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Solutions of Proportional Learning and Forgetting Models by Proportional Laplace Transform on Time Scales

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Abstract: Knowledge is acquired as a result of some activities but is forgotten over time. Much work has been done on this subject in fields such as mathematics, engineering and psychology. There are many learning and forgetting models in literature. In this study, a learning and forgetting model considered in classical analysis is redefined with the help of proportional derivative on time scales. The cases where the learning and forgetting rates are constant and the *Araştırma Makalesi Araştırma Makalesi* learning function shows exponential and hyperbolic functions properties are analyzed. These models are solved with the help of the proportional Laplace transform. Finally, the models considered with the general solution method of first-order proportional dynamic equations were examined on the time scale.

Key words: Proportional derivative, proportional Laplace transform, time scale, learning and forgetting model.

Zaman Skalasında Oransal Öğrenme ve Unutma Modellerinin Oransal Laplace Dönüşümü ile Çözümleri

Öz: Bilgi, bazı faaliyetler sonucunda kazanılır ancak zamanla unutulur. Matematik, mühendislik ve psikoloji gibi alanlarda bu konu üzerine pek çok çalışma yapılmıştır. Literatürde pek çok öğrenme ve unutma modeli bulunmaktadır. Bu çalışmada, klasik analizde ele alınan bir öğrenme ve unutma modeli, zaman skalasında oransal türev yardımıyla yeniden tanımlanmıştır. Öğrenme ve unutma oranlarının sabit olduğu ve öğrenme fonksiyonunun üstel ve hiperbolik fonksiyon özellikleri gösterdiği durumlar analiz edilmiştir. Bu modeller oransal Laplace dönüşümü yardımıyla çözülmektedir. Son olarak birinci mertebeden oransal dinamik denklemlerin genel çözüm yöntemi ile ele alınan modeler zaman skalasında incelenmiştir.

Anahtar kelimeler: Oransal türev, oransal Laplace dönüşümü, zaman skalası, öğrenme ve unutma modeli.

1. Introduction

Fractional calculus began to be studied shortly after the emergence of classical calculus. Fractional calculus is deeply related to the dynamics of complex problems. Many mathematical models can be expressed and solved by fractional order differential equations [1]. The fundamental theory of fractional calculus was developed by Grünwald, Letnikov, Liouville and Riemann [2]. There are many types of fractional derivatives. Riemann-Liouville, Caputo, Liouville-Grünwald, Marchaud, Hilfer, Conformable, and Proportional are a few of them. Today, fractional calculus has been applied and solved in many mathematical fields such as engineering, biology, physics, psychology, mechanics and economics [3-8]. Time scale theory, which is one of the important fields of study, is preferred in many fields of study. One of these areas is fractional calculus. The idea of developing fractional calculus on time scales originated with Ph.D. thesis of Bastos [9-10]. Now let us give the necessary information about time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary, non-empty, closed subset of $\mathbb{R}$. It was introduced by Hilger to unify continuous and discrete problems into a single theory [11]. In the following years, many important results have been obtained on this theory. It's basic concepts can be found in [12] and [13]. For an arbitrary time scale $\mathbb{T}$, following information can be given [12]. $\sigma: \mathbb{T} \to \mathbb{T}$ is forward jump operator described by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\},$$

for $t \in \mathbb{T}$. And, $\mu: \mathbb{T} \to [0, \infty)$ is graininess function as

$$\mu(t) = \sigma(t) - t.$$

We also need a set $\mathbb{T}^\kappa$, which is reproduced from $\mathbb{T}$ as follows: $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, provided that $\mathbb{T}$ has a left-scattered maximum m. In other cases, $\mathbb{T}^\kappa = \mathbb{T}$. Suppose $f: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then, one can define $f^\Delta(t)$ to be a number (if it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

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$$\left|\left[f(\sigma(t)) - f(s)\right] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s|,$$

for $\forall s \in U$. $f^{\Delta}(t)$ is Δ- derivative of f at t. On the other hand, there exists a function F that is pre-differentiable with a region of differentiation D and f is regulated. Then, indefinite delta integral of f is

$$\int f(t)\Delta t = F(t) + C,$$

where f and F satisfy $F^{\Delta}(t) = f(t)$ for $\forall t \in D$ and $C \in \mathbb{R}$. By the same reasoning, Cauchy integral of f on $[r, s]$ is

$$\int_r^s f(t)\Delta t = F(s) - F(r),$$

for $r, s \in \mathbb{T}$. Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$ where C_{rd} is set of all rd-continuous functions on $\mathbb{T}$. Moreover,

