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Curves of Constant Ratio with Quasi frame in $\mathbb{E}^3$

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Abstract: In the present study we handle a regular unit speed curve by means of the position vector given by the vectorial equation $\gamma\,(s) = m_0\, t(s) + m_1 n_q(s) + m_2 b_q(s)$ where $b_q(s)$, $n_q(s)$ and $t(s)$ are quasi frame vectors. Firstly, we analysis these curves and investigate to being constant ratio curve. Then, we give the parameterizations of T-constant and N- constant curve in accordance with quasi frame. Further, we get the conditions for a regular curve to correspond to be a W- curve *in* $\mathbb{E}^3$.

$\mathbb{E}^3$ Uzayında Quasi Çatısına Göre Sabit Oranlı Eğriler

Anahtar Kelimeler
Sabit oranlı eğri,
Pozisyon vektörü,
Quasi çatısı

Öz: Bu çalışmada, $b_q(s)$, $n_q(s)$ ve $t(s)$ quasi çatı vektörleri olmak üzere pozisyon vektörü $\gamma\,(s) = m_0\, t(s) + m_1 n_q(s) + m_2 b_q(s)$ vektörel denklemi ile verilen birim hızlı eğriyi ele aldık. İlk olarak eğriyi inceleyerek sabit oranlı olma durumunu araştırdık. Sonrasında T-sabit ve N-sabit eğrilerin parametrizasyonlarını verdik. Ayrıca, bir regüler eğrinin W-eğrisine karşılık gelme koşulunu elde ettik.

1. INTRODUCTION

In $3-$dimensional Euclidean space, the rectifying curves that located on rectifying plane are defined by B. Y. Chen [3]. The binormal vector field and tangent vector field spans the related plane. Chen also presents a simple classification in this paper. In study [5], the connection between centrodes (that is of great importance in kinematics, mechanics) and rectifying curves is mentioned. Moreover, in $3-$dimensional Minkowski space, the rectifying curves are examined in [8, 11, 12, 14].

In Euclidean $3-$space, the rectifying curves can be written as

$$\gamma(s) = \lambda(s)t(s) + \mu(s)b(s), \tag{1}$$

where $\lambda\,(s)$ and $\mu\,(s)$ are curvature functions [13].

Non $-$ degenerate and continuously $3-$times differentiable curves can be considered for creating the Frenet frame. Namely, there is a possibility that second derivative vanishes. In this station, instead of this, the new frame is needed.

Quasi frame is more useful than Frenet frame, Bishop frame, etc. For example, the quasi frame can also be defined on a straight line. The structure of the $q-$frame is the same whether the curve unit speed or not, and the $q-$frame can be easily determined [7].

For a regular curve given by a position vector, the sum of its normal and tangent component can be considered as

$$\gamma = \gamma^T + \gamma^N \tag{2}$$

(see, [1]). If $\frac{\|\gamma^T\|}{\|\gamma^N\|}$ is equal to a constant, such curves are called as curves of constant ratio [1]. Here $\|\gamma^N\|$ and $\|\gamma^T\|$, are the norms or the lengths of γ^N and γ^T, respectively. As can be seen from here, the constancy of the ratio $\frac{\|\gamma^T\|}{\|\gamma\|}$ also corresponds to the same definition [2].

In particular, since the expression $\|\,\mathrm{grad}\,(\|\gamma\|)\,\|$ is equal to the related ratio, satisfying the condition

$$\|\,\mathrm{grad}\,(\|\gamma\|)\,\| = c \tag{3}$$

on the curve means that it has a constant ratio. In addition to these explanations, a W- curve is known as a curve which has constant principle curvatures and was named by F. Klein and S. Lie in study [15]. It is also called as a helix or a screw line in $\mathbb{E}^n$.

