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Hermite-Hadamard Inequalities for a New Class of Generalized-(s, m) via Fractional Integral

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Abstract: This paper defines a new generalized (s,m)-σ convex function using the σ convex functions and provides some applications and exact results for this kind of functions. The new definition of the (s,m)-σ convex function class is used to obtain the Hermite Hadamard type integral inequalities existing in the literature, and new integral inequalities are obtained with the help of the σ-Riemann-Liouville fractional integral. Additionally, a new Hermite-Hadamard type fractional integral inequality is constructed using the σ-Riemann-Liouville fractional integral.

Genelleştirilmiş (s,m) Fonksiyonların Yeni Bir Sınıfı İçin Kesirli İntegral Yoluyla Hermite-Hadamard Eşitsizlikler

Anahtar Kelimeler
Hermite–Hadamard eşitsizliği, kesirli integral operatör, konveks fonksiyon, σ-konveks fonksiyon, (s,m) – σ konveks fonksiyon.

Öz: Bu makale, σ konveks fonksiyon sınıfını kullanarak yeni bir genelleştirilmiş (s,m)-σ konveks fonksiyonu tanımlamakta ve bu tür fonksiyonlar için bazı uygulamalar ve kesin sonuçlar sunmaktadır. Literatürde var olan Hermite Hadamard tipi integral eşitsizliklerini elde etmek için (s,m)-σ konveks fonksiyon sınıfının yeni tanımından yararlanılmış ve σ-Riemann-Liouville kesirli integrali yardımıyla yeni integral eşitsizlikleri elde edilmiştir. Ek olarak, σ-Riemann-Liouville kesirli integrali kullanılarak yeni bir Hermite-Hadamard tipi kesirli integral eşitsizliği oluşturuldu.

1. INTRODUCTION

Mathematics is a tool that serves both pure and applied sciences. Its history is as old as human history and it sheds light on how to express and solve problems. Mathematics employs various concepts and their relations for solving the problems. It defines spaces and algebraic structures built on spaces, creating structures that contribute to human life and nature. The concept of function is fundamental in mathematics, and researchers have focused on developing new function classes and classifying the space of functions. One of this classes of functions is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. A convex function is defined as a function where the line segment between any two points on the graph of the function lies above or on the graph. This definition ensures that the function is always 'curving upwards' and has no local maxima. The use of convex functions in various fields is due to their unique properties, such as their ability to model optimization problems and their connection to the theory of convex sets.

2. MATERIAL AND METHOD

Definition 2.1: [14,15] Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then, $\Omega: \mathcal{H} \to \mathbb{R}, \emptyset \neq \mathcal{H} \subseteq \mathbb{R}$ is said to be convex if

$$\Omega(\xi u + (1-\xi)v) \leq \xi\Omega(u) + (1-\xi)\Omega(v)$$

for all $u, v \in \mathcal{H}$ and $\xi \in [0,1]$.

Definition 2.2: [6] For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $\Omega: \mathcal{H} \subset [0,\infty) \to \mathbb{R}$ is said to be $(s,m)-$ convex in the second sense on $\mathcal{H}$ if

$$\Omega(\xi u + m(1-\xi)v) \leq \xi^s\Omega(u) + m(1-\xi)^s\Omega(v)$$

holds for all $u, v \in \mathcal{H}$ and $\xi \in [0,1]$.

Definition 2.3: [13] Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then, $\Omega: \mathcal{H} \to \mathbb{R}, \emptyset \neq \mathcal{H} \subseteq \mathbb{R}$ is said to be quasi convex if

$$\Omega(\xi u + (1-\xi)v) \leq sup\{\Omega(u), \Omega(v)\}$$

holds for all $u, v \in \mathcal{H}$ and $\xi \in [0,1]$.

The theory of convexity is important in various fields of pure and applied sciences. Therefore, the classical concepts of convex sets and convex functions have been extended in different directions. For further information, we refer [1,2,14]. The theory of convexity has also attracted many researchers due to its close relation with the theory of inequalities. The concept of convex functions can be used to derive many well-known inequalities. For further details, please refer to [3,15]. One of the most studied results among these inequalities is the Hermite-Hadamard inequality, which provides a necessary and sufficient condition for a function to be convex. The inequality reads as follows:

Definition 2.4: Let $\Omega: \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $\mathcal{H}$ of real numbers and $u, v \in \mathcal{H}$, with $u < v$. Then, one has

$$\Omega\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \Omega(x)\,dx \leq \frac{\Omega(u)+\Omega(v)}{2}. \quad (1)$$

This double inequality is called the Hermite-Hadamard inequality.

