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STATISTICAL INFERENCES ON SOME TRIANGULAR DISTRIBUTIONS: CASE OF BOUNDARY VALUES BEING PARAMETERS

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ABSTRACT

If continuous random variables X has a triangular distribution and if its boundary values θ_1 and/or θ_2 are unknown, then it may well be necessary to make some statistical inferences related to these parameters. For triangular distributions, with unknown boundary values, some estimators, as functions of ordered statistics, are proposed. The proposed estimators are compared based on their efficiencies. Based on efficiency criteria, the best estimator, among the proposed estimators, is determined. By the use of the best estimator, a confidence interval construction and the test of hypotheses procedures are developed. By means of a simulation process, matching accuracy between sampling results and theoretical findings is observed.

Keywords: Efficiencies of estimators, Estimators, Ordered statistics, Simulation, Triangular distribution.

1. INTRODUCTION

1.1 Type I Triangular Distribution

If a continuous random variable X has the following probability density function (pdf), its distribution will be named as Type I Triangular distribution.

$$f(x)=\frac{2}{(\theta-a)^2}(x-a), a<x<\theta \text{, (a is a known constant)} \quad (1)$$

As it is shown below, all the moments of this distribution are the functions of the same parameter θ(upper bound value of X). If θ is unknown, then it must be estimated.

$$E(X^k)=\int_a^\theta x^k\frac{2}{(\theta-a)^2}(x-a)dx=\frac{2}{(\theta-a)^2}\left[\frac{x^{k+2}}{k+2}-\frac{ax^{k+1}}{k+1}\right]\Bigg|_{x=a}^{\theta}$$

$$E(X^k)=\frac{2}{(\theta-a)^2}\left[\frac{\theta^{k+2}-a^{k+2}}{k+2}-\frac{a\theta^{k+1}-a^{k+2}}{k+1}\right] \quad (2)$$

If we let $Y=(X-a)$, then the distribution of Y turns out to be a special Type I triangular distribution. Y assumes values in the interval $(0,\lambda=(\theta-a))$ and its pdf will be as given in (4).

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$$F(y)=P(Y\le y)=P(X\le a+y)=\int_a^{a+y}\frac{2}{(\theta-a)^2}(x-a)dx=\frac{2}{(\theta-a)^2}\left[\frac{x^2}{2}-ax\right]\Big|_{x=a}^{a+y}$$

$$F(y)=\frac{y^2}{(\theta-a)^2},0<y<(\theta-a)=\lambda \quad (3)$$

$$f(y)=\frac{2y}{\lambda^2},0<y<\lambda \quad (4)$$

1.1.1. Estimation of the Parameter λ of the Random Variable $Y=(X-a)$ by the Largest Ordered Statistics $Y_{(n)}$

Let Ordered statistics obtained from a random sample of size n , taken from the pdf given in (4), be $Y_{(1)}<Y_{(2)}<...<Y_{(n)}$.

The pdf of $Y_{(n)}$ is as given below.

$$f_{Y_{(n)}}(y)=n[F(y)]^{n-1}f(y)=n\left[\frac{y^2}{\lambda^2}\right]^{n-1}\frac{2y}{\lambda^2}=\frac{2n}{\lambda^{2n}}y^{2n-1},0<y<\lambda$$

The expected value and variance of $Y_{(n)}$ are as follows.

$$E(Y_{(n)})=\frac{2n}{\lambda^{2n}}\int_0^{\lambda}yy^{2n-1}dy=\frac{2n}{\lambda^{2n}}\left[\frac{y^{2n+1}}{2n+1}\Big|_0^{\lambda}\right]=\frac{2n\lambda}{(2n+1)} \quad (5)$$

$$E(Y_{(n)}^2)=\frac{2n}{\lambda^{2n}}\int_0^{\lambda}y^2y^{2n-1}dy=\frac{2n}{\lambda^{2n}}\left[\frac{y^{2n+2}}{2n+2}\Big|_0^{\lambda}\right]=\frac{2n\lambda^2}{(2n+2)} \quad (6)$$

$$Var(Y_{(n)})=\frac{2n\lambda^2}{(2n+2)}-\frac{4n^2\lambda^2}{(2n+1)^2}=\frac{n\lambda^2}{(n+1)(2n+1)^2} \quad (7)$$

Since, $Y=(X-a)$ then

$$E(X_{(n)})=E(Y_{(n})+a=\frac{2n\lambda}{(2n+1)}+a=\frac{2n\theta+a}{(2n+1)} \quad (8)$$

$$Var(X_{(n)})=Var(Y_{(n)})==\frac{n\lambda^2}{(n+1)(2n+1)^2}=\frac{n(\theta-a)^2}{(n+1)(2n+1)^2} \quad (9)$$