Here, we consider a curve in Euclidean 3−space as a linear combination of the q−frame as

$$\gamma(s) = m_0 t(s) + m_1(s) n_q(s) + m_2(s) b_q(s). \tag{4}$$

In this equation, m_0, m_1, and m_2 are curvature functions. Based on the curvature functions, we investigate whether a unit−speed curve is of constant ratio, T−constant, N−constant or a W−curve in $\mathbb{E}^3$.

2. BASIC CONCEPTS

Suppose the unit speed regular curve is denoted by $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$.The tangent unit vector is known as $\gamma'(s) = t(s)$ also $\kappa_1 = \|\gamma''(s)\|$ is the first Frenet curvature. In the case $\kappa_1 \neq 0$,the unit normal vector field satisfies $n'(s) + \kappa_1(s) t(s) = \kappa_2(s) b(s)$. Here, b is the binormal vector field (the second principle normal) and the second Frenet curvature is indicated by κ_2 given by $b'(s) = -\kappa_2(s) n(s)$. Hence, the Serret − Frenet formulae is

$$t'(s) = \kappa_1(s) n(s)$$

$$n'(s) = -\kappa_1(s) t(s) + \kappa_2(s) b(s) \tag{5}$$

$$b'(s) = -\kappa_2(s) n(s)$$

(see, [9])

Moreover, a new frame Quasi is an alternative to Frenet frame consists by a projection vector *l*, the tangent vector *t(s)*, quasi-normal $n_q(s)$, and quasi-binormal $b_q(s)$. Then, the quasi frame vectors are given by

$$t(s) = \gamma'(s)$$

$$n_q(s) = \frac{t(s) \times l}{\|t(s) \times l\|}$$

$$b_q(s) = t(s) \times n_q(s)$$

where the projection vector l is chosen as $l = (1, 0, 0)$ (can also be chosen as (0, 1, 0), (0, 0, 1). Hence, the transition from Frenet frame vectors to Quasi frame vectors is

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{7}$$

where α is the angle between the quasi−normal vector field n_q and the principle normal vector field n. Inversely, we write

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} \tag{8}$$

Consequently, by the use of these relations, the quasi frame formulae is given by

$$\begin{bmatrix} t' \\ n_q' \\ b_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} \tag{9}$$

where $k_1 = -\langle t, n_q' \rangle$, $k_2 = -\langle t, b_q' \rangle$, $k_3 = \langle n_q', b_q \rangle$ are the quasi curvatures[6].

3. CLASSIFICATION OF CURVES WITH RESPECT TO Q - FRAME IN $\mathbb{E}^3$

Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$. Then the position vector can be considered as a combination of its quasi frame as

$$\gamma(s) = m_0(s) t(s) + m_1(s) n_q(s) + m_2(s) b_q(s) \tag{10}$$

where m_0, m_1 and m_2 are curvature functions of the curve γ (s), the derivative of the position vector is

$$\gamma'(s) = m_0' t(s) + m_0 t'(s) + m_1' n_q(s) + m_1 n_q'(s) + m_2' b_q(s) + m_2 b_q'(s) \tag{11}$$

Using Quasi frame formulas, we write

$$\gamma'(s) = m_0' t(s) + m_0(k_1 n_q(s) + k_2 b_q(s)) + m_1' n_q(s) + m_1(-k_1 t(s) + k_3 b_q(s)) + m_2' b_q(s) + m_2(-k_2 t(s) - k_3 n_q(s)) \tag{12}$$

It follows that

$$\begin{aligned} \gamma'(s) = {} & (m_0' - k_1 m_1 - k_2 m_2) t(s) \\ & + (m_1' + k_1 m_0 - k_3 m_2) n_q(s) \\ & + (m_2' + k_2 m_0 + k_3 m_1) b_q(s) \\ = {} & t(s) \end{aligned} \tag{13}$$

Hence, we obtain

$$m_0' - k_1 m_1 - k_2 m_2 = 1$$
$$m_1' + k_1 m_0 - k_3 m_2 = 0 \quad (14)$$
$$m_2' + k_2 m_0 + k_3 m_1 = 0$$

Lemma 3.1. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a unit speed regular curve in $\mathbb{E}^3$ with the vectorial equation(10). Then the position vector. γ satisfies the curvature conditions in the equation system (14).