This fragment of text discusses the Hermite-Hadamard inequality for convex functions and introduces a new class of convex sets and functions called σ-convex sets and $\sigma-$convex functions, respectively. The new class of convex sets and functions was introduced by Wu et al. in [4]. The definitions of σ-convex sets and σ-convex functions are explained as the followings:

Definition 2.5: A function $\Omega: \mathcal{H} \to \mathbb{R}$ is said to be σ-convex function with respect to strictly monotonic continuous function σ if

$$\Omega\left(\Psi_{[\sigma]}(u,v)\right) = \xi\Omega(v) + (1-\xi)\Omega(u) \;\; \forall u, v \in \mathcal{H}, \; \xi \in [0,1].$$

Definition 2.6: A set $\mathcal{H} \subseteq \mathbb{R}$ is said to be σ –convex set with respect to strictly monotonic continuous function σ if

$$\Psi_{[\sigma]}(u,v) = \sigma^{-1}\left(\xi\sigma(v) + (1-\xi)\sigma(u)\right) \in \mathcal{H} \;\; \forall u, v \in \mathcal{H}, \xi \in [0,1].$$

Note that the function Ψ is called strictly σ-convex on $\mathcal{H}$ if the above inequality is true as a strict inequality for each distinct u and $v \in \mathcal{H}$ and for each $\xi \in (0,1)$.
Fractional analysis has been known since ancient times. However, it has recently become a more popular subject in mathematical analysis and applied mathematics. The question of whether a solution exists when the order is fractional in a differential equation has led to the development of many derivative and integral operators. By defining the derivative and integral operators in fractional order, researchers have proposed more effective solutions for physical phenomena using new operators with general and strong kernels. This has provided mathematics and applied sciences with several operators that differ in terms of locality and singularity, as well as generalized operators with memory effect properties. The initial inquiry into the impact of a fractional order in a differential equation has now developed into the challenge of elucidating physical phenomena and identifying the most efficient fractional operators to offer practical solutions to real-world issues. Introducing fractional derivative and integral operators have made significant contributions to fractional analysis and these new operators have been effectively used in various fields by numerous researchers (see [17-18]).
The definition of the Riemann-Liouville fractional integral, as given in the literature, is:

Definition 2.7: Let $\Omega \in \mathcal{L}^1(u,v)$. The Riemann Liouville integrals $I_{u^+}^{\alpha}\Omega$ and $I_{v^-}^{\alpha}\Omega$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{u^+}^{\alpha}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_u^x \Omega(\xi)(x-\xi)^{\alpha-1}dt, \quad x > u$$

and

$$I_{v^-}^{\alpha}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_x^v \Omega(\xi)(\xi-x)^{\alpha-1}dt, \quad v > x.$$

Definition 2.8: Let $(u,v)\subseteq\mathbb{R}$, $\sigma(x)$ be an increasing and positive function on $(u,v]$ and $\sigma'(x)$ be continuous on (u,v). Then, the left-sided and right-sided σ –Riemann-Liouville fractional integrals of a function Ω with respect to the function $\sigma(x)$ on $[u,v]$ are respectively defined by [7,8]:

$$I_{u^+}^{\alpha:\varphi}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_u^x \Omega(\xi)\varphi'(\xi)\left(\varphi(x)-\varphi(\xi)\right)^{\alpha-1}dt$$

$$I_{v^-}^{\alpha:\varphi}\Omega(x) = \frac{1}{\Gamma(\alpha)}\int_x^v \Omega(\xi)\varphi'(\xi)\left(\varphi(\xi)-\varphi(x)\right)^{\alpha-1}dt, \qquad \alpha>0.$$

It can be observed that if φ is specialised by $\varphi(x)=x$, then the σ –Riemann-Liouville fractional integral operators are reduced to the classical Riemann-Liouville fractional integral operators.
The fractional Hermite-Hadamard integral inequalities [4,9] are given by:

$$\Omega\left(\frac{u+v}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[I_{u^+}^{\alpha}\Omega(v)+I_{v^-}^{\alpha}\Omega(u)\right]\le\frac{\Omega(u)+\Omega(v)}{2}.$$