By the use of equation (8) we can obtain an unbiased estimator for the parameter θ of the random variable X. Hence,

$$T_1=\frac{(2n+1)X_{(n)}-a}{2n} \quad (10)$$

$$Var(T_1)=\frac{(2n+1)^2}{4n^2}Var(X_{(n)})=\frac{(\theta-a)^2}{4n(n+1)} \quad (11)$$

1.2 Estimation of the Parameter θ of the Random Variable X by the Sample Mean $\overline{X}$

We know that, the pdf of Y is $f(y)=\frac{2y}{\lambda^2}, 0<y<\lambda$, then

$$E(Y)=\int_0^\lambda y\frac{2y}{\lambda^2}dy=\frac{2y^3}{3\lambda^2}\Big|_0^\lambda=\frac{2\lambda}{3} \tag{12}$$

$$E(Y^2)=\int_0^\lambda y^2\frac{2y}{\lambda^2}dy=\frac{2y^4}{4\lambda^2}\Big|_0^\lambda=\frac{\lambda^2}{2} \tag{13}$$

$$Var(Y)=\frac{\lambda^2}{2}-\frac{4\lambda^2}{9}=\frac{\lambda^2}{18} \tag{14}$$

Since $Y=(X-a)$, $E(X)=E(Y)+a$, and $Var(X)=Var(Y)$.

$$E(X)=\frac{2\lambda}{3}+a=\frac{2(\theta-a)}{3}+a=\frac{2\theta+a}{3} \tag{15}$$

$$Var(X)=\frac{\lambda^2}{18}=\frac{(\theta-a)^2}{18} \tag{16}$$

For any random variable X, $E(\overline{X})=E(X)$, and $Var(\overline{X})=\frac{Var(X)}{n}$. Then for above Type I Triangular distribution the following hold true.

$$E(\overline{X})=\frac{2\theta+a}{3}, \text{ and } Var(\overline{X})=\frac{(\theta-a)^2}{18n} . \tag{17}$$

An unbiased estimator for θ as a function of $\overline{X}$ is as given below.

$$T_2=\frac{3\overline{X}-a}{2} \quad \text{and its variance is } Var(T_2)=\frac{9}{4}Var(\overline{X})=\frac{(\theta-a)^2}{8n} .$$

1.3 Comparisons of Unbiased Estimators on the Basis of Efficiency

$$T_1=\frac{(2n+1)X_{(n)}-a}{2n} \quad , \quad Var(T_1)=\frac{(\theta-a)^2}{4n(n+1)}$$

$$T_2=\frac{3\overline{X}-a}{2} \quad , \quad Var(T_2)=\frac{(\theta-a)^2}{8n}$$

For any integer n>1, $Var(T_1)<Var(T_2)$. Hence $T_1=\frac{(2n+1)X_{(n)}-a}{2n}$ is preferred over $T_2=\frac{3\overline{X}-a}{2}$.

1.4 Confidence Interval for θ of the Random Variable X

The better unbiased estimator for θ is $T_1 = \dfrac{(2n+1)X_{(n)} - a}{2n}$ and it is a linear function of $X_{(n)}$.

Hence, in the construction of a confidence interval it may be reasonable to use the pdf $X_{(n)}$.

From the pdf of X, $f(x) = \dfrac{2}{(\theta-a)^2}(x-a), a < x < \theta$, the following are obtained.

$$F(x) = \int_a^x \frac{2}{(\theta-a)^2}(y-a)dy = \frac{2}{(\theta-a)^2}\left[\frac{x^2-a^2}{2} - ax + a^2\right] = \frac{(x-a)^2}{(\theta-a)^2} \tag{18}$$

The pdf of $X_{(n)}$ is as given below.

$$f_{X_{(n)}}(x) = n\left[\frac{(x-a)^2}{(\theta-a)^2}\right]^{n-1}\frac{2}{(\theta-a)^2}(x-a) = \frac{2n}{(\theta-a)^{2n}}(x-a)^{2n-1}, a < x < \theta \tag{19}$$

From the probability statement, $P(x_{nL} < X_{(n)} < x_{nU}) = 1-\alpha$, lower and upper confidence limits may be obtained.

$$P(X_{(n)} \le x_{nL}) = \frac{2n}{(\theta-a)^{2n}}\int_a^{x_{nL}}(x-a)^{2n-1}dx = \alpha/2 \tag{20}$$

In (20) let $x-a=t$: then $dt = dx$; $x = a \rightarrow t = 0$; $x = x_{nL} \rightarrow t = x_{nL} - a$.