I. If $[a, b]$ includes only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t), & if\ a < b \\ 0, & if\ a = b. \\ -\sum_{t\in[b,a)} \mu(t)f(t), & if\ a > b \end{cases}$$

II. If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t), & if\ a < b \\ 0, & if\ a = b\ . \\ -\sum_{t=b}^{a-1} f(t), & if\ a > b \end{cases}$$

Now let us mention about proportional derivative, which is a variant of fractional derivative preferred in this study. The advantage of the proportional derivative over other fractional derivatives is that proportional fractional integrals have the semigroup property and provide a generalization to derivatives and integrals in classical analysis [14]. Moreover, in the proportional derivative, the fact that D^0 satisfies the identity operator $(D^0 f(t) = f(t))$ and D^1 satisfies delta derivative operator $(D^1 f(t) = f^{\Delta}(t))$ on time scale $\mathbb{T}$ shows that it is conformable with the classical operator. Here, proportional derivative D^{α} is order $\alpha \in [0,1]$ [15-16]. The basic definitions and theorems for proportional derivative used in this study are given in the next section.

Many dynamic processes occurring in real life can be modelled by some ordinary or partial differential equations. As a result of taking the orders of these models outside the natural number, the theory of fractional differential equations emerges [17]. Some studies on this subject are in the related references [18-25].

One of the methods used in the solution of mathematical models is Laplace transform. There are various studies on this subject. The application of Laplace transform to fractional systems is given in [14]. Akgül et al. solved the proportional Caputo derivative by Laplace transform [26]. Also, Anderson and Georgiev defined the proportional Laplace transform on $\mathbb{T}$, which forms the basis of this work [16]. Detailed information about the proportional Laplace transform will be given in the next section.

In this study, we reconstruct some versions of classical learning and forgetting model of Edelstein-Keshet [27] using proportional derivatives. Let us now explain the learning and forgetting model in classical case. One of the fields of study between mathematics and psychology is cognitive science and memory research. The learning and forgetting model on $\mathbb{T}$ that we discuss in this study is an example of this research [28-29]. This model expresses the change in memory power as a function of time in a mathematical way. The classical case of the learning and forgetting model is as follows [27]:

$$\frac{dy}{dt} = S - fy(t), \tag{1}$$

where $y(t)$ is the amount of information a person has at time t (in years), $f \geq 0$ is rate of forgetting and $S \geq 0$ is rate of learning. The degree of learning requires a subjective interpretation of the material learnt. It is therefore different for each individual. Knowledge with a better learning degree is more retained in the future compared to less learnt knowledge. The act of forgetting provides an adaptation to environmental realities [30-32]. In this paper, the classical model (1) is redefined with the help of the proportional derivative. We analysed the following models obtained by taking the learning rates as a constant, a proportional exponential function and a proportional hyperbolic function with the forgetting rate fixed. In this study, we choose S in different forms as Q, $E_Q(t, 0)$, $Cosh_Q(t, 0)$ and $E_Q(t, 0) + Cosh_Q(t, 0)$, respectively where f and Q are fixed functions and $\pm Q \in \mathfrak{R}_c$. Let $\alpha \in$

[0,1]. Here, following different versions of the proportional learning-forgetting models on time scales will be solved.

$D^{\alpha}y(t) = Q - fy(t)$,
$D^{\alpha}y(t) = E_Q(t,0) - fy(t)$,
$D^{\alpha}y(t) = Cosh_Q(t,0) - fy(t)$,
$D^{\alpha}y(t) = E_Q(t,0) + Cosh_Q(t,0) - fy(t)$.

The most important difference of the study is the difficulties that the Laplace transform and time scale theory will create in the proportional derivative.

2. Preliminaries

Here, important concepts and theorems related to proportional derivative on $\mathbb{T}$, which form the basis of the study are given. Let $\mathbb{T}$ be a time scale with forward jump operator σ and delta differentiation operator Δ. Also, let $\alpha \in [0,1]$. Throughout this study, we suppose the below property (A1) holds.

