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Characterization of Two Specific Cases with New Operators in Ideal Topological Spaces

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Abstract — This research deals with new operators $\wedge_{\Gamma}$, $\veebar_{\Gamma}$, and $\bar{\wedge}_{\Gamma}$, defined using Γ-local closure function and Ψ_{Γ}-operator in ideal topological spaces. It investigates the main features of these operators and their relationships with each other. The paper also analyzes their behaviors in some special ideals. Besides, it explores whether these operators preserve some set operations. Then, the study researches the properties of some special sets using these operators and proposes their characterizations. Additionally, it interprets some characterizations of the case $\mathrm{cl}(\tau)\cap\Im = \{\emptyset\}$ and the closure compatibility by means of these new operators.

Keywords *Ideal, Γ-local closure function, Ψ_{Γ}-operator*

Mathematics Subject Classification (2020) 54A05, 54A99

1. Introduction

After the emergence of the concept of ideal in [1, 2], this topic has been discussed by many authors in the literature. The local function [1] obtained using ideals and the *-topology [2] obtained with the help of this function are the most tackled topics by researchers. One of the most notable studies about these topics is the work of Janković and Hamlett in [3]. Furthermore, Ψ-operator [4], extensions of ideal [5], I-open sets [6], $PC^{\star}$-closed sets [7], and weakly I_{rg}-open sets [8] are the other examples of these topics. Apart from these topics, Selim et al. [9, 10] and Modak and Selim [11] also studied various set operators acquired by local function and Ψ-operator.

Afterward, Al-Omari and Noiri [12] introduced the Γ-local closure function and presented various properties of this operator. Furthermore, they defined the Ψ_{Γ}-operator with the Γ-local closure function and formed two topologies called σ and σ_0 owing to the operator Ψ_{Γ}. Many new studies based on Al-Omari and Noiri's works have been produced. For instance, Pavlović [13] investigated the similarities and differences between local functions with Γ-local closure functions and researched the cases under which they coincide. Tunç and Özen Yıldırım [14] made additions to Pavlović's conditions, and hence they [15, 16] defined some special sets using local closure functions and Ψ_{Γ}-operators. Furthermore, they [17] obtained a Γ-boundary operator through local closure functions and researched its properties.

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In this study, we build new set operators termed $\wedge_\Gamma$, $\veebar_\Gamma$, and $\bar{\wedge}_\Gamma$ via local closure function and Ψ_Γ-operator using different methods in ideal topological spaces. We research their main properties and the relationships of these operators with each other. We characterize the case $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ and interpret the closure compatibility by means of these operators.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following sections. Throughout this study, (Y, τ) represents a topological space. In (Y, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and the interior of a subset A of Y, respectively. $P(Y)$ represents the family of all the subsets of Y. An ideal $\Im$ [1] on a topological space (Y, τ) is a nonempty collection of subsets of Y satisfying the following conditions:

i. if $A \in \Im$ and $B \subseteq A$, then $B \in \Im$ (heredity).

ii. if $A \in \Im$ and $B \in \Im$, then $A \cup B \in \Im$ (finite additivity).

An ideal topological space $(Y, \tau, \Im)$ is a topological space (Y, τ) with an ideal $\Im$ on Y.

Let $(Y, \tau, \Im)$ be an ideal topological space. For a subset A of Y,

$$\Gamma(A)(\Im, \tau) = \{x \in Y \mid A \cap \mathrm{cl}(U) \notin \Im, \text{ for all } U \in \tau(x)\}$$

is called the local closure function of A with respect to $\Im$ and τ where $\tau(x) = \{U \in \tau \mid x \in U\}$ [12]. It is shortly denoted by $\Gamma(A)$ instead of $\Gamma(A)(\Im, \tau)$. An operator $\Psi_\Gamma : P(Y) \mapsto \tau$ is defined by $\Psi_\Gamma(A) = Y \setminus \Gamma(Y \setminus A)$, for all $A \in P(Y)$ [12]. A subset A of Y is called $\Im_\Gamma$-perfect (respectively, Γ-dense-in-itself, L_Γ-perfect, R_Γ-perfect, and $\Im_\Gamma$-dense) if $A = \Gamma(A)$ (respectively, $A \subseteq \Gamma(A)$, $A \setminus \Gamma(A) \in \Im$, $\Gamma(A) \setminus A \in \Im$, and $\Gamma(A) = Y$) [15]. A subset A of Y is called C_Γ-perfect if A is both L_Γ-perfect and R_Γ-perfect [15]. A subset A of Y is called $\theta^{\Im}$-closed if $\Gamma(A) \subseteq A$ [18].

For a topological space (Y, τ) and a subset A of Y, $\mathrm{cl}_\theta(A) = \{x \in Y : \mathrm{cl}(U) \cap A \neq \emptyset \text{ for each } U \in \tau(x)\}$ is called the θ-closure of A [19]. The θ-interior of A [20], denoted $\mathrm{int}_\theta(A)$, consists of those points x of A such that $U \subseteq \mathrm{cl}(U) \subseteq A$ for some open set U containing x. Furthermore, $Y \setminus \mathrm{int}_\theta(A) = \mathrm{cl}_\theta(Y \setminus A)$ [21]. A subset A is called θ-closed [19] if $A = \mathrm{cl}_\theta(A)$. The complement of a θ-closed set is called θ-open [19]. The family of all θ-open sets in (Y, τ) is denoted by τ_θ. Moreover, τ_θ is a topology on Y and it is coarser than τ.

As mentioned above, Al-Omari and Noiri [12] have defined the two topologies on Y as follows: $\sigma = \{A \subseteq Y : A \subseteq \Psi_\Gamma(A)\}$ and $\sigma_0 = \{A \subseteq Y : A \subseteq \mathrm{int}(\mathrm{cl}(\Psi_\Gamma(A)))\}$. They have shown that $\tau_\theta \subseteq \sigma \subseteq \sigma_0$ in $(Y, \tau, \Im)$. A subset A of Y is called σ-open (σ_0-open) set if $A \in \sigma$ ($A \in \sigma_0$). The topology τ is said to be closure compatible with the ideal $\Im$, denoted by $\tau \sim_\Gamma \Im$, if the following condition is held, for all subset A of Y: if, for all $x \in A$, there exists a $U \in \tau(x)$ such that $\mathrm{cl}(U) \cap A \in \Im$, then $A \in \Im$ [12].

Theorem 2.1. [12] Let $(Y, \tau, \Im)$ be an ideal topological space and $A, B \subseteq Y$. Then,

i. $\Gamma(\emptyset) = \emptyset$

ii. If $A \in \Im$, then $\Gamma(A) = \emptyset$.

iii. $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$

iv. $\Psi_\Gamma(A \cap B) = \Psi_\Gamma(A) \cap \Psi_\Gamma(B)$

v. $\Gamma(A) = \mathrm{cl}(\Gamma(A)) \subseteq \mathrm{cl}_\theta(A)$

vi. If $A \subseteq B$, then $\Psi_\Gamma(A) \subseteq \Psi_\Gamma(B)$.

vii. If $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$.

Corollary 2.2. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. If $B \in \Im$, then $\Gamma(A \cup B) = \Gamma(A) = \Gamma(A \setminus B)$ in $(Y, \tau, \Im)$, for all $A, B \subseteq Y$.

Definition 2.3. [22] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, the θ-closure of A with respect to an ideal $\Im$ is defined by $\mathrm{cl}_{\Im_\theta}(A) = A \cup \Gamma(A)(\Im, \tau)$. If $A = \mathrm{cl}_{\Im_\theta}(A)$, then A is called to be $\Im_\theta$-closed. Moreover, $\mathrm{Int}_{\Im_\theta}(A)$ is defined by $\mathrm{Int}_{\Im_\theta}(A) = Y \setminus \mathrm{cl}_{\Im_\theta}(Y \setminus A)$. If $A = \mathrm{Int}_{\Im_\theta}(A)$, then A is called to be $\Im_\theta$-open.

Remark 2.4. [17] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then,

$$A \text{ is } \Im_\theta\text{-closed } \Leftrightarrow A = \mathrm{cl}_{\Im_\theta}(A) = A \cup \Gamma(A) \Leftrightarrow \Gamma(A) \subseteq A \Leftrightarrow A \text{ is } \theta^{\Im}\text{-closed}$$

Thus, the concept of $\Im_\theta$-closed set in [22] and the concept of $\theta^{\Im}$-closed set in [18] are identical.