$$P(X_{(n)} \le x_{nL}) = \frac{2n}{(\theta-a)^{2n}}\int_a^{x_{nL}}(x-a)^{2n-1}dx = \frac{2n}{(\theta-a)^{2n}}\int_0^{x_{nL}-a}(t)^{2n-1}dt =$$

$$\frac{2n}{(\theta-a)^{2n}}\frac{t^{2n}}{2n}\Big|_{t=0}^{x_{nL}-a} = \left(\frac{x_{nL}-a}{\theta-a}\right)^{2n} = \frac{\alpha}{2}$$

x_{nL} is computed as given in (21).

$$x_{nL} = a + (\theta-a)\left(\frac{\alpha}{2}\right)^{1/2n} \tag{21}$$

Similarly, the upper confidence limit is obtained as in (22).

$$P(X \le x_{nU}) = \frac{2n}{(\theta-a)^{2n}}\int_a^{x_{nU}}(x-a)^{2n-1}dx = \frac{2n}{(\theta-a)^{2n}}\int_0^{x_{nU}-a}(t)^{2n-1}dt =$$

$$\frac{2n}{(\theta-a)^{2n}}\frac{t^{2n}}{2n}\Big|_{t=0}^{x_{nU}-a} = \left(\frac{x_{nU}-a}{\theta-a}\right)^{2n} = 1-\frac{\alpha}{2}$$

$$x_{nU} = a + (\theta - a)\left(1 - \frac{\alpha}{2}\right)^{1/2n} \quad (22)$$

If (21) and (22) are substituted in the probability statement, $P(x_{nL} < X_{(n)} < x_{nU}) = 1 - \alpha$, the following is obtained.

$$P\left(a + (\theta - a)\left(\frac{\alpha}{2}\right)^{1/2n} < X_{(n)} < a + (\theta - a)\left(1 - \frac{\alpha}{2}\right)^{1/2n}\right) = 1 - \alpha \quad (23)$$

If the inequalities in (23) are solved for θ simultaneously, we reach the following expression.

$$P\left(a + \frac{X_{(n)} - a}{(1 - \alpha/2)^{1/2n}} < \theta < a + \frac{X_{(n)} - a}{(\alpha/2)^{1/2n}}\right) = 1 - \alpha \quad (24)$$

Hence, a $100 * (1 - \alpha)\%$ confidence interval for θ can be given by the use of the following formula.

$$\left(a + \frac{X_{(n)} - a}{(1 - \alpha/2)^{1/2n}}, \qquad a + \frac{X_{(n)} - a}{(\alpha/2)^{1/2n}}\right) \quad (25)$$

1.5 Tests of Hypotheses Related to the Parameter θ of Type I Triangular

In testing $H_0 : \theta = \theta_0$ against to any alternative hypothesis, $X_{(n)} = X_{Max}$ may be used as a proper test statistic. For the chosen level of significance α, the decision rules given in the following table are applicable.

$H_0 : \theta = \theta_0$ $H_1 : \theta \neq \theta_0$	$H_0 : \theta \leq \theta_0$ $H_1 : \theta > \theta_0$	$H_0 : \theta \geq \theta_0$ $H_1 : \theta < \theta_0$
If $x_{(n)} \geq x_{nU}$ or $x_{(n)} \leq x_{nL}$ H_0 is rejected	If $x_{(n)} \geq x_{nU}$ H_0 is rejected	If $x_{(n)} \leq x_{nL}$ H_0 is rejected
Don't reject H_0 otherwise	Don't reject H_0 otherwise	Don't reject H_0 otherwise
Where, $x_{nL} = a + (\theta_0 - a)\left(\frac{\alpha}{2}\right)^{1/2n}$ and $x_{nU} < a + (\theta_0 - a)\left(1 - \frac{\alpha}{2}\right)^{1/2n}$	Where, $x_{nU} < a + (\theta_0 - a)(1 - \alpha)^{1/2n}$	Where, $x_{nL} = a + (\theta_0 - a)(\alpha)^{1/2n}$

2. STATISTICAL INFERENCES RELATED TO TYPE II TRIANGULAR DISTRIBUTION

If a random variable X has the following pdf, it is said to have a Type II triangular distribution.

$$f(x) = \frac{2}{(a-\theta)^2}(a-x), \theta < x < a \qquad (26)$$

All the moments of this distribution are functions of the parameter θ. For this reason, θ need to be estimated.

$$E(X^k) = \int_\theta^a x^k \frac{2}{(a-\theta)^2}(a-x)dx = \frac{2}{(a-\theta)^2}\left[\frac{ax^{k+1}}{k+1} - \frac{x^{k+2}}{k+2}\right]\Bigg|_{x=\theta}^{a}$$

$$E(X^k) = \frac{2}{(a-\theta)^2}\left[\frac{a^{k+2} - a\theta^{k+1}}{k+1} - \frac{a^{k+2} - \theta^{k+2}}{k+2}\right] \qquad (27)$$

If we let $Y = (X - \theta)$, then the distribution of Y turns out to have a special Type II triangular distribution. Y assumes values in the interval $(0, \lambda = (a-\theta))$ and its pdf will be as given in (28).