In their recent work, Mohammed et al. [5] utilised this novel convex function for a fractional operator to present the new findings:

Theorem 2.1: Let $\Omega:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be an integrable σ -convex function and $\Omega\in\mathcal{L}^1(u,v)$ with $0\le u<v$. If the function σ is increasing and positive on $[u,v]$ and $\sigma'(x)$ is continuous on (u,v), then for $\alpha>0$

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+\sigma(v)}{2}\right)1\right)\le\frac{\Gamma(\alpha+1)}{2\left(\sigma(v)-\sigma(u)\right)^\alpha}\left[I_{u^+}^{\alpha:\varphi}\Omega(v)+I_{v^-}^{\alpha:\varphi}\Omega(u)\right]\le\frac{\Omega(u)+\Omega(v)}{2}.$$

Theorem 2.2: Let $\Omega:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be an integrable σ -convex function and $\Omega\in\mathcal{L}^1(u,v)$ with $0\le u<v$. If the function σ is increasing and positive on $[u,v]$ and $\sigma'(x)$ is continuous on (u,v), then for $\alpha>0$, we have

$$\Omega\left(\sigma^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sigma(v)-\sigma(u)\right)^\alpha}\left[I^{\alpha:\varphi}_{\sigma^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right)^+}\Omega(v)+I^{\alpha:\varphi}_{\sigma^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right)^-}\Omega(u)\right]\le\frac{\Omega(u)+\Omega(v)}{2}.$$

3. RESULTS

Definition 3.1: For some fixed $s\in(0,1]$ and $m\in[0,1]$ a mapping $\Omega:\mathcal{H}\subset[0,b]\to\mathbb{R}^\propto$ with $b>0$ is said to be generalized $\sigma-(s,m)-$ convex if

$$\begin{aligned}\Psi_{[\sigma]}(x,y)-(s,m) &= \Omega\left(\sigma^{-1}\left(\xi\sigma(x)+m(1-\xi)\sigma(y)\right)\right)\\ &\le \xi^{\propto s}\Omega(x)+m^\propto(1-\xi)^{\propto s}\Omega(y)\end{aligned}\qquad(2)$$

holds for all $x,y\in\mathcal{H}$ and $\xi\in[0,1]$.

Remark 3.1: If we take $\propto=1$ and $\sigma^{-1}(x)=x$ then, we get Definition 2.2 in [6].

Remark 3.2: If we take $\propto=1, s=1$ and $m=1$ then, we get Definition 2.5 in [4].

Remark 3.3: If we take $\propto=s=m=1$ and $\sigma(x)=x$ then, we get the concept of classical convex functions.

Moreover, if we take $\xi=\frac{1}{2}$ in (2), then the generalized $\Psi_{[\sigma]}(x,y)-(s,m)$ convex functions become Jensen-type generalized $\sigma-(s,m)$ convex functions as follows:

$$\Omega\left(\sigma^{-1}\left(\frac{\varphi(x)+m\varphi(y)}{2}\right)\right)\le\frac{\Omega(x)+m^\propto\Omega(y)}{2^{s\propto}}.$$

For all $x,y\in\mathcal{H}$ and $\xi\in[0,1]$ and for some fixed $s\in(0,1]$ and $m\in[0,1]$.

Some special cases are obtained as follows.

Case-1
If we take $\sigma^{-1}(x)=lnx$, then we get geometrically (s,m)-convex function as in [10]

$$\Omega(x^t y^{1-t})\le\xi^s\Omega(x)+m(1-\xi)^s\Omega(y)$$
holds for all $x,y\in\mathcal{H}$ and $t\in[0,1]$.

Cese-2
If we take $\sigma^{-1}(x)=\frac{1}{x}$, then we get harmonically (s,m)-convex function as in [11]

$$\Omega\left(\frac{mxy}{mtx+(1-\xi)y}\right)\le\xi^s\Omega(x)+m(1-\xi)^s\Omega(y)$$

holds for all $x,y\in\mathcal{H}$ and $\xi\in[0,1]$.