Proposition 2.5. [17] Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then,

i. A is $\Im_\theta$-open $\Leftrightarrow Y \setminus A$ is $\Im_\theta$-closed

ii. A is $\Im_\theta$-open $\Leftrightarrow A \subseteq \Psi_\Gamma(A)$

iii. A is σ-open $\Leftrightarrow A$ is $\Im_\theta$-open

Theorem 2.6. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ such that $\mathrm{cl}(\tau) = \{\mathrm{cl}(G) : G \in \tau\}$ if and only if $Y = \Gamma(Y)$.

Theorem 2.7. [12] Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $A \setminus \Gamma(A) \in \Im$

Theorem 2.8. [14] Let $(Y, \tau, \Im)$ be an ideal topological space such that $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$.

Theorem 2.9. [15] Let $(Y, \tau, \Im)$ be an ideal topological space. Every $\Im_\Gamma$-dense set is Γ-dense-in-itself.

3. The Operator $\wedge_\Gamma$

This section defines the operator $\wedge_\Gamma$ and investigates its basic properties.

Definition 3.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\wedge_\Gamma : P(Y) \to P(Y)$ is defined by $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A$, for all $A \subseteq Y$.

Proposition 3.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\Gamma(K) = \mathrm{cl}_\theta(K)$. Therefore,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \mathrm{cl}_\theta(Y \setminus K)) \setminus K = \mathrm{int}_\theta(K) \setminus K = \emptyset$$

ii. If $\Im = P(Y)$, then $\Gamma(Y \setminus K) = \emptyset$. Thus,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \Gamma(Y \setminus K)) \setminus K = Y \setminus K$$

Remark 3.3. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $M \subseteq N$ implies that neither $\wedge_\Gamma(M) \subseteq \wedge_\Gamma(N)$ nor $\wedge_\Gamma(N) \subseteq \wedge_\Gamma(M)$, for all $M, N \subseteq Y$.

Example 3.4. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{q\}, \{r\}, \{q, r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. Suppose that $M = \{p\}$, $N = \{p, r\}$, and $K = \{p, s\}$. Although, $M \subseteq N$ in the ideal topological space $(Y, \tau, \Im)$, $\wedge_\Gamma(M) \nsubseteq \wedge_\Gamma(N)$. Similarly, although $M \subseteq K$, $\wedge_\Gamma(K) \nsubseteq \wedge_\Gamma(M)$.

Theorem 3.5. Let $(Y, \tau, \Im)$ be an ideal topological space and $M, N \subseteq Y$. Then, the following are held.

i. $\wedge_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(Y) = \emptyset$

iii. If $\text{cl}(\tau) \cap \Im = \{\emptyset\}$ and $M \in \Im$, then $\wedge_\Gamma(M) = \emptyset$.

iv. $\wedge_\Gamma(M) = (Y \setminus M) \setminus \Gamma(Y \setminus M)$

v. $\wedge_\Gamma(M) \cap \wedge_\Gamma(N) \subseteq \wedge_\Gamma(M \cup N)$

vi. $\wedge_\Gamma(M \cap N) = (\wedge_\Gamma(M) \cap \Psi_\Gamma(N)) \cup (\wedge_\Gamma(N) \cap \Psi_\Gamma(M)) \subseteq \wedge_\Gamma(M) \cup \wedge_\Gamma(N)$

vii. $\wedge_\Gamma(\wedge_\Gamma(M)) \subseteq \wedge_\Gamma(\Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M))$

viii. $\Gamma(\wedge_\Gamma(M)) \subseteq \Gamma(\Psi_\Gamma(M))$

ix. $\wedge_\Gamma(M) \cap M = \emptyset$ and thus $\wedge_\Gamma(M) \subseteq Y \setminus M$

x. $\wedge_\Gamma(M) \cup \wedge_\Gamma(N) \subseteq (M \cap \wedge_\Gamma(N)) \cup \wedge_\Gamma(M \cup N) \cup (\wedge_\Gamma(M) \cap N)$

PROOF. Let $(Y, \tau, \Im)$ be an ideal topological space and $M, N \subseteq Y$.

i. $\wedge_\Gamma(\emptyset) = \Psi_\Gamma(\emptyset) \setminus \emptyset = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(Y) = \Psi_\Gamma(Y) \setminus Y = \emptyset$

iii. Let $\text{cl}(\tau) \cap \Im = \{\emptyset\}$ and $M \in \Im$. Then, by Corollary 2.2,

$$\wedge_\Gamma(M) = \Psi_\Gamma(M) \setminus M = (Y \setminus \Gamma(Y \setminus M) \setminus M = (Y \setminus \Gamma(Y)) \setminus M$$

Moreover, $\Gamma(Y) = Y$ from Theorem 2.6. Thus,

$$\wedge_\Gamma(M) = (Y \setminus Y) \setminus M = \emptyset$$

iv. $\wedge_\Gamma(M) = \Psi_\Gamma(M) \setminus M = (Y \setminus M) \cap (Y \setminus \Gamma(Y \setminus M)) = (Y \setminus M) \setminus \Gamma(Y \setminus M)$.

v.

$$\begin{aligned}\wedge_\Gamma(M) \cap \wedge_\Gamma(N) &= (\Psi_\Gamma(M) \setminus M) \cap (\Psi_\Gamma(N) \setminus N)\\ &= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \cap [(Y \setminus M) \cap (Y \setminus N)]\\ &= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \cap [Y \setminus (M \cup N)]\end{aligned}$$

From Theorem 2.1 (*iv.*),

$$\wedge_\Gamma(M) \cap \wedge_\Gamma(N) = \Psi_\Gamma(M \cap N) \cap [Y \setminus (M \cup N)] = \Psi_\Gamma(M \cap N) \setminus (M \cup N)$$

From Theorem 2.1 (*vi.*),

$$\Psi_\Gamma(M \cap N) \setminus (M \cup N) \subseteq \Psi_\Gamma(M \cup N) \setminus (M \cup N) = \wedge_\Gamma(M \cup N)$$

Therefore,

$$\wedge_\Gamma(M) \cap \wedge_\Gamma(N) \subseteq \wedge_\Gamma(M \cup N)$$

vi. By Theorem 2.1 (*iv.*),

$$\begin{aligned}\wedge_\Gamma(M \cap N) &= \Psi_\Gamma(M \cap N) \setminus (M \cap N)\\ &= (\Psi_\Gamma(M) \cap \Psi_\Gamma(N)) \setminus (M \cap N)\\ &= [\Psi_\Gamma(M) \cap \Psi_\Gamma(N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(M) \cap \Psi_\Gamma(N) \cap (Y \setminus N)]\end{aligned}$$

$$= ([\Psi_\Gamma(M) \cap (Y \setminus M)] \cap \Psi_\Gamma(N)) \cup (\Psi_\Gamma(M) \cap [\Psi_\Gamma(N) \cap (Y \setminus N)])$$

$$= (\wedge_\Gamma(M) \cap \Psi_\Gamma(N)) \cup (\wedge_\Gamma(N) \cap \Psi_\Gamma(M))$$

$$\subseteq \wedge_\Gamma(M) \cup \wedge_\Gamma(N)$$

vii. By Theorem 2.1 (*iv.*),

$$\begin{aligned}
\wedge_\Gamma(\wedge_\Gamma(M)) &= \Psi_\Gamma(\wedge_\Gamma(M)) \setminus \wedge_\Gamma(M) \\
&= \Psi_\Gamma(\Psi_\Gamma(M) \setminus M) \setminus (\Psi_\Gamma(M) \setminus M) \\
&= (\Psi_\Gamma(\Psi_\Gamma(M)) \cap \Psi_\Gamma(Y \setminus M)) \cap [(Y \setminus \Psi_\Gamma(M)) \cup M] \\
&\subseteq \Psi_\Gamma(\Psi_\Gamma(M)) \cap [(Y \setminus \Psi_\Gamma(M)) \cup M] \\
&= [\Psi_\Gamma(\Psi_\Gamma(M)) \cap (Y \setminus \Psi_\Gamma(M))] \cup (\Psi_\Gamma(\Psi_\Gamma(M)) \cap M) \\
&\subseteq (\Psi_\Gamma(\Psi_\Gamma(M)) \setminus \Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M)) \\
&= \wedge_\Gamma(\Psi_\Gamma(M)) \cup \Psi_\Gamma(\Psi_\Gamma(M))
\end{aligned}$$

viii. By Theorem 2.1 (*vii.*), $\Gamma(\wedge_\Gamma(M)) = \Gamma(\Psi_\Gamma(M) \setminus M) \subseteq \Gamma(\Psi_\Gamma(M))$.