(A1) $k_0, k_1: [0,1] \times \mathbb{T} \to [0,\infty)$ are continuous functions such that
$\lim_{x\to 0^+} k_1(\alpha,t) = 1,\ \lim_{x\to 1^-} k_1(\alpha,t) = 0$,
$\lim_{x\to 0^+} k_0(\alpha,t) = 0,\ \lim_{x\to 1^-} k_0(\alpha,t) = 1$,
$k_1(\alpha,t) \neq 0, k_0(\alpha,t) \neq 0$,
where $t \in \mathbb{T}$ and $, \alpha \in [0,1)$ [16].

Definition 2.1. [16] Let f be $\Delta -$ differentiable at $t \in \mathbb{T}^{\kappa}$. Proportional Δ−derivative of f at t is
$D^{\alpha}f(t) = k_1(\alpha,t)f(t) + k_0(\alpha,t)f^{\Delta}(t),\ \alpha \in [0,1]$.
This equation shows the relationship between proportional derivative on $\mathbb{T}$ and the delta derivative on $\mathbb{T}$.

Definition 2.2. [16] $f: \mathbb{T} \to \mathbb{R}$ is proportional regressive if
$k_0(\alpha,t) - \mu(t)k_1(\alpha,t) \neq 0$,
and
$k_0(\alpha,t) - \mu(t)(f(t) - k_1(\alpha,t)) \neq 0$,
for any $\alpha \in (0,1]$ and $t \in \mathbb{T}$. The set of all proportional regressive functions on $\mathbb{T}$ is denoted by $\mathfrak{R}_c$.

Definition 2.3. [16] Suppose that $\alpha \in (0,1]$, $p \in \mathfrak{R}_c$. For $t, t_0 \in \mathbb{T}$, proportional exponential function is as follows
$E_p(t,t_0) = e_{\frac{p-k_1}{k_0}}(t,t_0)$,

where $e_{\frac{p-k_1}{k_0}}(t,t_0)$ is exponential function on $\mathbb{T}$. Proportional exponential function is calculated using exponential function structure on $\mathbb{T}$.

Remark 2.4. [12] If $p \in \mathcal{R}$, where $\mathcal{R}$ is the set of all rd-continuous, regressive functions on $\mathbb{T}$, then exponential function on $\mathbb{T}$ is defined by

$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),\ s,t \in \mathbb{T}$.

Definition 2.5. [16] Let $\pm f \in \mathfrak{R}_c$. Proportional hyperbolic functions $Cosh_f$ and $Sinh_f$ are defined by

$Cosh_f = \frac{E_f + E_{-f}}{2}$,

and

$Sinh_f = \frac{E_f - E_{-f}}{2}$.

Definition 2.6. [16] Let $\alpha \in (0,1]$, and fix $t \in \mathbb{T}$. For $h > 0$, multi-valued proportional cylinder transformation $\zeta_h^c: \mathbb{C}_h^c \to \mathbb{C}$ is

$$\zeta_h^c(z) = \begin{cases} \frac{1}{h}\log\left(1 + h\left(\frac{z-k_1(\alpha,t)}{k_0(\alpha,t)}\right)\right) & for\ h \neq 0 \\ \frac{z-k_1(\alpha,t)}{k_0(\alpha,t)} & for\ h = 0, \end{cases}$$

where $\mathbb{C}_h^c$ is given by

$$\mathbb{C}_h^c = \left\{z \in \mathbb{C}: z \neq k_1(\alpha,t) - \frac{k_0(\alpha,t)}{h}\right\},$$

and log is the multi-valued complex logarithm function.

Definition 2.7. [16] Let $f \in C_{rd}(\mathbb{T})$ where $C_{rd}(\mathbb{T})$ is set of all rd-continuous functions on $\mathbb{T}$ and $k_0(\alpha,t) - \mu(t)k_1(\alpha,t) \neq 0$. Proportional antiderivative of f is

$$\int D^\alpha f(t)\Delta_\alpha t = f(t) + cE_0(t,t_0),\ \ c \in \mathbb{R},\quad t \in \mathbb{T},$$

and, proportional $\Delta -$ integral of f on $[a,b]$ is

$$\int_a^t f(s)\,\Delta_{\alpha,t}s,\ \ t \in [a,b],$$

where $\Delta_{\alpha,t}s = \frac{E_0(t,\sigma(s))}{k_0(\alpha,s)}\Delta s$. Here, proportional integral is calculated by reducing it to delta integral on $\mathbb{T}$.