Estimation of $\lambda = (a-\theta)$ will enable to estimate θ.

$$F(y) = P(Y \le y) = P(X \le \theta + y) = \int_\theta^{\theta+y} \frac{2}{(a-\theta)^2}(a-x)dx = \frac{2}{(a-\theta)^2}\left[ax - \frac{x^2}{2}\right]\Bigg|_{x=\theta}^{\theta+y}$$

$$F(y) = \frac{2(a-\theta)y - y^2}{(a-\theta)^2} = \frac{2\lambda y - y^2}{\lambda^2}, 0 < y < (a-\theta) = \lambda \qquad (28)$$

$$f(y) = \frac{2(a-\theta) - 2y}{\lambda^2} = \frac{2(\lambda - y)}{\lambda^2}, 0 < y < \lambda \qquad (29)$$

2.1 Estimation of the Parameter λ of the Random Variable by the First Ordered Statistics $Y_{(1)}$

Let Ordered statistics obtained from a random sample of size n, taken from the pdf given in (29) be $Y_{(1)} < Y_{(2)} < ... < Y_{(n)}$.

The pdf of $Y_{(1)}$ is as given below.

$$f_{Y_{(1)}}(y) = n[1 - F(y)]^{n-1} f(y) = n\left[1 - \frac{2\lambda y - y^2}{\lambda^2}\right]^{n-1} \frac{2(\lambda - y)}{\lambda^2} = \frac{2n}{\lambda^{2n}}(\lambda - y)^{2n-1}, 0 < y < \lambda \qquad (30)$$

The expected value and variance of $Y_{(1)}$ are as follows.

$$E(Y_{(1)}) = \int_0^\lambda y \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy \tag{31}$$

In (32), if we let $(\lambda - y) = t$:

$dt = -dy;\ y = 0 \to t = \lambda;\ y = \lambda \to t = 0$

$$E(Y_{(1)}) = \int_0^\lambda y \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy = \frac{2n}{\lambda^{2n}} \int_0^\lambda (\lambda - t) t^{2n-1} dt =$$

$$\frac{2n}{\lambda^{2n}} \left[\frac{\lambda t^{2n}}{2n} - \frac{t^{2n+1}}{2n+1} \Big|_{t=0}^{\lambda} \right] = 2n\lambda \left[\frac{1}{2n} - \frac{1}{2n+1} \right] = \frac{\lambda}{2n+1}$$

$$E(Y_{(1)}) = \frac{\lambda}{2n+1} \tag{32}$$

$$E(Y_{(1)}^2) = \int_0^\lambda y^2 \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy = \frac{2n}{\lambda^{2n}} \int_0^\lambda (\lambda - t)^2 t^{2n-1} dt =$$

$$\frac{2n}{\lambda^{2n}} \left[\frac{\lambda t^{2n}}{2n} - \frac{2\lambda t^{2n+1}}{2n+1} + \frac{t^{2n+2}}{2n+2} \Big|_{t=0}^{\lambda} \right] = 2n\lambda^2 \left[\frac{1}{2n} - \frac{2}{2n+1} + \frac{1}{2n+2} \right] = \frac{\lambda^2}{(2n+1)(n+1)}$$

$$E(Y_{(1)}^2) = \frac{\lambda^2}{(2n+1)(n+1)} \tag{33}$$

By the use of the results in (32)and (33) we obtain

$$Var(Y_{(1)}) = \frac{\lambda^2}{(n+1)(2n+1)} - \frac{\lambda^2}{(2n+1)^2} = \frac{n\lambda^2}{(n+1)(2n+1)^2} \tag{34}$$

Since $Y = (X - \theta)$,

$$E(X_{(1)}) = E(Y_{(1)}) + a = \frac{\lambda}{(2n+1)} + \theta = \frac{2n\theta + a}{(2n+1)} \tag{35}$$

$$Var(X_{(1)}) = Var(Y_{(1)}) = \frac{n\lambda^2}{(n+1)(2n+1)^2} = \frac{n(a-\theta)^2}{(n+1)(2n+1)^2} \tag{36}$$

From the equation (35), we can obtain an unbiased estimator for θ as a function of $X_{(1)}$ as given in (37) and the variance of T_1 are obtained as in (38).