ix. $\wedge_\Gamma(M) \cap M = (\Psi_\Gamma(M) \setminus M) \cap M = \emptyset$ and thus $\wedge_\Gamma(M) \subseteq Y \setminus M$.

x.

$$\begin{aligned}
\wedge_\Gamma(M) &= \Psi_\Gamma(M) \setminus M \\
&= \Psi_\Gamma(M) \cap (Y \setminus M) \cap [(Y \setminus N) \cup N] \\
&= [\Psi_\Gamma(M) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N]
\end{aligned}$$

By Theorem 2.1 (*vi.*),

$$\wedge_\Gamma(M) \subseteq [\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N] \tag{3.1}$$

Moreover,

$$\begin{aligned}
\wedge_\Gamma(N) &= \Psi_\Gamma(N) \setminus N \\
&= \Psi_\Gamma(N) \cap (Y \setminus N) \cap [(Y \setminus M) \cup M] \\
&= [\Psi_\Gamma(N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M]
\end{aligned}$$

By Theorem 2.1 (*vi.*),

$$\wedge_\Gamma(N) \subseteq [\Psi_\Gamma(M \cup N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M] \tag{3.2}$$

From (3.1) and (3.2),

$$\begin{aligned}
\wedge_\Gamma(M) \cup \wedge_\Gamma(N) \subseteq\ & ([\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N]) \cup \\
& ([\Psi_\Gamma(M \cup N) \cap (Y \setminus N) \cap (Y \setminus M)] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M]) \\
=\ & [\Psi_\Gamma(M \cup N) \cap (Y \setminus M) \cap (Y \setminus N)] \cup [\Psi_\Gamma(M) \cap (Y \setminus M) \cap N] \cup [\Psi_\Gamma(N) \cap (Y \setminus N) \cap M] \\
=\ & [\Psi_\Gamma(M \cup N) \setminus (M \cup N)] \cup [(\Psi_\Gamma(M) \setminus M) \cap N] \cup [(\Psi_\Gamma(N) \setminus N) \cap M] \\
=\ & \wedge_\Gamma(M \cup N) \cup (\wedge_\Gamma(M) \cap N) \cup (\wedge_\Gamma(N) \cap M)
\end{aligned}$$

□

Theorem 3.6. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\text{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if $\wedge_\Gamma(\emptyset) = \emptyset$.

The proof of Theorem 3.6 is obvious by Theorem 2.6 and Theorem 3.5 (*i.*).

Theorem 3.7. Let $(Y, \tau, \Im)$ be an ideal topological space. If A is a θ-closed (or an $\Im_\theta$-closed) subset of Y, then $\wedge_\Gamma(A) \subseteq Y \setminus \Gamma(Y)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and A be a θ-closed (or an $\Im_\theta$-closed) subset of Y in $(Y, \tau, \Im)$. Then, $\Gamma(A) \subseteq A = \text{cl}_\theta(A)$ by Theorem 2.1 (*v.*) (or $\Gamma(A) \subseteq A$). It follows

$$\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A \subseteq \Psi_\Gamma(A) \setminus \Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A)$$

By Theorem 2.1 (*iv.*),

$$\wedge_\Gamma(A) \subseteq \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = \Psi_\Gamma(A \cap (Y \setminus A)) = \Psi_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$$

□

Remark 3.8. The inverses of the above requirements may not be true in general.

Example 3.9. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{p\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, suppose that $A = \{p\}$ and $B = \{q\}$. Then, $\wedge_\Gamma(A) = \emptyset \subseteq Y \setminus \Gamma(Y)$ but A is not θ-closed. Similarly, $\wedge_\Gamma(B) = \emptyset \subseteq Y \setminus \Gamma(Y)$ but B is not $\Im_\theta$-closed.

Theorem 3.10. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\wedge_\Gamma(K) \in \tau$, for all closed (or θ-closed) $K \subseteq Y$.

The proof of Theorem 3.10 is obvious by Theorem 2.1 (*v.*).

Corollary 3.11. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $\text{cl}(\tau) \cap \Im = \{\emptyset\}$. If A is θ-closed (or $\Im_\theta$-closed), then $\wedge_\Gamma(A) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, A be a θ-closed (or an $\Im_\theta$-closed) subset of Y, and $\text{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, by Theorem 3.7, $\wedge_\Gamma(A) \subseteq Y \setminus \Gamma(Y)$. It implies that $\wedge_\Gamma(A) = \emptyset$ from Theorem 2.6. □

Theorem 3.12. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $Y \setminus K$ is Γ-dense-in-itself if and only if $\wedge_\Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned} Y \setminus K \text{ is } \Gamma\text{-dense-in-itself} &\Leftrightarrow Y \setminus K \subseteq \Gamma(Y \setminus K) \\ &\Leftrightarrow Y \setminus \Gamma(Y \setminus K) \subseteq K \\ &\Leftrightarrow \Psi_\Gamma(K) \subseteq K \\ &\Leftrightarrow \Psi_\Gamma(K) \setminus K = \wedge_\Gamma(K) = \emptyset \end{aligned}$$

□

Corollary 3.13. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If $Y \setminus A$ is $\Im_\Gamma$-dense, then $\wedge_\Gamma(A) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $Y \setminus A$ be $\Im_\Gamma$-dense in $(Y, \tau, \Im)$. Then, by Theorem 2.9, $Y \setminus A$ is Γ-dense-in-itself. Thus, $\wedge_\Gamma(A) = \emptyset$ by Theorem 3.12. □

Remark 3.14. The reverse of the above requirement may not be true in general.

Example 3.15. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{q, s\}$, then $\wedge_\Gamma(A) = \emptyset$ but $Y \setminus A$ is not $\Im_\Gamma$-dense.

Theorem 3.16. Let $(Y, \tau, \Im)$ be an ideal topological space.

i. If K is $\Im_\Gamma$-perfect, then $\wedge_\Gamma(K) = Y \setminus \Gamma(Y)$, for all $K \subseteq Y$.

ii. If $Y \setminus K$ is $\Im_\Gamma$-perfect, then $\wedge_\Gamma(K) = \emptyset$, for all $K \subseteq Y$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

i. Let K be an $\Im_\Gamma$-perfect set. Then,

$$\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = (Y \setminus \Gamma(Y \setminus K)) \setminus K = Y \setminus (K \cup \Gamma(Y \setminus K)) = Y \setminus (\Gamma(K) \cup \Gamma(Y \setminus K))$$

From Theorem 2.1 (*iii.*), $\Gamma(K) \cup \Gamma(Y \setminus K) = \Gamma(Y)$. As a result, $\wedge_\Gamma(K) = Y \setminus \Gamma(Y)$.

ii. Let $Y \setminus K$ be an $\Im_\Gamma$-perfect set. Then,

$$\Psi_\Gamma(K) = Y \setminus \Gamma(Y \setminus K) = Y \setminus (Y \setminus K) = K$$

As a result, $\wedge_\Gamma(K) = \Psi_\Gamma(K) \setminus K = \emptyset$.

□

Remark 3.17. The reverse of the above requirements may not be true in general.

Example 3.18. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{p\}$, then $\wedge_\Gamma(A) = \emptyset = Y \setminus \Gamma(Y)$ but A and $Y \setminus A$ are not $\Im_\Gamma$-perfect.

Theorem 3.19. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, A is L_Γ-perfect $\Leftrightarrow$ $\wedge_\Gamma(Y \setminus A) \in \Im$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. From Theorem 3.5 (*iv.*), $\wedge_\Gamma(Y \setminus A) = A \setminus \Gamma(A)$. Thus, the proof is obvious. □

Corollary 3.20. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If A is C_Γ-perfect, then $\wedge_\Gamma(Y \setminus A) \in \Im$.

The proof is obvious by Theorem 3.19.

Remark 3.21. In an ideal topological space, the reverse of the above requirement may not be true in general.

Example 3.22. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{p\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{r\}$, then $\wedge_\Gamma(Y \setminus A) = \emptyset \in \Im$. However, A is not R_Γ-perfect and thus it is not C_Γ-perfect.