Remark 2.8. [16] Let $\alpha \in (0,1]$, k_0 and k_1 satisfy $(A1)$, $k_0 \in C_{rd}^1(\mathbb{T})$ where $C_{rd}^1(\mathbb{T})$ is the set of delta differentiable functions whose delta derivatives are rd-continuous and $|E_0(\infty,0)| < \infty$. Let $h \in C_{rd}^1(\mathbb{T})$ and $g \in \Re_c$ be such that

$$zh^\sigma E_g^\sigma(.,0) = -gE_g(.,0), \tag{2}$$

$$D^\alpha h - zhh^\sigma + (z-k_1)h^\sigma - k_1 h = 0,$$
$$h(0) = 1,$$

for $z \in \mathcal{H}_c(h)$, where $\mathcal{H}_c(h)$ consists of all complex numbers $z \in \Re_c$ for which $z - k_1 \in \Re_c$ and

$$k_0 + h^\sigma z(\mu - k_1) \neq 0.$$

Note that there exists a unique $h \in C_{rd}^1$ that satisfies the second and third equations of the system (2). Hence, there exists a unique $g \in \Re_c$ that satisfies the first equation of (2).

Definition 2.9. [16] Let $f: \mathbb{T} \to \mathbb{C}$ be regulated. Then, proportional Laplace transform of f is defined by

$$\mathcal{L}_c(f)(z) = \int_0^\infty f(t)h^\sigma(t)E_g^\sigma(t,0)\Delta_{\alpha,\infty}t,$$

for $z \in \mathfrak{D}_c(f)$, where $\mathfrak{D}_c(f)$ consists of all complex number $z \in \mathcal{H}_c(h)$ when the proportional improper integral exists. Here, $\mathcal{L}_c^{-1}$ is called proportional inverse Laplace transform which satisfies the property $\mathcal{L}_c^{-1}(F)(z) = f(.,0)$ with $\mathcal{L}_c(f(.,0))(z) = F(z)$.

Remark 2.10. [12] $f: \mathbb{T} \to \mathbb{R}$ is regulated if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Theorem 2.11. [16] Let $f, g: \mathbb{T} \to \mathbb{C}$ be regulated functions and $a, b \in \mathbb{C}$. Then,

$$\mathcal{L}_c(af + bg)(z) = a\mathcal{L}_c(f)(z) + b\mathcal{L}_c(g)(z),$$

for $z \in \mathfrak{D}_c(f) \cap \mathfrak{D}_c(g)$.

Here, it will be explained how the Laplace transform of proportional derivative is calculated, which will form the basis of the study.

Theorem 2.12. [16] Let $n \in \mathbb{N}$, $f: \mathbb{T} \to \mathbb{C}$ be such that $(D^\alpha)^k f$, $k \in \{0,1,\dots,n\}$, are regulated. Then

$\mathcal{L}_c((D^\alpha)^n f)(z) = z^n \mathcal{L}_c(f)(z) - E_0(\infty,0)(f(0)z^{n-1} + D^\alpha f(0) z^{n-2} + \cdots + (D^\alpha)^{n-1} f(0))$

for any

$z \in \mathfrak{D}_c(f) \cap \mathfrak{D}_c(D^\alpha(f)) \cap \ldots \cap \mathfrak{D}_c\left((D^\alpha)^n(f)\right),$

where

$\lim_{t\to\infty}\left((D^\alpha)^k f(t) h(t) E_g(t,0)\right) = 0,\ k \in \{0,1,\ldots,n\}.$

Theorem 2.13. [16] The proportional Laplace transforms of some common functions are as follows.

i. $\mathcal{L}_c(1)(z) = \frac{1}{z} E_0(\infty,0),\ z \in \mathfrak{D}_c(1).$

ii. $\mathcal{L}_c\left(E_f(.,0)\right)(z) = \frac{E_0(\infty,0)}{z-f}, f \in \mathfrak{R}_c$ and for those $z \in \mathfrak{D}_c\left(E_f(.,0)\right)$ where $\lim_{t\to\infty}\left(E_f(t,0)h(t)E_g(t,0)\right) = 0$.

iii. $\mathcal{L}_c\left(Cosh_f(.,0)\right)(z) = \frac{zE_0(\infty,0)}{z^2-f^2}, f \in \mathfrak{R}_c$ and for those $z \in \mathfrak{D}_c\left(E_f(.,0) \cap E_{-f}(.,0)\right)$ where $\lim_{t\to\infty}\left(E_f(t,0)h(t)E_g(t,0)\right) = \lim_{t\to\infty}\left(E_{-f}(t,0)h(t)E_g(t,0)\right) = 0.$

Now let's calculate the Laplace transform of the proportional sin-function.