$$T_1 = \frac{(2n+1)X_{(1)} - a}{2n} \tag{37}$$

$$Var(T_1) = \frac{(2n+1)^2}{4n^2} Var(X_{(1)}) == \frac{(a-\theta)^2}{4n(n+1)} \tag{38}$$

2.2 Estimation of the Parameter θ of X by the Sample Mean $\overline{X}$

$$f(y)=\frac{2(a-\theta)-2y}{\lambda^2}=\frac{2(\lambda-y)}{\lambda^2},0<y<\lambda$$

$$E(Y)=\int_0^{\lambda} y\frac{2(\lambda-y)}{\lambda^2}dy=\frac{2}{\lambda^2}\left[\frac{\lambda y^2}{2}-\frac{y^3}{3}\Big|_{y=0}^{\lambda}\right]=\frac{2}{\lambda^2}\left[\frac{\lambda^3}{2}-\frac{\lambda^3}{3}\right]=\frac{\lambda}{3} \tag{39}$$

$$E(Y^2)=\int_0^{\lambda} y^2\frac{2(\lambda-y)}{\lambda^2}dy=\frac{2}{\lambda^2}\left[\frac{\lambda y^3}{3}-\frac{y^4}{4}\Big|_{y=0}^{\lambda}\right]=\frac{2}{\lambda^2}\left[\frac{\lambda^4}{3}-\frac{\lambda^4}{4}\right]=\frac{\lambda^2}{6} \tag{40}$$

$$Var(Y)=\frac{\lambda^2}{6}-\frac{\lambda^2}{9}=\frac{\lambda^2}{18} \tag{41}$$

Since $Y=(X-\theta)$, $E(X)=E(Y)+\theta$, and $Var(X)=Var(Y)$.

$$E(X)=\frac{\lambda}{3}+\theta=\frac{(a-\theta)}{3}+\theta=\frac{2\theta+a}{3} \tag{42}$$

$$Var(X)=\frac{\lambda^2}{18}=\frac{(a-\theta)^2}{18} \tag{43}$$

For any random variable X, $E(\overline{X})=E(X)$, and $Var(\overline{X})=\frac{Var(X)}{n}$. Then for the above Type II Triangular distribution the followings hold true.

$$E(\overline{X})=E(X)=\frac{2\theta+a}{3}, \text{ and } \quad Var(\overline{X})=\frac{1}{n}Var(X)=\frac{(a-\theta)^2}{18n}.$$

An unbiased estimator for θ, as a function of $\overline{X}$, and its variance are as given below.

$$T_2=\frac{3\overline{X}-a}{2} \quad . \tag{44}$$

$$Var(T_2)=\frac{9}{4}Var(\overline{X})=\frac{(a-\theta)^2}{8n} \tag{45}$$

2.4 Comparisons of Unbiased Estimators on the Basis of Efficiency

$$T_1=\frac{(2n+1)X_{(1)}-a}{2n} \quad , \quad Var(T_1)=\frac{(\theta-a)^2}{4n(n+1)}$$

$$T_2=\frac{3\overline{X}-a}{2} \quad , \quad Var(T_2)=\frac{(\theta-a)^2}{8n}$$

For any n>1 $Var(T_1)<Var(T_2)$. $T_1=\dfrac{(2n+1)X_{(1)}-a}{2n}$ is a better unbiased estimator.

2.5 Confidence Interval for θ of the Type II Triangular Distribution

Since we let $Y=(X-\theta)$ and its pdf is obtained in (30) as given below

$f_{Y_{(1)}}(y)=\frac{2n}{\lambda^{2n}}(\lambda-y)^{2n-1}, 0<y<\lambda$.

From the probability statement, $P(x_{1L}<X_{(1)}<x_{1U})=1-\alpha$, lower and upper confidence limits may be obtained.

$$P(x_{1L}<X_{(1)}<x_{1U})=P(x_{1L}<Y_{(1)}+\theta<x_{1U})=P(x_{1L}-\theta<Y_{(1)}<x_{1U}-\theta)=1-\alpha$$

If we let $(x_{1L}-\theta)=y_{1L}$ *and* $(x_{1U}-\theta)=y_{1U}$

$$P(Y_{(1)}<y_{1L})=\frac{2n}{\lambda^{2n}}\int_0^{y_{1L}}(\lambda-y)^{2n-1}dy=\alpha/2 \tag{46}$$

$$P(Y_{(1)}<y_{1U})=\frac{2n}{\lambda^{2n}}\int_0^{y_{1U}}(\lambda-y)^{2n-1}dy=1-\alpha/2 \tag{47}$$

In (46) and (47), let $(\lambda-y)=t$:

