = 0$.

(ii) *If* $B \in (\ell_1(\mu) : c)$*, in that case*

$$\frac{1}{2}\limsup_r \left(\sup_s |\phi_{rs} - a_s|\right) \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \left(\sup_s |\phi_{rs} - a_s|\right)$$

and $\mathscr{L}_B$ *is compact if and only if* $\lim_r \left(\sup_s |\phi_{rs} - a_s|\right) = 0$.

(iii) *If* $B \in (\ell_1(\mu) : \ell_\infty)$*, in that case* $0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \left(\sup_s |\phi_{rs}|\right)$ *and* $\mathscr{L}_B$ *is compact if* $\lim_r \left(\sup_s |\phi_{rs}|\right) = 0$.

Proof. It can be acquired in a analogous procedure of Theorem 5.7. Thence, it is omitted, too. □

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