Theorem 2.14. Let $f \in \mathfrak{R}_c$ be a constant. Then, $\mathcal{L}_c\left(Sinh_f(.,0)\right)(z) = \frac{fE_0(\infty,0)}{z^2-f^2}$.

Proof. By the definition of $Sinh_f$, we have

$$\begin{aligned}
\mathcal{L}_c\left(Sinh_f(.,0)\right)(z) &= \mathcal{L}_c\left(\frac{E_f(.,0)-E_{-f}(.,0)}{2}\right)(z) \\
&= \frac{1}{2}\mathcal{L}_c\left(E_f(.,0)\right)(z) - \frac{1}{2}\mathcal{L}_c\left(E_{-f}(.,0)\right)(z) \\
&= \frac{E_0(\infty,0)}{2(z-f)} - \frac{E_0(\infty,0)}{2(z+f)} \\
&= \frac{fE_0(\infty,0)}{z^2-f^2},
\end{aligned}$$

for those $z \in \mathfrak{D}_c\left(E_f(.,0) \cap E_{-f}(.,0)\right)$ where $\lim_{t\to\infty}\left(E_f(t,0)h(t)E_g(t,0)\right) = \lim_{t\to\infty}\left(E_{-f}(t,0)h(t)E_g(t,0)\right) = 0.$

Proportional Laplace transform is applied to proportional IVP's as follows.

Theorem 2.15. [16] Consider the following proportional IVP

$$(D^\alpha)^n y + a_{n-1}(D^\alpha)^{n-1} y + \cdots + a_1 D^\alpha y + a_0 y = f(t),\ \ t > 0, \tag{3}$$

$$\begin{aligned}
&(D^\alpha)^{n-1} y(0) = b_{n-1}, \\
&\vdots \\
&D^\alpha y(0) = b_1, \\
&y(0) = b_0,
\end{aligned} \tag{4}$$

where $a_i, b_i \in \mathbb{C}, i \in \{1,\ldots,n-1\}, f: \mathbb{T} \to \mathbb{C}$ is regulated. Apply proportional Laplace transform to both sides of (3) and use initial conditions (4), we get

$$\begin{aligned}
\mathcal{L}_c(f)(z) &= \mathcal{L}_c((D^\alpha)^n y + a_{n-1}(D^\alpha)^{n-1} y + \cdots + a_1 D^\alpha y + a_0 y)(z) \\
&= \mathcal{L}_c((D^\alpha)^n y)(z) + a_{n-1}\mathcal{L}_c((D^\alpha)^{n-1} y)(z) + \cdots + a_1 \mathcal{L}_c(D^\alpha y)(z) + a_0 \mathcal{L}_c(y)(z) \\
&= z^n \mathcal{L}_c(y)(z) - E_0(\infty,0)\left(y(0)z^{n-1} + D^\alpha y(0) z^{n-2} + \cdots + (D^\alpha)^{n-1} y(0)\right) \\
&\quad + a_{n-1}\left(z^{n-1}\mathcal{L}_c(y)(z) - E_0(\infty,0)\left(y(0)z^{n-2} + D^\alpha y(0) z^{n-3} + \cdots + (D^\alpha)^{n-2} y(0)\right)\right)
\end{aligned}$$

$$+a_1\big(z\mathcal{L}_c(y)(z) - E_0(\infty,0)y(0)\big) + a_0\mathcal{L}_c(y)(z)$$
$$= (z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)\mathcal{L}_c(y)(z) - E_0(\infty,0)(b_0 z^{n-1} + (b_1+b_2)z^{n-2} + (b_2+b_1+b_0)z^{n-3}$$
$$+\cdots + (b_{n-2} + b_{n-1} + \cdots + b_1 + b_0) + b_{n-1} + b_{n-2} + \cdots + b_1 + b_0).$$

$$l(z) = E_0(\infty,0)(b_0 z^{n-1} + (b_1+b_2)z^{n-2} + (b_2+b_1+b_0)z^{n-3}$$
$$+\cdots + (b_{n-2} + b_{n-1} + \cdots + b_1 + b_0) + b_{n-1} + b_{n-2} + \cdots + b_1 + b_0).$$