$dt=-dy;\quad y=0\rightarrow t=\lambda;\ \ y=y_{1L}\rightarrow t=\lambda-y_{1L}$

$$P(Y_{(1)}<y_{1L})=\frac{2n}{\lambda^{2n}}\int_0^{y_{1L}}(\lambda-y)^{2n-1}dy=\frac{2n}{\lambda^{2n}}\int_{\lambda-y_{1L}}^{\lambda}t^{2n-1}dt=\left(\frac{\lambda}{\lambda-y_{1L}}\right)^{2n}=\alpha/2 \tag{48}$$

Similarly,

$$P(Y_{(1)}<y_{1U})=\frac{2n}{\lambda^{2n}}\int_0^{y_{1U}}(\lambda-y)^{2n-1}dy=\frac{2n}{\lambda^{2n}}\int_{\lambda-y_{1U}}^{\lambda}t^{2n-1}dt=\left(\frac{\lambda}{\lambda-y_{1U}}\right)^{2n}=1-\alpha/2 \tag{49}$$

From the equations (48) and (49), y_{1L} and y_{1U} can be computed as given below.

$$y_{1L}=\lambda-(\lambda)\left(\frac{\alpha}{2}\right)^{-1/2n}=(a-\theta)-(a-\theta)\left(\frac{\alpha}{2}\right)^{-1/2n} \tag{50}$$

$$y_{1U}=\lambda-(\lambda)\left(1-\frac{\alpha}{2}\right)^{-1/2n}=(a-\theta)-(a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n} \tag{51}$$

Since we have $(x_{1L}-\theta)=y_{1L}$ *and* $(x_{1U}-\theta)=y_{1U}$, then

$x_{1L}=y_{1L}+\theta$ and $x_{1U}=y_{1U}+\theta$.

$$x_{1L}=(a)-(a-\theta)\left(\frac{\alpha}{2}\right)^{1/2n} \tag{52}$$

$$x_{1U}=(a)-(a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n} \tag{53}$$

Substituting the results of (52) and (53) in the following probability expression yields (54).

$$P(x_{1L} < X_{(1)} < x_{1U}) = 1-\alpha$$
$$P\left(a-(a-\theta)\left(\frac{\alpha}{2}\right)^{-1/2n} < X_{(1)} < a-(a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n} \right) = 1-\alpha \quad (54)$$

By solving inequalities in (54) for θ simultaneously we obtain the following probability statement.

$$P\left(a-(a-X_{(1)})(1-\alpha/2)^{1/2n} < \theta < a-(a-X_{(1)})(\alpha/2)^{1/2n}\right) = 1-\alpha \quad (55)$$

Hence, a $100*(1-\alpha)\%$ confidence interval for θ can be given by the use of the following formula.

$$\left(a-(a-X_{(1)})(1-\alpha/2)^{1/2n}, \quad < a-(a-X_{(1)})(\alpha/2)^{1/2n}\right)$$

2.6 Tests of Hypotheses Related to the Parameter θ of Type II Triangular Distribution

In testing $H_0 : \theta = \theta_0$ against to any alternative hypothesis, $X_{(1)} = X_{Min}$ may be used as a proper test statistic. For the chosen level of significance α, the decision rules given in the following table are applicable.

$H_0 : \theta = \theta_0$ $H_1 : \theta \neq \theta_0$	$H_0 : \theta \leq \theta_0$ $H_1 : \theta > \theta_0$	$H_0 : \theta \geq \theta_0$ $H_1 : \theta < \theta_0$
If $x_{(1)} \geq x_{1U}$ or $x_{(1)} \leq x_{1L}$ H_0 is rejected.	If $x_{(1)} \geq x_{1U}$ H_0 is rejected.	If $x_{(1)} \leq x_{1L}$ H_0 is rejected.
Don't reject H_0 otherwise.	Don't reject H_0 otherwise.	Don't reject H_0 otherwise.
Where, $x_{1L} = a-(a-\theta_0)\left(\frac{\alpha}{2}\right)^{1/2n}$ and $x_{1U} < a-(a-\theta_0)\left(1-\frac{\alpha}{2}\right)^{1/2n}$	Where, $x_{1U} < a-(a-\theta_0)(1-\alpha)^{1/2n}$	Where, $x_{1L} = a-(a-\theta_0)(\alpha)^{1/2n}$

3. SIMULATION

For the purpose of simulations from both Type I, and Type II triangular distributions inverse transformations are used on the respective F(x) functions.

For the type I triangular distribution over the interval (5,10) [$\theta = 5$ *and* $a = 10$] the pdf is

$f(x)=\dfrac{2(x-5)}{25}, 5<x<10$. Hence we obtain

$F(x)=\dfrac{(x-5)^2}{25},\ 5<x<10$.