Theorem 3.23. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $\wedge_\Gamma(Y \setminus K) = K$ if and only if $K \cap \Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned} \wedge_\Gamma(Y \setminus K) = K &\Leftrightarrow \Psi_\Gamma(Y \setminus K) \cap K = K \\ &\Leftrightarrow K \subseteq \Psi_\Gamma(Y \setminus K) = Y \setminus \Gamma(K) \\ &\Leftrightarrow K \cap \Gamma(K) = \emptyset \end{aligned}$$

□

Theorem 3.24. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $\wedge_\Gamma(A) \in \Im$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space. Then,

$$\begin{aligned} \wedge_\Gamma(A) \in \Im, \text{ for all subset } A \text{ of } Y &\Leftrightarrow \wedge_\Gamma(Y \setminus A) \in \Im, \text{ for all subset A of } Y \\ &\Leftrightarrow \Psi_\Gamma(Y \setminus A) \setminus (Y \setminus A) = A \setminus \Gamma(A) \in \Im, \text{ for all subset A of } Y \\ &\Leftrightarrow \tau \sim_\Gamma \Im \text{ from Theorem 2.7} \end{aligned}$$

□

4. The Operator $\veebar_\Gamma$

This section propounds the operator $\veebar_\Gamma$ and analyzes its basic properties.

Definition 4.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\veebar_\Gamma : P(Y) \to P(Y)$ is defined by $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A)$, for all $A \subseteq Y$.

Theorem 4.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$. Then, the following are held.

i. $\veebar_\Gamma(C) = Y \setminus \Gamma(Y)$

ii. $\veebar_\Gamma(C) \in \tau$

iii. $\veebar_\Gamma(C) = \Psi_\Gamma(C) \setminus \Gamma(C)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$.

i. By Theorem 2.1 (*iv.*), $\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C \cap (Y \setminus C)) = \Psi_\Gamma(\emptyset) = Y \setminus \Gamma(Y)$.

ii. By Theorem 2.1 (*v.*), $Y \setminus \Gamma(Y)$ is in τ. As a result, from (*i.*), $\veebar_\Gamma(C) \in \tau$.

iii. $\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C) \cap (Y \setminus \Gamma(C)) = \Psi_\Gamma(C) \setminus \Gamma(C)$

□

Proposition 4.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $D \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\veebar_\Gamma(D) = Y \setminus \Gamma(Y) = Y \setminus \text{cl}_\theta(Y) = \emptyset$.

ii. If $\Im = P(Y)$, then $\veebar_\Gamma(D) = Y \setminus \Gamma(Y) = Y \setminus \emptyset = Y$.

The proof is obvious by Theorem 4.2 (*i.*).

Corollary 4.4. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\Psi_\Gamma(A) \subseteq \Gamma(A)$ if and only if $\veebar_\Gamma(A) = \emptyset$.

The proof is obvious by Theorem 4.2 (*iii.*).

Corollary 4.5. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\text{cl}(\tau) \cap \Im = \{\emptyset\}$

ii. $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$

iii. $\veebar_\Gamma(A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$, for all $A \subseteq Y$

iv. $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

(*i.*) $\Rightarrow$ (*ii.*) Let $A \subseteq Y$ and $\text{cl}(\tau) \cap \Im = \{\emptyset\}$. By Theorem 2.8, $\Psi_\Gamma(A) \subseteq \Gamma(A)$. From Corollary 4.4, $\veebar_\Gamma(A) = \emptyset$.

(*ii.*) $\Rightarrow$ (*i.*) Let $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$. Then, by Theorem 4.2 (*i.*), $\veebar_\Gamma(Y) = Y \setminus \Gamma(Y) = \emptyset$. It implies that $Y = \Gamma(Y)$ and thus $\text{cl}(\tau) \cap \Im = \{\emptyset\}$ from Theorem 2.6.

(i.) $\Rightarrow$ *(iii.)* Let $\text{cl}(\tau) \cap \Im = \{\emptyset\}$ and $A \subseteq Y$. Then, $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$ from Theorem 2.8.

(iii.) $\Rightarrow$ *(i.)* Let $\veebar_\Gamma(A) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$, for all subset A of Y. We know that $\veebar_\Gamma(A) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = (Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A))$ and by the hypothesis $(Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A)) \subseteq \Gamma(A) \cap \Gamma(Y \setminus A)$. Therefore, $(Y \setminus \Gamma(Y \setminus A)) \cap (Y \setminus \Gamma(A))$ must be an empty set, namely $\veebar_\Gamma(A) = \emptyset$. From the equivalence *(i.)* $\Leftrightarrow$ *(ii.)*, $\text{cl}(\tau) \cap \Im = \{\emptyset\}$

(i.) $\Rightarrow$ *(iv.)* It is obvious by Theorem 2.8.

(iv.) $\Rightarrow$ *(ii.)* Let $\Psi_\Gamma(A) \subseteq \Gamma(A)$, for all $A \subseteq Y$. Then, $\veebar_\Gamma(A) = \emptyset$, for all $A \subseteq Y$, from Corollary 4.4. □

Theorem 4.6. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. $\veebar_\Gamma(A) = Y \Leftrightarrow$ there exists a $U \in \tau(x)$ such that $\text{cl}(U) \in \Im$, for all $x \in Y$.

ii. If there exists a nonempty set A such that $\veebar_\Gamma(A) = A$, then $\text{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i.

$$\begin{aligned}\veebar_\Gamma(A) = Y &\Leftrightarrow Y \setminus \Gamma(Y) = Y \text{ from Theorem 4.2 } (i.)\\ &\Leftrightarrow \Gamma(Y) = \emptyset\\ &\Leftrightarrow x \notin \Gamma(Y), \text{ for all } x \in Y\\ &\Leftrightarrow \text{there exists a } U \in \tau(x) \text{ such that } \text{cl}(U) \cap Y = \text{cl}(U) \in \Im, \text{ for all } x \in Y\end{aligned}$$

ii. Let A be a nonempty set such that $\veebar_\Gamma(A) = A$. Then, from Theorem 4.2 *(i.)*, $A = Y \setminus \Gamma(Y)$ and thus $Y \setminus \Gamma(Y) \neq \emptyset$. It implies that $\Gamma(Y) \neq Y$. By Theorem 2.6, $\text{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

□

Theorem 4.7. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$. Then, the following are held.

i. If C is $\Im_\theta$-open, then $C \setminus \Gamma(C) \subseteq \veebar_\Gamma(C)$.

ii. If C is θ-open, then $C \setminus \Gamma(C) \subseteq \veebar_\Gamma(C)$.

iii. $\veebar_\Gamma(C) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$

iv. $\veebar_\Gamma(C) \cap C = \text{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \text{Int}_{\Im_\theta}(C) \setminus \Gamma(C)$

v. $\veebar_\Gamma(C) \setminus C = \text{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C) = \Psi_\Gamma(C) \setminus \text{cl}_{\Im_\theta}(C)$

vi. If $C \in \Im$, then $\veebar_\Gamma(C) = \Psi_\Gamma(C)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $C \subseteq Y$.

i. Let C be $\Im_\theta$-open. Then, $C \subseteq \Psi_\Gamma(C)$ from Proposition 2.5 *(ii.)*. Thus,

$$C \cap (Y \setminus \Gamma(C)) \subseteq \Psi_\Gamma(C) \cap (Y \setminus \Gamma(C))$$

namely

$$C \setminus \Gamma(C) \subseteq \Psi_\Gamma(C) \cap (Y \setminus \Gamma(Y \setminus (Y \setminus C))) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \veebar_\Gamma(C)$$

ii. Let C be θ-open. Then, it is σ-open as $\tau_\theta \subseteq \sigma$. By Proposition 2.5 *(iii.)* and from *(i.)* in this theorem, the proof is obvious.

iii. By Theorem 2.1 (*vi.*), as $\emptyset \subseteq \veebar_\Gamma(C)$, $\Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$. Then, by Theorem 2.1 (*iv.*),

$$\veebar_\Gamma(C) = \Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C) = \Psi_\Gamma(C \cap (Y \setminus C)) = \Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\veebar_\Gamma(C))$$

iv.

$$\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \mathrm{Int}_{\Im_\theta}(C) \cap (Y \setminus \Gamma(C)) = \mathrm{Int}_{\Im_\theta}(C) \setminus \Gamma(C)$$