Then

$$(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)\mathcal{L}_c(y)(z) = \mathcal{L}_c(f)(z) + l(z)$$

or

$$\mathcal{L}_c(y)(z)(y)(z) = \frac{1}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}\big(\mathcal{L}_c(f)(z) + l(z)\big).$$

Hence,

$$y(t) = \mathcal{L}_c^{-1}\left(\frac{1}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}\big(\mathcal{L}_c(f)(z) + l(z)\big)\right), t \geq 0.$$

Theorem 2.16. [16] Let $p, q \in \mathcal{C}_{rd}(\mathbb{T})$. Consider

$$D^\alpha y = (p(t) + k_1(\alpha,t))y + q(t), \quad t \in \mathbb{T}^\kappa, \tag{5}$$

$$y(t_0) = y_0, \tag{6}$$

where $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$. Suppose that

$$k_0(\alpha,t) + \mu(t)p(t) \neq 0, \ \ \alpha \in (0,1], \ \ t \in \mathbb{T}.$$

Then the problem $(5)-(6)$ has a unique solution represented in the form

$$y(t) = y_0 E_{p+k_1}(t,t_0) + \int_{t_0}^t q(s)E_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad s,t \in \mathbb{T}^\kappa,$$

where

$$g = \frac{p(\mu k_1 - k_0)}{k_0 + \mu p}.$$

Theorem 2.17. [16] Let $p \in \mathcal{C}_{rd}(\mathbb{T}) \cap \mathfrak{R}_c$ and $q \in \mathcal{C}_{rd}(\mathbb{T})$. Consider

$$D^\alpha y = p(t)y + q(t), \quad t \in \mathbb{T}^\kappa, \tag{7}$$

$$y(t_0) = y_0, \tag{8}$$

where $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$. We can rewrite the equation (7) in the form

$$D^\alpha y = \big(p(t) - k_1(\alpha,t) + k_1(\alpha,t)\big)y + q(t) \quad t \in \mathbb{T}^\kappa.$$

Because $p \in \mathfrak{R}_c$, we have

$$1 + \mu(t)\big(p(t) - k_1(\alpha,t)\big) \neq 0, \ \ \alpha \in (0,1], \ \ t \in \mathbb{T}^\kappa.$$

Therefore, the solution y of (7) can be represented in the form

$$y(t) = y_0 E_p(t,t_0) + \int_{t_0}^t q(s)E_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad t \in \mathbb{T}^\kappa,$$

where

$$g = \frac{(p-k_1)(\mu k_1 - k_0)}{k_0 + \mu(p - k_1)}.$$

3. Main Results

In this section, we solve proportional versions of learning and forgetting models with proportional Laplace transform on time scales. What these solutions mean in psychology can be examined by scientists interested in that subject. Here, $k_0(\alpha,t)$ and $k_1(\alpha,t)$ satisfy condition $(A1)$, f and Q are constants and $\alpha \in [0,1]$.

Since differential equations are transformed into polynomials that are easier to solve with the Laplace transform, it is used in modeling time-independent linear systems and solving differential equations, in various problems such as the initial value theorem, final value theorem and boundary value problem, in probability theory, and because it clearly shows the frequency characteristic of the relevant function. It is also used in processing. There are various methods used when solving proportional dynamic equations on the time scale. Since the Laplace transform has a detailed literature on the proportional time scale and its known functionality in the classical case, the problem addressed in this study will be solved using proportional Laplace transform on time scale.

The model in the classical case has not only been generalized to proportional derivative on the time scale, but also the functions in the classical case have been changed in various ways and solutions have been obtained with the following theorems.

In the following theorems, proportional learning-forgetting models will be solved in four different situations on $\mathbb{T}$. Here, in addition to the Laplace transform, solutions were obtained for each model with two theorems used for first-order proportional dynamic equations on time scales.