Corollary 3.2. Suppose that, a regular unit speed curve in $\mathbb{E}^3$ is denoted by $\gamma : I \longrightarrow \mathbb{E}^3$. Then, γ corresponds to a W−curve if and only if it satisfes the differential equation

$$m_0''(s) = -(k_1^2 + k_2^2) m_0(s) + k_3(k_1 m_2(s) - k_2 m_1(s))$$

where the principle curvatures k_i, $i = 1, 2, 3$ are real constants.

Proof. Assume, the regular and unit speed curve γ satisfies the equation system (14). If k_1, k_2 and k_3 are chosen real constants in (14), using the first equation, one can write

$$m_0''(s) = k_1 m_1' + k_2 m_2' \quad (15)$$

Combining (15) with the second and third equation of (14), we obtain the result.

3.1. Constant −ratio curves with quasi frame

Definition 3.3. Suppose that a regular unit speed curve in $\mathbb{E}^3$ is denoted by $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$. Then the relation

$$\gamma = \gamma^T + \gamma^N \quad (16)$$

is valid. In case of $\|\gamma^T\| : \|\gamma^N\|$ is a real constant, then γ is defined as a constant ratio curve.In other words, for these curves, the related constant ratio can be considered as $\|\gamma^T\| : \|\gamma\|$ [1].

In addition, $grad(\|\gamma(s)\|)$ is calculated by

$$\mathrm{grad}(\|\gamma(s)\|) = \frac{d(\|\gamma(s)\|)}{ds}\gamma'(s) = \frac{<\gamma(s),\gamma'(s)>}{\|\gamma(s)\|}\gamma'(s) \quad (17)$$

where $\gamma' = T$ is the tangent vector field of γ [2].

The next consequence classifies constant - ratio curves.

Theorem 3.4. Suppose, a regular unit speed curve is denoted by $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$ *in* $\mathbb{E}^n$. Then γ is of constant ratio satisfying $\|\gamma^T\| : \|\gamma\| = c \Leftrightarrow \|grad\rho\| = c$ where c is a real constant.

Especially, we know that for a curve of constant ratio we have $\|grad(\|\gamma(s)\|)\| = c \leq 1$.

The next theorem, which is a result of equation (17), is important in classifying constant ratio curves.

Theorem 3.5.[4] For a regular unit speed curve $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$, $\|grad(\|\gamma(s)\|)\| = c \ (c = const.)$ if and only if the following three statements are valid:

(i) $\|grad\rho\| = 0 \Leftrightarrow \gamma(I)$ is included in a hypersphere whose center is the origin.
(ii) $\|grad\rho\| = 1 \Leftrightarrow \gamma(I)$ is congruent to line segment (pass through the origin).
(iii) $\|grad\rho\| = c \Leftrightarrow \rho = \|\gamma(s)\| = cs$, $0 < c < 1$

The following result provides a classification of constant ratio curves according to quasi frame in$\mathbb{E}^3$:

Proposition 3.6. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a unit speed regular curve according to q − frame in $\mathbb{E}^3$. In case of the curve is of constant ratio then its position vector is given by

$$\gamma(s) = c^2 s t(s) + \frac{(c^2-1)k_1 \pm k_2\sqrt{(k_1^2+k_2^2)(1-c^2)c^2s^2-(c^2-1)^2}}{k_1^2+k_2^2} n_q(s) + \frac{(c^2-1)k_2 \pm k_1\sqrt{(k_1^2+k_2^2)(1-c^2)c^2s^2-(c^2-1)^2}}{k_1^2+k_2^2} b_q(s) \quad (18)$$

Proof. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a unit speed regular curve given by the vectorial equation (10). Then the equation system(14) holds. If the curve is of constant ratio, then

$$\frac{\|\gamma^T\|}{\|\gamma\|} = c = \text{const.}$$
$$\frac{m_0}{cs} = c$$
$$m_0 = c^2 s$$