Proposition 3.1: For $s\in(0,1]$ and $m\in[0,1]$ if $\Omega,\mathcal{L}:\mathcal{H}\to\mathbb{R}^\propto$ are generalized $\sigma-(s,m)-$ convex functions, then we have the following statements:

$\Omega+\mathcal{L}$ is a generalized $\sigma-(s,m)-$ convex functions.
$\zeta^\propto\,\Omega$ is a generalized $\sigma-(s,m)-$ convex functions.

Proof: Since Ω and $\mathcal{L}$ are generalized $-(s,m)-$ convex functions on $\mathcal{H}$ and $\xi \in [0,1]$, we have

$$\begin{aligned}(\Omega+\mathcal{L})\left(\sigma^{-1}\big(\xi\sigma(x)+m(1-\xi)\sigma(y)\big)\right) &= \Omega\left(\sigma^{-1}\big(\xi\sigma(x)+m(1-\xi)\sigma(y)\big)\right)\\ &+\mathcal{L}\left(\sigma^{-1}\big(\xi\sigma(x)+m(1-\xi)\sigma(y)\big)\right)\\ &\le \xi^{\propto s}\Omega(x)+m^{\propto}(1-\xi)^{\propto s}\Omega(y)\\ &+\xi^{\propto s}\mathcal{L}(x)+m^{\propto}(1-\xi)^{\propto s}\mathcal{L}(y)\\ &= \xi^{\propto s}(\Omega+\mathcal{L})(x)\\ &+m^{\propto}(1-\xi)^{\propto s}(\Omega+\mathcal{L})(y)\end{aligned}$$

Hence, $\Omega+\mathcal{L}$ is a generalized $-(s,m)-$ convex functions on $\mathcal{H}$.
Since, Ω and $\mathcal{L}$ are generalized $-(s,m)-$ convex functions on $\mathcal{H}$ and $\xi \in [0,1]$, $\zeta \in \mathbb{R}_+$, we have

$$\begin{aligned}\zeta^{\propto}\Omega\left(\sigma^{-1}\big(\xi\sigma(x)+m(1-\xi)\sigma(y)\big)\right) &\le \zeta^{\propto}[\,\xi^{\propto s}f(x)+m^{\propto}(1-\xi)^{\propto s}f(y)]\\ &= \xi^{\propto s}(\zeta^{\propto}\Omega)(x)+(\zeta^{\propto}\Omega)(y)\end{aligned}$$

and, so $\zeta^{\propto}\Omega$ is a generalized $-(s,m)-$ convex functions on $\mathcal{H}$.

3.1 New Results on Generalized $\sigma-(s,m)$-Convexity

This section is devoted to establish some generalized Hermite– Hadamard type fractional integral inequalities via generalized $\sigma-(s,m)$ –convex.

Theorem 3.1: Let $\Omega : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be an integrable $\sigma-(s,m)$-convex function and $\Omega \in \mathcal{L}^1(u,v)$ with $0 \le u < v$, $m \in (0,1]$. If the function σ is increasing and positive on $[u,v]$ and $\varphi'(x)$ is continuous on (u,v), then for $\alpha > 0$, we have

$$\begin{aligned}&\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right)\\ &\le \frac{1}{2^{\propto s}\big(\sigma(u)-m\sigma(v)\big)}\left[\int_{mv}^{u}\Omega(x)\,\sigma'(x)dx\right.\\ &\left.+m^{\propto}\int_{v}^{mu}\Omega(x)\,\sigma'(x)dx\right]\\ &\le \frac{[\Omega(u)+\Omega(v)+m^{\propto}(\Omega(u/m)+\Omega(v/m))]}{2^{\propto s}(\propto s+1)}\end{aligned}\tag{3}$$

Proof: To prove the first inequality of (3), assume that Ω is a $\sigma-(s,m)$ convex function

$$\Omega\left(\sigma^{-1}\big(\xi\sigma(x)+m(1-\xi)\sigma(y)\big)\right) \le \xi^{\propto s}\Omega(x)+m^{\propto}(1-\xi)^{\propto s}\Omega(y).$$

If we take $\xi = \frac{1}{2}$, we obtain

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(x)+m\sigma(y)}{2}\right)\right) \le \frac{\Omega(x)+m^{\propto}\Omega(y)}{2^{s\propto}}.\tag{4}$$