Theorem 3.1. Consider

$$D^{\alpha}y(t) = Q - fy(t), \tag{9}$$

$$y(0) = \beta,\ \beta \in \mathbb{R}.$$

The solution of this proportional IVP is

$$y(t) = \beta E_{-f}(t,0) - \frac{Q}{f}E_{-f}(t,0) + \frac{Q}{f}.$$

Proof. If we apply proportional Laplace transform to both sides of (9) and consider Theorem 2.14, we get

$$\mathcal{L}_c\big(D^{\alpha}y(t)\big)(z) = Q\mathcal{L}_c(1) - f\mathcal{L}_c\big(y(t)\big)(z),$$

$$z\mathcal{L}_c\big(y(t)\big)(z) - E_0(\infty,0)y(0) = Q\frac{E_0(\infty,0)}{z} - f\mathcal{L}_c\big(y(t)\big)(z),$$

$$(z+f)\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = Q\frac{E_0(\infty,0)}{z} + E_0(\infty,0)y(0),$$

$$\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = Q\frac{E_0(\infty,0)}{z(z+f)} + \frac{E_0(\infty,0)}{z+f}y(0).$$

By applying inverse proportional Laplace transform to last equation and set $y(0) = \beta$, we get

$$y(t) = \beta E_{-f}(t,0) - \frac{Q}{f}E_{-f}(t,0) + \frac{Q}{f}.$$

Conclusion 3.2. From Theorem 2.17., the solution to problem (9) is

$$y(t) = \beta E_{-f}(t,0) + \int_0^t QE_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad t \in \mathbb{T}^{\kappa},$$

where

$$g = \frac{(p-k_1)(\mu k_1 - k_0)}{k_0 + \mu(p-k_1)}.$$

Theorem 3.3. Consider

$$D^{\alpha}y(t) = E_Q(t,0) - fy(t), \tag{10}$$

$$y(0) = \beta\ ,\ \beta \in \mathbb{R}.$$

The solution of this proportional IVP is

$$y(t) = \beta E_{-f}(t,0) + \frac{1}{f+Q} E_Q(t,0) - \frac{1}{f+Q} E_{-Q}(t,0).$$

Proof. Considering proportional Laplace transform for equation (10) and using Theorem 2.14, we can perform the following operations;

$$\mathcal{L}_c\big(D^\alpha y(t)\big)(\mathrm{z}) = \mathcal{L}_c\left(E_Q(t,0)\right) - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z}),$$

$$z\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) - E_0(\infty,0)y(0) = \frac{E_0(\infty,0)}{z-Q} - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z}),$$

$$(z+f)\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = \frac{E_0(\infty,0)}{z-Q} + E_0(\infty,0)y(0),$$

$$\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = \frac{E_0(\infty,0)}{(z+f)(z-Q)} + \frac{E_0(\infty,0)}{z+f} y(0).$$

If inverse proportional Laplace transform is applied to perform necessary calculations, we get

$$y(t) = \beta E_{-f}(t,0) + \frac{1}{f+Q} E_Q(t,0) - \frac{1}{f+Q} E_{-f}(t,0).$$

Conclusion 3.4. From Theorem 2.17., the solution to problem (10) is

$$y(t) = \beta E_{-f}(t,0) + \int_0^t E_Q(s,0) E_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad t \in \mathbb{T}^\kappa,$$

where

$$g = \frac{(p-k_1)(\mu k_1 - k_0)}{k_0 + \mu(p-k_1)}.$$

Theorem 3.5. Consider

$$D^\alpha y(t) = Cosh_Q(t,0) - fy(t), \tag{11}$$

$$y(0) = \beta, \ \beta \in \mathbb{R}.$$

The solution of this proportional IVP is

$$y(t) = \beta E_{-f}(t,0) + \frac{f}{Q^2-f^2} E_{-f}(t,0) - \frac{f}{Q^2-f^2} Cosh_Q(t,0) + \frac{Q}{Q^2-f^2} Sinh_Q(t,0).$$

Proof. Now let us solve equation (11) by proportional Laplace transform. By applying proportional Laplace transform to both sides of (11), we get

$$\mathcal{L}_c\big(D^\alpha y(t)\big)(\mathrm{z}) = \mathcal{L}_c\left(Cosh_Q(t,0)\right) - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z})$$

$$z\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) - E_0(\infty,0)y(0) = \frac{zE_0(\infty,0)}{z^2-Q^2} - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z}),$$

$$(z+f)\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = E_0(\infty,0)y(0) + \frac{zE_0(\infty,0)}{z^2-Q^2},$$

$$\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = \frac{E_0(\infty,0)}{z+f} y(0) + \frac{zE_0(\infty,0)}{(z+f)(z^2-Q^2)}.0$$