If a random sample is taken from the $Uniform[0,1]$ we obtain the corresponding X value in the
type I triangular distribution by the use of the following formula

$x=5(1+\sqrt{F(x)})$.

The median, m, for this Type I triangular distribution is:

$$F(m)=\frac{(m-5)^2}{25}=\frac{1}{2}\rightarrow\frac{(m-5)}{5}=\frac{1}{\sqrt{2}}\rightarrow m=5+\frac{5}{\sqrt{2}}\cong 8.535$$

Similarly, **f**or the type II triangular distribution over the interval (5, 10) **[** $a=5\ \ and\ \ \theta=10]$ the pdf is

$f(x)=\dfrac{2(10-x)}{25}, 5<x<10$, and the respective distribution function is

$F(x)=1-\dfrac{(10-x)^2}{25},\ 5<x<10$.

If a random sample is taken from the $Uniform[0,1]$ we obtain the corresponding X value in the
type II triangular distribution by the use of the following formula

$x=10-5\sqrt{1-F(x)}$.

The median, m, for this Type II triangular distribution is:

$$F(m)=1-\frac{(10-m)^2}{25}=\frac{1}{2}\rightarrow\frac{(10-m)}{5}=\frac{1}{\sqrt{2}}\rightarrow m=10-\frac{5}{\sqrt{2}}\cong 6.464\ .$$

Table 1. 50 Random Samples of Sizes n=30, From Uniform (0, 1)

u_1	u_2	u_3	u_4	u_5	...	u_{48}	u_{49}	u_{50}
0,517788	0,498149	0,73755	0,231313	0,906975	...	0,357554	0,955998	0,259729
0,746209	0,260626	0,239369	0,473726	0,815237		0,171827	0,454524	0,764033
0,377066	0,449645	0,178527	0,453634	0,773392		0,483062	0,988522	0,711061
0,692188	0,387486	0,778406	0,122221	0,626263		0,261265	0,929536	0,805118
0,606739	0,406041	0,60322	0,43981	0,234233		0,759478	0,567909	0,919013
0,421063	0,729669	0,367489	0,722403	0,2267		0,750751	0,153411	0,369019
0,80357	0,124422	0,543059	0,808808	0,504451		0,324161	0,27868	0,516978

Table 2. 50 Random Samples of Sizes n=30, From TYPE I Triangular Distribution and Estimators:
$x_i = 5(1+\sqrt{u_i})$

	X1	X2	X3	X4	X5	....	X49	X50		
	8,597876	8,528984	9,294036	7,404752	9,761761	....	9,888757	7,548182		
	9,31917	7,552579	7,446267	8,441389	9,514525	.....	8,370919	9,37045		
	8,070286	8,352779	7,112625	8,367617	9,397135	.....	9,971222	9,216223		
	8,24447	9,271034	8,031045	9,249713	7,380651	.....	6,958389	8,037347		
	9,482104	6,763672	8,684627	9,496689	8,551237	.....	7,639509	8,595061	Mean	Var
Mean	**8,36463**	**8,312875**	**8,433408**	**8,721029**	**8,75695**	.....	**8,426896**	**8,771766**	**8,4317**	**0,0460**
Var	**1,296027**	**1,174865**	**0,980048**	**1,171179**	**0,798496**	.....	**1,739837**	**0,767044**	**1,2601**	**0,0940**
Xmax	**9,94276**	**9,89737**	**9,945686**	**9,860279**	**9,930852**	.....	**9,971222**	**9,864761**	**9,921**	**0,0044**
Xmin	**5,166093**	**6,20382**	**6,297411**	**5,459905**	**6,89126**	.....	**5,809413**	**6,676654**	**5,8748**	**0,2427**
T1	**10,02514**	**9,978993**	**10,02811**	**9,941284**	**10,01303**	......	**10,05408**	**9,94584**	**10,003**	**0,0045**
T2	**10,04695**	**9,969312**	**10,15011**	**10,58154**	**10,63542**	......	**10,14034**	**10,65765**	**10,147**	**0,1037**
Med	**8,672594**	**8,436987**	**8,651681**	**9,223838**	**8,953614**	.....	**8,842737**	**8,854906**	**8,6634**	**0,0948**
Tmed	**10,19383**	**9,860634**	**10,16426**	**10,97341**	**10,59125**	......	**10,43445**	**10,45166**	**10,180**	**0,1897**

The above simulation results are in accordance with the theoretical findings, for that the estimator T_1 is seen to estimate the true parameter θ of the Type I triangular distribution very accurately.