Moreover,

$$\begin{aligned}
\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) &= (Y \setminus \mathrm{cl}_{\Im_\theta}(Y \setminus C)) \cap \Psi_\Gamma(Y \setminus C) \\
&= (Y \setminus [\Gamma(Y \setminus C) \cup (Y \setminus C)]) \cap \Psi_\Gamma(Y \setminus C) \\
&= [(Y \setminus \Gamma(Y \setminus C)) \cap C] \cap \Psi_\Gamma(Y \setminus C) \\
&= (\Psi_\Gamma(C) \cap C) \cap \Psi_\Gamma(Y \setminus C) \\
&= (\Psi_\Gamma(C) \cap \Psi_\Gamma(Y \setminus C)) \cap C \\
&= \veebar_\Gamma(C) \cap C
\end{aligned}$$

Consequently, $\mathrm{Int}_{\Im_\theta}(C) \cap \Psi_\Gamma(Y \setminus C) = \mathrm{Int}_{\Im_\theta}(C) \setminus \Gamma(C) = \veebar_\Gamma(C) \cap C$.

v. From (*iv.*),

$$\veebar_\Gamma(Y \setminus C) \cap (Y \setminus C) = \mathrm{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C)$$

From Theorem 4.2 (*i.*),

$$\veebar_\Gamma(Y \setminus C) \cap (Y \setminus C) = \veebar_\Gamma(C) \cap (Y \setminus C) = \veebar_\Gamma(C) \setminus C$$

and thus

$$\veebar_\Gamma(C) \setminus C = \mathrm{Int}_{\Im_\theta}(Y \setminus C) \cap \Psi_\Gamma(C) = (Y \setminus \mathrm{cl}_{\Im_\theta}(C)) \cap \Psi_\Gamma(C) = \Psi_\Gamma(C) \setminus \mathrm{cl}_{\Im_\theta}(C)$$

vi. If $C \in \Im$, then $\Gamma(C) = \emptyset$ by Theorem 2.1 (*ii.*). From Theorem 4.2 (*iii.*),

$$\veebar_\Gamma(C) = \Psi_\Gamma(C) \setminus \Gamma(C) = \Psi_\Gamma(C) \setminus \emptyset = \Psi_\Gamma(C)$$

□

Theorem 4.8. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then, $\veebar_\Gamma(K) = \Psi_\Gamma(Y \setminus K)$ if and only if $Y \setminus \Gamma(K) \subseteq \Psi_\Gamma(K)$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Then,

$$\begin{aligned}
\veebar_\Gamma(K) = \Psi_\Gamma(Y \setminus K) &\Leftrightarrow \Psi_\Gamma(K) \cap \Psi_\Gamma(Y \setminus K) = \Psi_\Gamma(Y \setminus K) \\
&\Leftrightarrow Y \setminus \Gamma(K) = \Psi_\Gamma(Y \setminus K) \subseteq \Psi_\Gamma(K)
\end{aligned}$$

□

Theorem 4.9. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If A is $\Im_\Gamma$-perfect, then $\veebar_\Gamma(A) = \wedge_\Gamma(A)$.

Proof. Let $A \subseteq Y$ in $(Y, \tau, \Im)$. If A is $\Im_\Gamma$-perfect, then $\Gamma(A) = A$. Hence, $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus A = \Psi_\Gamma(A) \setminus \Gamma(A)$. Consequently, from Theorem 4.2 (*iii.*), $\wedge_\Gamma(A) = \Psi_\Gamma(A) \setminus \Gamma(A) = \veebar_\Gamma(A)$. □

Remark 4.10. The reverse of Theorem 4.9 may not be true in general.

Example 4.11. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = \{p\}$, then $\wedge_\Gamma(A) = \emptyset = \veebar_\Gamma(A)$ but A is not $\Im_\Gamma$-perfect.

Theorem 4.12. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If K is Γ-dense-in-itself, then $\veebar_\Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$.

ii. If K is $\Im_\Gamma$-dense, then $\veebar_\Gamma(K) = \emptyset$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$.

i. If K is Γ-dense-in-itself, then $K \subseteq \Gamma(K)$. Therefore, by Theorem 4.2 (*iii.*), $\veebar_\Gamma(K) = \Psi_\Gamma(K) \setminus \Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$.

ii. If K is $\Im_\Gamma$-dense, then $\Gamma(K) = Y$. Thus, by Theorem 4.2 (*iii.*), $\veebar_\Gamma(K) = \Psi_\Gamma(K) \setminus \Gamma(K) = \Psi_\Gamma(K) \setminus Y = \emptyset$.

□

Remark 4.13. The reverse of the above requirements may not be true in general.

Example 4.14. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{r\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $K = \{r\}$, then $\veebar_\Gamma(K) = \emptyset = \Gamma(K)$. Although $\emptyset = \veebar_\Gamma(K) \subseteq \Psi_\Gamma(K) \setminus K$, K is neither Γ-dense-in-itself nor $\Im_\Gamma$-dense.

Corollary 4.15. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if there is an $\Im_\Gamma$-dense set $A \subseteq Y$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space.

$(\Rightarrow)$: Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. From Theorem 2.6, $\Gamma(Y) = Y$. Consequently, Y is $\Im_\Gamma$-dense.

$(\Leftarrow)$: Let there be an $\Im_\Gamma$-dense set $A \subseteq Y$. Hence, $\veebar_\Gamma(A) = \emptyset$ by Theorem 4.12 (*ii.*). It is known that $\veebar_\Gamma(A) = Y \setminus \Gamma(Y)$ by Theorem 4.2 (*i.*). Thereby, $Y \setminus \Gamma(Y) = \emptyset$ and thus $Y = \Gamma(Y)$. Consequently, by Theorem 2.6, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. □

5. The Operator $\barwedge_\Gamma$

This section proposes the operator $\barwedge_\Gamma$ and researches its basic properties.

Definition 5.1. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the operator $\barwedge_\Gamma : P(Y) \to P(Y)$ is defined by $\barwedge_\Gamma(A) = A \setminus \Gamma(A)$, for all $A \subseteq Y$.

Theorem 5.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$. Then, the following are held.

i. $\wedge_\Gamma(Y \setminus F) = \barwedge_\Gamma(F)$

ii. $\wedge_\Gamma(F) \cap \barwedge_\Gamma(F) = \emptyset$

iii. $\wedge_\Gamma(F) \cap \wedge_\Gamma(Y \setminus F) = \emptyset$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$.

i. $\wedge_\Gamma(Y \setminus F) = \Psi_\Gamma(Y \setminus F) \setminus (Y \setminus F) = \Psi_\Gamma(Y \setminus F) \cap F = (Y \setminus \Gamma(F)) \cap F = F \setminus \Gamma(F) = \barwedge_\Gamma(F)$

ii. $\wedge_\Gamma(F) \cap \barwedge_\Gamma(F) = (\Psi_\Gamma(F) \cap (Y \setminus F)) \cap (F \cap (Y \setminus \Gamma(F)) = (F \cap (Y \setminus F)) \cap (\Psi_\Gamma(F) \cap (Y \setminus \Gamma(F))) = \emptyset$

iii. It is obvious from Theorem 5.2 (*i.*) and (*ii.*).

□

Proposition 5.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. If $\Im = \{\emptyset\}$, then $\barwedge_\Gamma(A) = \emptyset$.

ii. If $\Im = P(Y)$, then $\barwedge_\Gamma(A) = A$.

Theorem 5.4. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. If an element x of Y is in $\overline{\curlywedge}_\Gamma(K)$, then $\{x\} \in \Im$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K \subseteq Y$. Suppose that an element x of Y is in $\overline{\curlywedge}_\Gamma(K)$, i.e., $x \in K \setminus \Gamma(K)$. Then, $x \in K$ but $x \notin \Gamma(K)$. Therefore, there exists a $G \in \tau(x)$ such that $\mathrm{cl}(G) \cap K \in \Im$. It implies that $x \in \mathrm{cl}(G) \cap K \in \Im$. Hence, $\{x\} \in \Im$ by the heredity of the ideal. □

Theorem 5.5. Let $(Y, \tau, \Im)$ be an ideal topological space and $x \in Y$. Then, $x \in \overline{\curlywedge}_\Gamma(\{x\})$ if and only if $\{x\} \in \Im$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $x \in Y$.

$(\Rightarrow)$: It is obvious by Theorem 5.4.