Let us set $x=\sigma^{-1}\big(\xi\sigma(u)+m(1-\xi)\sigma(v)\big)$ and $y=\sigma^{-1}\big(\xi\sigma(v)+m(1-\xi)\sigma(u)\big)$ in (4), one has

$$\begin{aligned}2^{\propto s}\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) &\le \Omega\left(\sigma^{-1}\big(\xi\sigma(u)+m(1-\xi)\sigma(v)\big)\right)\\ &+m^{\propto}\Omega\left(\sigma^{-1}\big(\xi\sigma(v)\right.\\ &\left.+m(1-\xi)\sigma(u)\big)\right).\end{aligned}$$

Integrating this inequality with respect to ξ over $[0,1]$, we have

$$\begin{aligned}&\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right)\\ &\le \int_0^1\Omega\left(\sigma^{-1}\big(\xi\sigma(u)+m(1-\xi)\sigma(v)\big)\right)d\xi\\ &+m^{\propto}\int_0^1\Omega\left(\sigma^{-1}\big(\xi\sigma(v)\right.\\ &\left.+m(1-\xi)\sigma(u)\big)\right)d\xi\\ &=\frac{1}{2^{\propto s}}\left[\frac{1}{\sigma(u)-m\sigma(v)}\int_{mv}^{u}\Omega(x)\,\sigma'(x)dx\right.\\ &\left.+\frac{m^{\propto}}{m\sigma(u)-\sigma(v)}\int_{v}^{mu}\Omega(x)\,\sigma'(x)dx\right].\end{aligned}\tag{5}$$

The first inequality has been proved. To prove the second inequality, we will use the definition of the $\sigma-(s,m)$ convex function as follow:

$$\Omega\left(\sigma^{-1}\big(\xi\sigma(u)+m(1-\xi)\sigma(v)\big)\right) \le \xi^{\propto s}\Omega(u)+m^{\propto}(1-\xi)^{\propto s}\Omega(v/m)$$

and

$$\Omega\left(\sigma^{-1}\big(\xi\sigma(v)+m(1-\xi)\sigma(u)\big)\right) \le \xi^{\propto s}\Omega(v)+m^{\propto}(1-\xi)^{\propto s}\Omega\left(\frac{u}{m}\right).$$

By addition, we have

$$\begin{aligned}&\Omega\left(\sigma^{-1}\big(\xi\sigma(u)+m(1-\xi)\sigma(v)\big)\right)\\ &+\Omega\left(\sigma^{-1}\big(\xi\sigma(v)+m(1-\xi)\sigma(u)\big)\right)\\ &\le [\Omega(u)+\Omega(v)]\,\xi^{\propto s}\\ &+\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]m^{\propto}(1-\xi)^{\propto s}.\end{aligned}$$

Integrating this inequality with respect to ξ over $[0,1]$, we have

$$\begin{aligned}&\left[\frac{1}{\sigma(u)-m\sigma(v)}\int_{mv}^{u}\Omega(x)\,\sigma'(x)dx\right.\\ &\left.+\frac{m^{\propto}}{m\sigma(u)-\sigma(v)}\int_{v}^{mu}\Omega(x)\,\sigma'(x)dx\right]\\ &\le[\Omega(u)+\Omega(v)]\frac{1}{\propto s+1}\\ &+\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]\frac{m^{\propto}}{\propto s+1}.\end{aligned}\tag{6}$$

By combining the last two inequalities (5) and (6), the desired result is obtained as:

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) \leq \frac{1}{2^{\propto s}}\left[\frac{1}{\sigma(u)-m\sigma(v)}\int_{mv}^{u}\Omega(x)\,\sigma'(x)dx + \frac{m^{\propto}}{m\sigma(u)-\sigma(v)}\int_{v}^{mu}\Omega(x)\,\sigma'(x)dx\right] \leq [\Omega(u)+\Omega(v)]\frac{1}{\propto s+1}+\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]\frac{m^{\propto}}{\propto s+1}.$$

This completes the proof.

Corollary 3.1: If we take $m = 1$ and $\propto = 1$, then we obtain Theorem 5 in [12].

Corollary 3.2: If we take $\sigma(x) = x$, $s = m = \propto = 1$, then inequality (3) reduces to inequality (1).