Putting $m_0 = c^2 s$ in (14), multiplying the second equation of (14) with m_1 and multiplying the third equation of (14) with m_2, we get

$$c^2 s(k_1 m_1 + k_2 m_2) + m_1' m_1 + m_2' m_2 = 0$$

Using the frst equation of (14), we yield

$$m_1^2 + m_2^2 = c^2 s^2 (1 - c^2) \quad (19)$$

By the use of the first equation of (14) and substituting $m_2 = \frac{c^2-1-k_1 m_1}{k_2}$ into (19), we obtain

$$m_1 = \frac{(c^2-1)k_1 \pm k_2\sqrt{(k_1^2+k_2^2)(1-c^2)c^2s^2-(c^2-1)^2}}{k_1^2+k_2^2}$$

and

$$m_2 = \frac{(c^2-1)k_2 \pm k_1\sqrt{(k_1^2+k_2^2)(1-c^2)c^2s^2-(c^2-1)^2}}{k_1^2+k_2^2}$$

This completes the proof.

3.2. T − Constant curves with quasi frame

Defnition 3.7. Suppose, a regular unit speed curve in $\mathbb{E}^3$ is denoted by $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$. If the length of the tangent component of the curve ($\|\gamma^T\|$) is constant, then the curve is called T − constant curve [2]. Especially, if $\|\gamma^T\| = 0$, then the curve is called as T − constant curve of first kind, if not, second kind [10].

Corollary 3.8. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve according to q-frame in Euclidean 3 − space. Then, it is a T-constant curve of first kind if and only if the curvature functions m_1 and m_2 satisfies:

$$m_1^2 + m_2^2 = c_1 \tag{20}$$

Proof. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed reguler curve given by the vectorial equation (10). The equation system (14) is hold if γ is T − constant curve of first kind, then $m_0 = 0$ in (14):

$$k_1 m_1 + k_2 m_2 = -1 \tag{21}$$

$$m_1' = k_3 m_2 \tag{22}$$

$$m_2' = -k_3 m_1 \tag{23}$$

By the use of (22) and (23), we write

$$\frac{m_1'}{m_1} = -\frac{m_2'}{m_2} = k_3$$

$$m_1' m_1 + m_2' m_2 = 0$$

Therefore, we obtain

$$m_1^2 + m_2^2 = c_1$$

This completes the proof.

Proposition 3.9. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve according to q − frame in Euclidean 3 − space. It is congruent to T − constant curve of first kind if and only if it has the parameterization

$$\gamma(s) = \frac{-k_1 \pm k_2\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2} n_q(s) + \frac{-k_2 \pm k_1\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2} b_q(s) \tag{24}$$

where c_1 is a real constant.

Proof. Let γ be a T − constant curve of first kind. Then, the equation (20) is satisfied:

$$m_2^2 = c_1 - m_1^2 \tag{25}$$

With the help of (21), one can put $m_2 = -\frac{1 + k_1 m_1}{k_2}$ into (25). Then, the curvature functions are

$$m_1 = \frac{-k_1 \pm k_2\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2} \tag{26}$$

and

$$m_2 = \frac{-k_2 \pm k_1\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2} \tag{27}$$

This completes the proof.

Theorem 3.10. Let γ be a T − constant curve of first kind. Then the quasi curvatures k_1, k_2, k_3 satisfies the relation:

$$\left[\frac{-k_1 \pm k_2\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2}\right]' = k_3 \left[\frac{-k_2 \pm k_1\sqrt{(k_1^2+k_2^2)c_1 - 1}}{k_1^2+k_2^2}\right] \tag{28}$$

Proof. By putting (26) and (27) into (22), we obtain the desired result.