$(\Leftarrow)$: It is known that $Y \in \tau(x)$. Let $\{x\} \in \Im$. Then, $\mathrm{cl}(Y) \cap \{x\} = \{x\} \in \Im$, for $Y \in \tau(x)$, and thus $x \notin \Gamma(\{x\})$. Consequently, $x \in \{x\} \setminus \Gamma(\{x\}) = \overline{\curlywedge}_\Gamma(\{x\})$. □

Remark 5.6. In an ideal topological space $(Y, \tau, \Im)$, it is obvious that from Example 3.4 and Theorem 5.2 (*i.*), if $M \subseteq N \subseteq Y$, then neither $\overline{\curlywedge}_\Gamma(M) \subseteq \overline{\curlywedge}_\Gamma(N)$ nor $\overline{\curlywedge}_\Gamma(N) \subseteq \overline{\curlywedge}_\Gamma(M)$.

Theorem 5.7. Let $(Y, \tau, \Im)$ be an ideal topological space.

i. $\overline{\curlywedge}_\Gamma(G) \in \tau$, for all $G \in \tau$ (or $G \in \tau_\theta$)

ii. $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\} \Leftrightarrow \overline{\curlywedge}_\Gamma(Y) = \emptyset$

The proofs are obvious by Theorem 2.1 (*v.*) and Theorem 2.6, respectively.

Theorem 5.8. Let $(Y, \tau, \Im)$ be an ideal topological space and $K, L \subseteq Y$. Then, the following are held.

i. $\overline{\curlywedge}_\Gamma(\emptyset) = \emptyset$

ii. If K is in $\Im$, then $\overline{\curlywedge}_\Gamma(K) = K$.

iii. $\overline{\curlywedge}_\Gamma(\overline{\curlywedge}_\Gamma(K)) \subseteq \overline{\curlywedge}_\Gamma(K)$

iv. $\overline{\curlywedge}_\Gamma(K) \cap \Gamma(K) = \emptyset$

v. $\overline{\curlywedge}_\Gamma(K \cup L) = (\overline{\curlywedge}_\Gamma(K) \setminus \Gamma(L)) \cup (\overline{\curlywedge}_\Gamma(L) \setminus \Gamma(K))$

vi. $\overline{\curlywedge}_\Gamma(\overline{\curlywedge}_\Gamma(K)) \subseteq K$

vii. $\overline{\curlywedge}_\Gamma(K) \cap \overline{\curlywedge}_\Gamma(L) = (K \cap L) \setminus \Gamma(K \cup L)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $K, L \subseteq Y$.

i. $\overline{\curlywedge}_\Gamma(\emptyset) = \emptyset \setminus \Gamma(\emptyset) = \emptyset$

ii. Let $K \in \Im$. Then, from Theorem 2.1 (*ii.*), $\overline{\curlywedge}_\Gamma(K) = K \setminus \Gamma(K) = K \setminus \emptyset = K$.

iii. $\overline{\curlywedge}_\Gamma(\overline{\curlywedge}_\Gamma(K)) = \overline{\curlywedge}_\Gamma(K) \setminus \Gamma(\overline{\curlywedge}_\Gamma(K)) \subseteq \overline{\curlywedge}_\Gamma(K)$

iv. $\overline{\curlywedge}_\Gamma(K) \cap \Gamma(K) = (K \setminus \Gamma(K)) \cap \Gamma(K) = \emptyset$

v. From Theorem 2.1 (*iii.*),

$$\begin{aligned}
\overline{\curlywedge}_\Gamma(K \cup L) &= (K \cup L) \setminus \Gamma(K \cup L) \\
&= (K \cup L) \setminus (\Gamma(K) \cup \Gamma(L)) \\
&= (K \cup L) \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L)) \\
&= [K \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L))] \cup [L \cap (Y \setminus \Gamma(K)) \cap (Y \setminus \Gamma(L))] \\
&= [\overline{\curlywedge}_\Gamma(K) \cap (Y \setminus \Gamma(L))] \cup [\overline{\curlywedge}_\Gamma(L) \cap (Y \setminus \Gamma(K))] \\
&= (\overline{\curlywedge}_\Gamma(K) \setminus \Gamma(L)) \cup (\overline{\curlywedge}_\Gamma(L) \setminus \Gamma(K))
\end{aligned}$$

vi. The proof is obvious by (*iii.*) in this theorem.

vii. From Theorem 2.1 (*iii.*),

$$\begin{aligned}\overline{\curlywedge}_\Gamma(K)\cap\overline{\curlywedge}_\Gamma(L) &= (K\setminus\Gamma(K))\cap(L\setminus\Gamma(L))\\ &= (K\cap L)\cap[(Y\setminus\Gamma(K))\cap(Y\setminus\Gamma(L))]\\ &= (K\cap L)\cap[Y\setminus(\Gamma(K)\cup\Gamma(L))]\\ &= (K\cap L)\cap(Y\setminus\Gamma(K\cup L))\\ &= (K\cap L)\setminus\Gamma(K\cup L)\end{aligned}$$

□

Theorem 5.9. Let $(Y,\tau,\Im)$ be an ideal topological space and $K\subseteq Y$. Then, the following are equivalent.

i. $\overline{\curlywedge}_\Gamma(K)=\emptyset$

ii. $\mathrm{cl}_{\Im_\theta}(K)=\Gamma(K)$

iii. K is Γ-dense-in-itself.

Proof. Let $(Y,\tau,\Im)$ be an ideal topological space and $K\subseteq Y$.

(*i.*) ⇔ (*ii.*) $\overline{\curlywedge}_\Gamma(K)=\emptyset\Leftrightarrow K\setminus\Gamma(K)=\emptyset\Leftrightarrow K\subseteq\Gamma(K)\Leftrightarrow\mathrm{cl}_{\Im_\theta}(K)=K\cup\Gamma(K)=\Gamma(K)$.

(*i.*) ⇔ (*iii.*): It is obvious from Theorem 3.12 and Theorem 5.2 (*i.*). □

Theorem 5.10. Let $(Y,\tau,\Im)$ be an ideal topological space and $A\subseteq Y$. Then, the following are held.

i. If A is $\Im_\Gamma$-perfect, then $\overline{\curlywedge}_\Gamma(A)=\emptyset$.

ii. If A is $\Im_\Gamma$-dense, then $\overline{\curlywedge}_\Gamma(A)=\emptyset$.

Proof. Let $(Y,\tau,\Im)$ be an ideal topological space and $A\subseteq Y$.

i. It is obvious from Theorem 3.16 (*ii.*) and Theorem 5.2 (*i.*).

ii. It is obvious from Corollary 3.13 and Theorem 5.2 (*i.*).

□

Theorem 5.11. Let $(Y,\tau,\Im)$ be an ideal topological space and $A\subseteq Y$. Then, A is L_Γ-perfect ⇔ $\overline{\curlywedge}_\Gamma(A)\in\Im$.

The proof is obvious from Theorem 5.2 (*i.*) and Theorem 3.19.

Corollary 5.12. Let $(Y,\tau,\Im)$ be an ideal topological space and $A\subseteq Y$. If A is C_Γ-perfect, then $\overline{\curlywedge}_\Gamma(A)\in\Im$.

The proof is obvious by Theorem 5.11.

Remark 5.13. It is obvious that the reverse of the above requirement may not be true from Example 3.22 and Theorem 5.2 (*i.*).

Theorem 5.14. Let $(Y,\tau,\Im)$ be an ideal topological space and $K\subseteq Y$. Then, $\overline{\curlywedge}_\Gamma(K)=K$ if and only if $K\cap\Gamma(K)=\emptyset$.

The proof is obvious from Theorem 3.23 and Theorem 5.2 (*i.*).

Theorem 5.15. Let $(Y, \tau, \Im)$ be an ideal topological space. Then, the following are equivalent.

i. $\tau \sim_\Gamma \Im$

ii. For all subset A of Y, $\barwedge_\Gamma(A) \in \Im$

The proof is obvious from Theorem 2.7.

6. Various Relations

This section investigates various relations between the operators defined herein.