Theorem 3.2: Let $\Omega : [u, v] \subseteq \mathbb{R} \to \mathbb{R}$ be an integrable $\sigma - (s, m)$-convex function and $\Omega \in \mathcal{L}^1(u, v)$ with $0 \leq u < v$, $m \in (0, 1]$. If the function σ is increasing and positive on $[u, v]$ and $\sigma'(x)$ is continuous on (u, v), then for $\alpha > 0$, we have

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) \leq \frac{\Gamma(\alpha+1)}{2^{\propto s}}\left[\frac{1}{\left(m\sigma(v)-\sigma(u)\right)^{\alpha}}I_{(u)^+}^{\alpha:\varphi}\Omega(mv) + \frac{1}{\left(\sigma(v)-m\sigma(u)\right)^{\alpha}}I_{(v)^-}^{\alpha:\varphi}\Omega(mu)\right] \leq \frac{[\Omega(u)+\Omega(v)]}{2^{\propto s}(\propto s+\alpha)} + \frac{\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]m^{\propto}}{2^{\propto s}}\frac{\Gamma(\alpha+1)\Gamma(\propto s+1)}{\Gamma(\alpha+\propto s+1)}. \quad (7)$$

Proof: To prove the first inequality of (7), assume that Ω is $\sigma - (s, m)$ convex function, then we can write

$$\Omega\left(\sigma^{-1}\left(\xi\sigma(x)+m(1-\xi)\sigma(y)\right)\right) \leq \xi^{\propto s}\Omega(x)+m^{\propto}(1-\xi)^{\propto s}\Omega(y).$$

If we take $\xi = \frac{1}{2}$, we obtain

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(x)+m\sigma(y)}{2}\right)\right) \leq \frac{\Omega(x)+m^{\propto}\Omega(y)}{2^{s\propto}}. \quad (8)$$

Substituting $x = \sigma^{-1}\left(\xi\sigma(u)+m(1-\xi)\sigma(v)\right)$ and $y = \sigma^{-1}\left(\xi\sigma(v)+m(1-\xi)\sigma(u)\right)$ into (8), we get

$$2^{\propto s}\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) \leq \Omega\left(\sigma^{-1}\left(\xi\sigma(u)+m(1-\xi)\sigma(v)\right)\right) + m^{\propto}\Omega\left(\sigma^{-1}\left(\xi\sigma(v)+m(1-\xi)\sigma(u)\right)\right). \quad (9)$$

Multiplying both sides of (9) by $\xi^{\alpha-1}$, then integrating the resulting inequality with respect to t over [0,1], we get

$$\frac{2^{\propto s}}{\alpha}\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) \leq \int_0^1 \xi^{\alpha-1}\Omega\left(\sigma^{-1}\left(\xi\sigma(u)+m(1-\xi)\sigma(v)\right)\right)d\xi + \int_0^1 \Omega\left(\sigma^{-1}\left(\xi\sigma(v)+m(1-\xi)\sigma(u)\right)\right)\xi^{\alpha-1}d\xi.$$

By changing the variables, $\lambda = \sigma^{-1}\left(\xi\sigma(u)+m(1-\xi)\sigma(v)\right)$ and $\eta = \sigma^{-1}\left(\xi\sigma(v)+m(1-\xi)\sigma(u)\right)$, then the last inequality becomes

$$\Omega\left(\sigma^{-1}\left(\frac{\sigma(u)+m\sigma(v)}{2}\right)\right) \leq \frac{1}{\left(m\sigma(v)-\sigma(u)\right)^{\alpha}}\int_u^{mv}\Omega(\lambda)\varphi'(\lambda)\left(\mathrm{m}\varphi(v)-\varphi(\lambda)\right)^{\alpha-1}d\lambda + \frac{1}{\left(\sigma(v)-m\sigma(u)\right)^{\alpha}}\int_{mu}^{v}\Omega(\eta)\varphi'(\eta)\left(\varphi(\eta)-m\varphi(u)\right)^{\alpha-1}d\eta = \frac{\Gamma(\alpha+1)}{2^{\propto s}}\left[\frac{1}{\left(m\sigma(v)-\sigma(u)\right)^{\alpha}}I_{(u)^+}^{\alpha:\varphi}\Omega(mv) + \frac{1}{\left(\sigma(v)-m\sigma(u)\right)^{\alpha}}I_{(v)^-}^{\alpha:\varphi}\Omega(mu)\right]. \quad (10)$$