Corollary 3.11. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve according to q − frame in Euclidean 3 − space. Then it is a T − constant curve of second kind if and only if the curvature functions m_1 and m_2 satisfies:

$$2c_1 s + c_2 = m_1^2 + m_2^2 \tag{29}$$

Proof. Let γ be a unit speed regular curve given by the vectorial equation(10). If it is T − constant curve of second kind, $m_0 = c_1$. Hence, the equation system (14) turns into

$$k_1 m_1 + k_2 m_2 = -1 \tag{30}$$

$$m_1' = k_3 m_2 - k_1 c_1 \tag{31}$$

$$m_2' = -k_3 m_1 - k_2 c_1 \tag{32}$$

Multipiying (31) with m_1 and multiplying (32) with m_2, sum of them are

$$m_1 + m_2' m_2 = -c_1(k_1 m_1 + k_2 m_2) \tag{33}$$

Therefore, by the use of (30), we yield

$$m_1' m_1 + m_2' m_2 = c_1$$

and

$$m_1^2 + m_2^2 = 2c_1 s + c_2$$

Proposition 3.12. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve according to q − frame in Euclidean 3 − space. It is congruent to T − constant curve of second kind if and only if it has the parameterization

$$\gamma(s) = c_1 t(s) + \frac{-k_1 \pm k_2\sqrt{(k_1^2+k_2^2)(2c_1 s + c_2) - 1}}{k_1^2+k_2^2} n_q(s) + \frac{-k_2 \pm k_1\sqrt{(k_1^2+k_2^2)(2c_1 s + c_2) - 1}}{k_1^2+k_2^2} b_q(s) \tag{34}$$

where c_1, c_2 are real constants.

Proof. Let γ be a T − constant curve of second kind ($m_0 = c_1$). Then, the equation (29) is satisfied. With the help of (21), one can put $m_2 = -\frac{1 + k_1 m_1}{k_2}$ into (29). Then, the curvature functions are

$$\mathrm{m}_1 = \frac{-\mathrm{k}_1 \pm \mathrm{k}_2\sqrt{(k_1^2+k_2^2)(2c_1 s + c_2)-1}}{k_1^2+k_2^2} \tag{35}$$

and

$$\mathrm{m}_2 = \frac{-\mathrm{k}_2 \pm \mathrm{k}_1\sqrt{(k_1^2+k_2^2)(2c_1 s + c_2)-1}}{k_1^2+k_2^2} \tag{36}$$

This completes the proof.

3.3. N − Constant curves with quasi frame

Definition 3.13. Suppose, a regular unit speed curve in $\mathbb{E}^3$ is denoted by $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$. If the length of the normal component of the curve $\|\gamma^N\|$ is constant, then the curve is called N − constant curve [2]. Especially, if $\|\gamma^N\| = 0$, then the curve is called as N−constant curve of first kind, if not, second kind.

So, for a N – constant curve γ in $\mathbb{E}^3$

$$\left\|\gamma^{N}(s)\right\|^2 = \mathrm{m}_1^2(s) + \mathrm{m}_2^2(s) \tag{37}$$

becomes a constant function. Therefore, by differentiation

$$m_1'\mathrm{m}_1 + m_2'\mathrm{m}_2 = 0 \tag{38}$$

For the N − constant curves of first kind we give the following result.

Lemma 3.14. : Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve according to q − frame in Euclidean 3 − space. If it is congruent to N − constant curve, then the following equation system is hold:

$$\begin{aligned} m_0' - \mathrm{k}_1\mathrm{m}_1 - \mathrm{k}_2\mathrm{m}_2 &= 1,\\ m_1' + \mathrm{k}_1\mathrm{m}_0 - \mathrm{k}_3\mathrm{m}_2 &= 0,\\ m_2' + \mathrm{k}_2\mathrm{m}_0 + \mathrm{k}_3\mathrm{m}_1 &= 0,\\ m_1'\mathrm{m}_1 + m_2'\mathrm{m}_2 &= 0. \end{aligned} \tag{39}$$

Theorem 3.15. Let γ be a N − constant curve of first kind. Then it corresponds to a straight line.