Theorem 6.1. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$. Then, the following are held.

i. $\veebar_\Gamma(F) \cap \wedge_\Gamma(F) = \veebar_\Gamma(F) \setminus F$

ii. $\veebar_\Gamma(F) \cap \barwedge_\Gamma(F) = \Psi_\Gamma(F) \cap \barwedge_\Gamma(F)$

iii. $\veebar_\Gamma(F) \setminus \wedge_\Gamma(F) = \veebar_\Gamma(F) \cap F$

iv. $\veebar_\Gamma(F) \setminus \barwedge_\Gamma(F) = \veebar_\Gamma(F) \setminus F$

v. $\barwedge_\Gamma(F) \setminus \veebar_\Gamma(F) = \barwedge_\Gamma(F) \setminus \Psi_\Gamma(F)$

vi. $\wedge_\Gamma(F) \cup \veebar_\Gamma(F) = \Psi_\Gamma(F) \setminus (F \cap \Gamma(F))$

vii. $\veebar_\Gamma(F) \cup \barwedge_\Gamma(F) = (\Psi_\Gamma(F) \cup F) \setminus \Gamma(F)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $F \subseteq Y$.

i.

$$\begin{aligned}
\veebar_\Gamma(F) \cap \wedge_\Gamma(F) &= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (\Psi_\Gamma(F) \cap (Y \setminus F)) \\
&= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (Y \setminus F) \\
&= \veebar_\Gamma(F) \setminus F
\end{aligned}$$

ii. By Theorem 4.2 (*iii.*),

$$\begin{aligned}
\veebar_\Gamma(F) \cap \barwedge_\Gamma(F) &= (\Psi_\Gamma(F) \cap (Y \setminus \Gamma(F))) \cap (F \cap (Y \setminus \Gamma(F))) \\
&= (Y \setminus \Gamma(F)) \cap (F \cap \Psi_\Gamma(F)) \\
&= \Psi_\Gamma(F) \cap (F \cap (Y \setminus \Gamma(F))) \\
&= \Psi_\Gamma(F) \cap \barwedge_\Gamma(F)
\end{aligned}$$

iii.

$$\begin{aligned}
\veebar_\Gamma(F) \setminus \wedge_\Gamma(F) &= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap [Y \setminus (\Psi_\Gamma(F) \cap (Y \setminus F))] \\
&= (\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap [(Y \setminus \Psi_\Gamma(F)) \cup F] \\
&= [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap (Y \setminus \Psi_\Gamma(F))] \cup [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap F] \\
&= (\Psi_\Gamma(Y \setminus F) \cap [\Psi_\Gamma(F) \cap (Y \setminus \Psi_\Gamma(F))]) \cup [(\Psi_\Gamma(F) \cap \Psi_\Gamma(Y \setminus F)) \cap F] \\
&= (\Psi_\Gamma(Y \setminus F) \cap \emptyset) \cup (\veebar_\Gamma(F) \cap F) \\
&= \veebar_\Gamma(F) \cap F
\end{aligned}$$

iv.

$$\begin{aligned}
\underline{\vee}_\Gamma(F)\setminus\overline{\wedge}_\Gamma(F) &= \underline{\vee}_\Gamma(F)\cap(Y\setminus\overline{\wedge}_\Gamma(F))\\
&= \underline{\vee}_\Gamma(F)\cap[Y\setminus(F\setminus\Gamma(F))]\\
&= \underline{\vee}_\Gamma(F)\cap(Y\setminus[F\cap(Y\setminus\Gamma(F))])\\
&= \underline{\vee}_\Gamma(F)\cap[(Y\setminus F)\cup\Gamma(F)]\\
&= [\underline{\vee}_\Gamma(F)\cap(Y\setminus F)]\cup(\underline{\vee}_\Gamma(F)\cap\Gamma(F))\\
&= [\underline{\vee}_\Gamma(F)\cap(Y\setminus F)]\cup[(\Psi_\Gamma(F)\cap\Psi_\Gamma(Y\setminus F))\cap\Gamma(F)]\\
&= [\underline{\vee}_\Gamma(F)\cap(Y\setminus F)]\cup[\Psi_\Gamma(F)\cap(\Psi_\Gamma(Y\setminus F)\cap\Gamma(F))]\\
&= [\underline{\vee}_\Gamma(F)\cap(Y\setminus F)]\cup(\Psi_\Gamma(F)\cap[(Y\setminus\Gamma(F))\cap\Gamma(F)])\\
&= [\underline{\vee}_\Gamma(F)\cap(Y\setminus F)]\cup(\Psi_\Gamma(F)\cap\emptyset)\\
&= \underline{\vee}_\Gamma(F)\cap(Y\setminus F)\\
&= \underline{\vee}_\Gamma(F)\setminus F
\end{aligned}$$

v.

$$\begin{aligned}
\overline{\wedge}_\Gamma(F)\setminus\underline{\vee}_\Gamma(F) &= \overline{\wedge}_\Gamma(F)\cap(Y\setminus\underline{\vee}_\Gamma(F))\\
&= \overline{\wedge}_\Gamma(F)\cap[Y\setminus(\Psi_\Gamma(F)\cap\Psi_\Gamma(Y\setminus F))]\\
&= \overline{\wedge}_\Gamma(F)\cap[(Y\setminus\Psi_\Gamma(F))\cup(Y\setminus\Psi_\Gamma(Y\setminus F))]\\
&= [\overline{\wedge}_\Gamma(F)\cap(Y\setminus\Psi_\Gamma(F))]\cup[\overline{\wedge}_\Gamma(F)\cap(Y\setminus\Psi_\Gamma(Y\setminus F))]\\
&= [(\overline{\wedge}_\Gamma(F)\setminus\Psi_\Gamma(F))\cup([F\cap(Y\setminus\Gamma(F))]\cap(Y\setminus\Psi_\Gamma(Y\setminus F)))\\
&= (\overline{\wedge}_\Gamma(F)\setminus\Psi_\Gamma(F))\cup[(F\cap\Psi_\Gamma(Y\setminus F))\cap(Y\setminus\Psi_\Gamma(Y\setminus F))]\\
&= (\overline{\wedge}_\Gamma(F)\setminus\Psi_\Gamma(F))\cup(F\cap[\Psi_\Gamma(Y\setminus F)\cap(Y\setminus\Psi_\Gamma(Y\setminus F))])\\
&= (\overline{\wedge}_\Gamma(F)\setminus\Psi_\Gamma(F))\cup(F\cap\emptyset)\\
&= \overline{\wedge}_\Gamma(F)\setminus\Psi_\Gamma(F)
\end{aligned}$$

vi.

$$\begin{aligned}
\wedge_\Gamma(F)\cup\underline{\vee}_\Gamma(F) &= (\Psi_\Gamma(F)\setminus F)\cup(\Psi_\Gamma(F)\cap\Psi_\Gamma(Y\setminus F))\\
&= [\Psi_\Gamma(F)\cap(Y\setminus F)]\cup(\Psi_\Gamma(F)\cap\Psi_\Gamma(Y\setminus F))\\
&= \Psi_\Gamma(F)\cap[(Y\setminus F)\cup\Psi_\Gamma(Y\setminus F)]\\
&= \Psi_\Gamma(F)\cap[(Y\setminus F)\cup(Y\setminus\Gamma(F))]\\
&= \Psi_\Gamma(F)\cap[Y\setminus(F\cap\Gamma(F))]\\
&= \Psi_\Gamma(F)\setminus(F\cap\Gamma(F))
\end{aligned}$$

vii. By Theorem 4.2 (*iii.*),

$$\begin{aligned}
\underline{\vee}_\Gamma(F)\cup\overline{\wedge}_\Gamma(F) &= (\Psi_\Gamma(F)\setminus\Gamma(F))\cup(F\setminus\Gamma(F))\\
&= [\Psi_\Gamma(F)\cap(Y\setminus\Gamma(F))]\cup[F\cap(Y\setminus\Gamma(F))]\\
&= (\Psi_\Gamma(F)\cup F)\cap(Y\setminus\Gamma(F))\\
&= (\Psi_\Gamma(F)\cup F)\setminus\Gamma(F)
\end{aligned}$$

□

Theorem 6.2. Let $(Y, \tau, \Im)$ be an ideal topological space and $H \subseteq Y$. Then, the following are held.

i. $\barwedge_\Gamma(\veebar_\Gamma(H)) = Y \setminus \Gamma(Y)$

ii. $\wedge_\Gamma(\veebar_\Gamma(H)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$

iii. $\wedge_\Gamma(\veebar_\Gamma(H)) = \emptyset$ if $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$

iv. $\barwedge_\Gamma(\wedge_\Gamma(H)) = \wedge_\Gamma(H)$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $H \subseteq Y$.

i. By Theorem 4.2 (*i.*),

$$\begin{aligned}\barwedge_\Gamma(\veebar_\Gamma(H)) &= \barwedge_\Gamma(Y \setminus \Gamma(Y)) \\ &= (Y \setminus \Gamma(Y)) \setminus \Gamma(Y \setminus \Gamma(Y)) \\ &= (Y \setminus \Gamma(Y)) \cap (Y \setminus \Gamma(Y \setminus \Gamma(Y))) \\ &= Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y)))\end{aligned}$$