If the necessary calculations are made and $y(0) = \beta$ is taken into consideration, we get

$$y(t) = \beta E_{-f}(t,0) + \frac{f}{Q^2-f^2} E_{-f}(t,0) - \frac{f}{Q^2-f^2} Cosh_Q(t,0) + \frac{Q}{Q^2-f^2} Sinh_Q(t,0).$$

Conclusion 3.6. From Theorem 2.17., the solution to problem (11) is

$$y(t) = \beta E_{-f}(t,0) + \int_0^t Cosh_Q(s,0) E_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad t \in \mathbb{T}^\kappa,$$

where

$g = \frac{(p-k_1)(\mu k_1 - k_0)}{k_0 + \mu(p-k_1)}$.

Theorem 3.7. Consider

$$D^\alpha y(t) = Cosh_Q(t,0) + E_Q(t,0) - fy(t), \tag{12}$$

$y(0) = \beta,\ \beta \in \mathbb{R}$.

The solution of this proportional IVP is

$$y(t) = \beta E_{-f}(t,0) + \frac{f}{Q^2-f^2}E_{-f}(t,0) - \frac{f}{Q^2-f^2}Cosh_Q(t,0) + \frac{Q}{Q^2-f^2}Sinh_Q(t,0) + \frac{1}{f+Q}E_Q(t,0) - \frac{1}{f+Q}E_{-f}(t,0).$$

Proof. Let us take the proportional Laplace transform in equation (12). Considering $y(0) = \beta$ and Theorem 2.14, the following calculations can be made;

$$\mathcal{L}_c\big(D^\alpha y(t)\big)(\mathrm{z}) = \mathcal{L}_c\left(Cosh_Q(t,0)\right) + \mathcal{L}_c\left(E_Q(t,0)\right) - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z}),$$
$$z\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) - E_0(\infty,0)y(0) = \frac{zE_0(\infty,0)}{z^2-Q^2} + \frac{E_0(\infty,0)}{z-Q} - f\mathcal{L}_c\big(y(t)\big)(\mathrm{z}),$$
$$(z+f)\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = E_0(\infty,0)y(0) + \frac{zE_0(\infty,0)}{z^2-Q^2} + \frac{E_0(\infty,0)}{z-Q},$$
$$\mathcal{L}_c\big(y(t)\big)(\mathrm{z}) = \frac{E_0(\infty,0)}{z+f}y(0) + \frac{zE_0(\infty,0)}{(z+f)(z^2-Q^2)} + \frac{E_0(\infty,0)}{(z+f)(z-Q)}.$$

If inverse proportional Laplace transform is applied to last equation, we get

$$y(t) = \beta E_{-f}(t,0) + \frac{f}{Q^2-f^2}E_{-f}(t,0) - \frac{f}{Q^2-f^2}Cosh_Q(t,0) + \frac{Q}{Q^2-f^2}Sinh_Q(t,0) + \frac{1}{f+Q}E_Q(t,0) - \frac{1}{f+Q}E_{-f}(t,0).$$

Conclusion 3.8. From Theorem 2.17., the solution to problem (12) is

$$y(t) = \beta E_{-f}(t,0) + \int_0^t \big(Cosh_Q(s,0) + E_Q(s,0)\big)E_g(\sigma(s),t)\Delta_{\alpha,t}s, \quad t \in \mathbb{T}^\kappa,$$

where

$g = \frac{(p-k_1)(\mu k_1 - k_0)}{k_0 + \mu(p-k_1)}$.

4. Conclusion

Learning is the phenomenon obtained by the individual as a result of his/her experiences. Forgetting is the change that occurs in learning over time. In this study, some proportional learning and forgetting models, which are very important fields of study in psychology, is discussed. This model is reconstructed with the help of proportional derivative and handled with various learning functions. These models are solved with the help of proportional Laplace transform. After using the Laplace transform for each of the models considered, solutions are presented with the help of two theorems that give the solutions of first-order proportional dynamic equations on the time scale. These solutions can be evaluated especially by scientists working in the field of psychology. With this study, the results obtained in the classical case have been generalized. Specific choices of time scale and alpha value give the results obtained in the classical case.

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