Proof. Let γ be a N − constant curve of first kind. Then

$$m_1^2 + m_2^2 = 0$$

Hence, we write $\mathrm{m}_1 = 0$, $\mathrm{m}_2 = 0$. The position vector is given by $\gamma(s) = \mathrm{m}_0 \mathrm{t}(s)$. Since the curve is along to its tangent, it corresponds to a straight line.

Proposition 3.16. Let $\gamma : I \to \mathbb{E}^3$ be a unit speed regular curve with respect to quasi frame in $\mathbb{E}^3$. Then it is congruent to N − constant curve of second kind if and only if it is a T − constant curve of first kind or the position vector is given by

$$\gamma(s) = (s+b)\mathrm{t}(s) + \frac{k_1^2+k_2^2}{k_1'\mathrm{k}_2 - \mathrm{k}_1 k_2' - (k_1^2+k_2^2)\mathrm{k}_3}\mathrm{n}_q(s) + \frac{-\mathrm{k}_1(k_1^2+k_2^2)}{\mathrm{k}_2(k_1'\mathrm{k}_2 - \mathrm{k}_1 k_2') - (k_1^2+k_2^2)\mathrm{k}_3}\mathrm{b}_q(s)$$

Proof. Let γ be a N − constant curve of second kind. Then the equation system (39) is satisfied. Multiplying the second equation of (39) with m_1 and multiplying the third equation of (39) with m_2, the sum of these relations are

$$\mathrm{m}_0(\mathrm{k}_1\mathrm{m}_1 + \mathrm{k}_2\mathrm{m}_2) = 1$$

Using the first equation of (39), we obtain $\mathrm{m}_0\,(m_0' - 1) = 0$. If $\mathrm{m}_0 = 0$, then it corresponds to a T − constant curve of first kind.

If $m_0' = 1$, then putting $\mathrm{m}_0 = s + b$ into (39) we get

$$\mathrm{k}_1\mathrm{m}_1 + \mathrm{k}_2\mathrm{m}_2 = 0$$

Substituting $\mathrm{m}_2 = \frac{-\mathrm{k}_1\mathrm{m}_1}{\mathrm{k}_2}$ into the second equation of (39), we write

$$m_1' = \frac{-\mathrm{k}_1\mathrm{k}_3}{\mathrm{k}_2}\mathrm{m}_1 - \mathrm{k}_1(s+b) \tag{40}$$

Also, substituting $\mathrm{m}_2 = \frac{-\mathrm{k}_1\mathrm{m}_1}{\mathrm{k}_2}$ into the third equation of (39), we write

$$\left[\frac{-\mathrm{k}_1\mathrm{m}_1}{\mathrm{k}_2}\right]' + \mathrm{k}_2(s+b) + \mathrm{k}_3\mathrm{m}_1 = 0$$

$$\left[\frac{-\mathrm{k}_1}{\mathrm{k}_2}\right]'\mathrm{m}_1 - \frac{\mathrm{k}_1}{\mathrm{k}_2}m_1' + \mathrm{k}_2(s+b) + \mathrm{k}_3\mathrm{m}_1 = 0 \tag{41}$$

Combining (40) and (41), we yield

$$\mathrm{m}_1 = \frac{k_1^2 + k_2^2}{k_1'\mathrm{k}_2 - \mathrm{k}_1 k_2' - (k_1^2 + k_2^2)\mathrm{k}_3}$$

Since $\mathrm{m}_1 = -\frac{\mathrm{k}_1}{\mathrm{k}_2}\mathrm{m}_1$, then

$$\mathrm{m}_2 = \frac{-\mathrm{k}_1(k_1^2 + k_2^2)}{\mathrm{k}_2(k_1'\mathrm{k}_2 - \mathrm{k}_1 k_2' - (k_1^2 + k_2^2)\mathrm{k}_3)}$$

This completes the proof.

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