From Theorem 2.1 (*iii.*),

$$Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y))) = Y \setminus \Gamma(Y \cup (Y \setminus \Gamma(Y)) = Y \setminus \Gamma(Y)$$

As a result, $\barwedge_\Gamma(\veebar_\Gamma(H)) = Y \setminus \Gamma(Y)$.

ii. By Theorem 4.2 (*i.*), $\wedge_\Gamma(\veebar_\Gamma(H)) = \wedge_\Gamma(Y \setminus \Gamma(Y))$. Then, by Theorem 3.5 (*iv.*),

$$\wedge_\Gamma(\veebar_\Gamma(H)) = \wedge_\Gamma(Y \setminus \Gamma(Y)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$$

iii. Let $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, from Theorem 2.6 and Theorem 6.2 (*ii.*),

$$\wedge_\Gamma(\veebar_\Gamma(H)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) = \Gamma(Y) \setminus \Gamma(Y) = \emptyset$$

iv. By Theorem 2.1 (*iii.*),

$$\begin{aligned}\barwedge_\Gamma(\wedge_\Gamma(H)) &= \wedge_\Gamma(H) \setminus \Gamma(\wedge_\Gamma(H)) \\ &= (\Psi_\Gamma(H) \setminus H) \setminus \Gamma(\Psi_\Gamma(H) \setminus H) \\ &= [(Y \setminus \Gamma(Y \setminus H)) \cap (Y \setminus H)] \cap (Y \setminus \Gamma(\Psi_\Gamma(H) \setminus H)) \\ &= (Y \setminus H) \cap [(Y \setminus \Gamma(Y \setminus H)) \cap (Y \setminus \Gamma(\Psi_\Gamma(H) \setminus H))] \\ &= (Y \setminus H) \cap [Y \setminus (\Gamma(Y \setminus H) \cup \Gamma(\Psi_\Gamma(H) \setminus H))] \\ &= (Y \setminus H) \cap (Y \setminus \Gamma((Y \setminus H) \cup (\Psi_\Gamma(H) \setminus H)))\end{aligned}$$

Then,

$$\begin{aligned}\barwedge_\Gamma(\wedge_\Gamma(H)) &= (Y \setminus H) \cap (Y \setminus \Gamma((Y \setminus H) \cup (\Psi_\Gamma(H) \cap (Y \setminus H)))) \\ &= (Y \setminus H) \cap (Y \setminus \Gamma(Y \setminus H)) \\ &= (Y \setminus H) \cap \Psi_\Gamma(H) \\ &= \Psi_\Gamma(H) \setminus H \\ &= \wedge_\Gamma(H)\end{aligned}$$

□

Remark 6.3. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Although $\wedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$, $\mathrm{cl}(\tau) \cap \Im$ may not be equal to $\{\emptyset\}$.

Example 6.4. Let $Y = \{p, q, r, s\}$, $\Im = \{\emptyset, \{q\}, \{s\}, \{q, s\}\}$, and $\tau = \{\emptyset, \{s\}, \{p, r\}, \{p, r, s\}, Y\}$. In the ideal topological space $(Y, \tau, \Im)$, if $A = Y$, then $\wedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$ but $\text{cl}(\tau) \cap \Im \neq \{\emptyset\}$.

Theorem 6.5. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\barwedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$ if and only if $\text{cl}(\tau) \cap \Im = \{\emptyset\}$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

$(\Rightarrow)$: Let $\barwedge_\Gamma(\veebar_\Gamma(A)) = \emptyset$. Then,

$$\veebar_\Gamma(A) \setminus \Gamma(\veebar_\Gamma(A)) = (Y \setminus \Gamma(Y)) \setminus \Gamma(Y \setminus \Gamma(Y)) = \emptyset$$

from Theorem 4.2 (*i.*). Therefore,

$$\veebar_\Gamma(A) \setminus \Gamma(\veebar_\Gamma(A)) = Y \setminus (\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y))) = \emptyset$$

and thus $\Gamma(Y) \cup \Gamma(Y \setminus \Gamma(Y)) = Y$. From Theorem 2.1 (*iii.*), $\Gamma(Y \cup (Y \setminus \Gamma(Y))) = \Gamma(Y) = Y$. By Theorem 2.6, $\text{cl}(\tau) \cap \Im = \{\emptyset\}$.

$(\Leftarrow)$: The proof is obvious by Theorem 6.2 (*i.*) and Theorem 2.6. □

Theorem 6.6. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\veebar_\Gamma(A) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$.

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. It is obvious that $\Psi_\Gamma(\emptyset) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$ by Theorem 2.1 (*vi.*). Since from Theorem 2.1 (*iv.*),

$$\Psi_\Gamma(\emptyset) = \Psi_\Gamma(A \cap (Y \setminus A)) = \Psi_\Gamma(A) \cap \Psi_\Gamma(Y \setminus A) = \veebar_\Gamma(A)$$

and thus $\veebar_\Gamma(A) \subseteq \Psi_\Gamma(\wedge_\Gamma(A))$. □

Theorem 6.7. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, the following are held.

i. $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y)) = \veebar_\Gamma(A) = Y \setminus \Gamma(Y)$

ii. $\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$.

i. $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y))$ by Theorem 4.2 (*i.*). Moreover, $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = Y \setminus \Gamma(Y)$ by Theorem 6.2 (*i.*). As a result, from Theorem 4.2 (*i.*),

$$\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A))) = \barwedge_\Gamma(Y \setminus \Gamma(Y)) = \veebar_\Gamma(A) = Y \setminus \Gamma(Y)$$

ii. By Theorem 6.2 (*iv.*), $\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(A))$. Moreover, from Theorem 6.2 (*ii.*), $\wedge_\Gamma(\veebar_\Gamma(A)) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$. Thus,

$$\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.1}$$

and

$$\wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.2}$$

from Theorem 6.2 (*ii.*). From Theorem 6.2 (*i.*),

$$\wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A)) = \wedge_\Gamma(Y \setminus \Gamma(Y)) = \Psi_\Gamma(Y \setminus \Gamma(Y)) \setminus (Y \setminus \Gamma(Y)) = (Y \setminus \Gamma(\Gamma(Y))) \cap \Gamma(Y) = \Gamma(Y) \setminus \Gamma(\Gamma(Y)) \tag{6.3}$$

Consequently, from (6.1)-(6.3),

$$\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A))) = \wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A))) = \wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A))) = \Gamma(Y) \setminus \Gamma(\Gamma(Y))$$

□

Corollary 6.8. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. Then, $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$ if and only if each of the following is empty.

i. $\veebar_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A)))$

ii. $\veebar_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A)))$

iii. $\veebar_\Gamma(\barwedge_\Gamma(\wedge_\Gamma(A)))$

iv. $\veebar_\Gamma(\wedge_\Gamma(\barwedge_\Gamma(A)))$

v. $\barwedge_\Gamma(\veebar_\Gamma(\wedge_\Gamma(A)))$

vi. $\barwedge_\Gamma(Y \setminus \Gamma(Y))$

vii. $\veebar_\Gamma(A)$

The proof is obvious by Theorem 6.7 and Corollary 4.5.

Corollary 6.9. Let $(Y, \tau, \Im)$ be an ideal topological space and $A \subseteq Y$. If $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$, then each of the following is empty.

i. $\barwedge_\Gamma(\wedge_\Gamma(\veebar_\Gamma(A)))$

ii. $\wedge_\Gamma(\veebar_\Gamma(\barwedge_\Gamma(A)))$

iii. $\wedge_\Gamma(\barwedge_\Gamma(\veebar_\Gamma(A)))$

iv. $\Gamma(Y) \setminus \Gamma(\Gamma(Y))$

Proof. Let $(Y, \tau, \Im)$ be an ideal topological space, $A \subseteq Y$, and $\mathrm{cl}(\tau) \cap \Im = \{\emptyset\}$. Then, from Theorem 2.6, $Y = \Gamma(Y)$ and thus

$$\Gamma(Y) \setminus \Gamma(\Gamma(Y)) = Y \setminus \Gamma(Y) = \emptyset$$

Hence, the proof is obvious by Theorem 6.7 (*ii.*). □

7. Conclusion

In this study, new set operators were presented via Ψ_Γ-operator and Γ-local closure function in ideal topological spaces, and their behavioral properties were analyzed. It was investigated whether these set operators preserve some set operations. In future studies, different set operators can be presented, and their relations with these newly studied operators can be researched.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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