Table 3. 50 Random samples 0f sizes n=30, from TYPE II Triangular Distribution and Estimators:
$x_i = 10 - 5\sqrt{1-u_i}$

	X1	X2	X3	X4	....	X48	X49	X50		
	6,527927	6,457928	7,438506	5,616261	....	5,992363	8,951165	5,69805		
	7,481117	5,700658	5,639291	6,372764	.....	5,449799	6,307182	7,571179		
	6,053693	6,290705	5,468244	6,30417	.....	6,405081	9,464323	7,312349		
					.....					
	6,195606	7,400334	6,023473	7,365624	.....	7,50376	5,399488	6,028285		
	7,783981	5,321382	6,62013	7,813727	.....	5,88953	5,753473	6,525013	Mean	Var
Mean	**6,651137**	**6,596583**	**6,683158**	**7,051849**	.....	**6,528217**	**6,903038**	**7,029369**	**6,767**	**0,0464**
Var	**1,088856**	**1,187989**	**1,158918**	**1,211741**	.....	**1,605636**	**1,807143**	**1,11921**	**1,335**	**0,0732**
Xmax	**9,245597**	**8,992148**	**9,265025**	**8,82625**	.....	**9,072844**	**9,464323**	**8,844968**	**9,194**	**0,1313**
Xmin	**5,002759**	**5,147082**	**5,171261**	**5,021196**	......	**5,017774**	**5,06595**	**5,289498**	**5,102**	**0,0105**
T1	**5,002805**	**5,149533**	**5,174115**	**5,02155**	......	**5,018071**	**5,067049**	**5,294323**	**5,104**	**0,0109**
T2	**4,976705**	**4,894875**	**5,024737**	**5,577773**	.....	**4,792326**	**5,354557**	**5,544054**	**5,151**	**0,1045**
Median	**6,607878**	**6,368598**	**6,584901**	**7,32477**	......	**6,198722**	**6,803171**	**6,815711**	**6,630**	**0,1153**
Tmed	**5,202815**	**4,864422**	**5,170321**	**6,216653**		**4,624181**	**5,479002**	**5,496736**	**5,234**	**0,2306**

The above simulation results are in accordance with the theoretical findings, for that the estimator T_1 is seen to estimate the true parameter θ of the Type II triangular distribution very accurately.

4. REFERENCES

Balakrishnan, N., Cohen, A. C., 1991. Order Statistics and Inference. New York: Academic Press.

Balakrishnan, N., Rao, C. R. (Eds.), 1998. Handbook of Statistics, Vol. 16: Order Statistics: Theory and Methods. Amsterdam, Netherlands: Elsevier.

Balakrishnan, N., Rao, C. R. (Eds.), 1998. Order Statistics: Applications. Amsterdam, Netherlands: Elsevier.

David, H. A., Order Statistics, 2nd ed., 1981. New York: Wiley.

Gibbons, J. D., Chakraborti, S. (Eds.), 1992. Nonparametric Statistic Inference, 3rd ed. exp. rev. New York: Dekker.

Hogg, R. V., Craig, A. T., 1970. Introduction to Mathematical Statistics, 3rd ed. New York: Macmillan.

Rose, C., Smith, M. D., 2002. Order Statistics. §9.4 in Mathematical Statistics with Mathematica. New York: Springer-Verlag, pp. 311-322.

Rose, C., Smith, M. D., 2005. Computational Order Statistics. Mathematica J. 9, 790-802.

BAZI ÜÇGENSEL DAĞILIMLAR ÜZERİNE İSTATİSTİKSEL ÇIKARIMLAR: SINIR DEĞERLERİ PARAMETRE OLAN DURUM

ÖZET

Eğer sürekli rastgele değişkenler üçgensel dağılıma sahipse ve eğer θ_1 ve/veya θ_2 sınır değerleri bilinmiyorsa, o zaman bu parametrelerle ilgili bazı istatistiksel çıkarımlar yapmak gerekli olabilir. Bilinmeyen sınır değerleri olan üçgensel dağılımlar için, sıralı istatistiklerin bir fonksiyonu olarak bazı tahmin ediciler önerilmiştir. Önerilen tahmin ediciler etkinliklerine dayalı olarak karşılaştırılmıştır. Önerilen tahmin ediciler arasındaki en iyi tahmin edici, etkinlik kriterine göre belirlenmiştir. En iyi tahmin edicinin kullanılmasıyla, bir güven aralığı oluşturma ve hipotez tesi izlekleri geliştirilmiştir. Simülasyon prosesi yoluyla, örneklem sonuçları ile teorik bulgular arasında eşleşen bir doğruluk gözlenmiştir.

Anahtar Kelimeler: Tahmin edicilerin etkinlikleri, Tahmin ediciler, Sıralı istatistikler, Simülasyon, Üçgensel dağılım.