In this way, the first inequality is proved.
To prove the second inequality, we use the definition of the $\sigma - (s, m)$ convex function as:

$$\Omega\left(\sigma^{-1}\left(\xi\sigma(u)+m(1-\xi)\sigma(v)\right)\right) \leq \xi^{\propto s}\Omega(u)+m^{\propto}(1-\xi)^{\propto s}\Omega(v/m)$$

and

$$\Omega\left(\sigma^{-1}\left(\xi\sigma(v)+m(1-\xi)\sigma(u)\right)\right) \leq \xi^{\propto s}\Omega(v)+m^{\propto}(1-\xi)^{\propto s}\Omega\left(\frac{u}{m}\right).$$

By addition, we have

$$\Omega\left(\sigma^{-1}(\xi\sigma(u)+m(1-\xi)\sigma(v))\right)+\Omega\left(\sigma^{-1}(\xi\sigma(v)+m(1-\xi)\sigma(u))\right)\leq[\Omega(u)+\Omega(v)]\,\xi^{\propto s}+\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]m^{\propto}(1-\xi)^{\propto s}. \quad (11)$$

Multiplying both sides of (11) by $\xi^{\alpha-1}$, then integrating the resulting inequality with respect to ξ over [0,1], we can obtain

$$\int_0^1 \xi^{\alpha-1}\Omega\left(\sigma^{-1}(\xi\sigma(u)+m(1-\xi)\sigma(v))\right)d\xi+\int_0^1\Omega\left(\sigma^{-1}(\xi\sigma(v)+m(1-\xi)\sigma(u))\right)\xi^{\alpha-1}d\xi\leq\int_0^1[\Omega(u)+\Omega(v)]\,\xi^{\propto s+\alpha-1}d\xi+\int_0^1[\Omega(u/m)+\Omega(v/m)]m^{\propto}(1-\xi)^{\propto s}d\xi.$$

Hence,

$$\frac{\Gamma(\alpha+1)}{2^{\propto s}}\left[\frac{1}{(m\sigma(v)-\sigma(u))^{\alpha}}I^{\alpha:\varphi}_{(u)^+}\Omega(v)+\frac{1}{(\sigma(v)-m\sigma(u))^{\alpha}}I^{\alpha:\varphi}_{(v)^-}\Omega(u)\right]\leq\frac{[\Omega(u)+\Omega(v)]}{2^{\propto s}(\propto s+\alpha)}+\frac{\left[\Omega\left(\frac{u}{m}\right)+\Omega\left(\frac{v}{m}\right)\right]m^{\propto}}{2^{\propto s}}\frac{\Gamma(\alpha+1)\Gamma(\propto s+1)}{\Gamma(\alpha+\propto s+1)}.$$

This completes the proof.

Corollary 3.3: If we take $m=1$ and $\propto=1$, then we obtain Theorem 8 in [12].

Corollary 3.4: If we take $\sigma(x)=x, m=1$ and $\alpha=\propto=1$, then we get the classical Hermite–Hadamard inequality under s-convex function proved by Dragomir and Fitzpatrick [16].

Corollary 3.5: If we take $\sigma(x)=x,\ s=\alpha=m=\propto=1$, then inequality (7) reduces to inequality (1).

4. DISCUSSION AND CONCLUSION

This paper introduces a new class of generalized $(s,m)-\sigma$ convex functions, extending the concept of σ-convexity within the framework of fractional calculus. By employing the σ-Riemann-Liouville fractional integral, we derived novel Hermite-Hadamard type inequalities, which generalize existing results and introduce new fractional inequalities. These findings provide valuable insights into the relationship between convexity and fractional operators. The flexibility of the new $(s,m)-\sigma$ convex functions enables further exploration in fractional calculus, particularly with different fractional operators. Future research could investigate the extension of these results to other operators, such as the Atangana-Baleanu integral, expanding the applications of these inequalities. In conclusion, the results obtained here represent a meaningful contribution to fractional analysis and its applications, offering a foundation for further studies in this field.

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