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A NOTE OF THE COMBINATORIAL INTERPRETATION OF THE PERRIN AND TETRARRIN SEQUENCE

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Abstract. The present study carries out an investigation around the Perrin and Tetrarrin numbers, allowing a combinatorial interpretation for these sequences. Furthermore, it is possible to establish a study around the respective polynomial numbers of Perrin and Tetrarrin, using the bracelet method. With this, we have the definition of combinatorial models of these numbers, contributing to the evolution of these sequences with their respective combinatorial approaches. As a conclusion, there is a discussion of theorems referring to the combinatorial models of these sequences, allowing the study of the mathematical advancement of these numbers.

1. Introduction

The present word aims to introduce new interpretations for the Perrin sequence, its extension and polynomial forms. In fact, works in the literature containing the existence of recent works are identified, involving new combinatorial approaches of recurrent numerical sequences [1, 2, 3, 4, 8]. With this, it is possible to observe forms of visualization of the terms of these respective studied sequences.

Based on this, a combinatorial interpretation is performed for the sequence of Perrin, Tetrarrin and polynomial forms, based on the works of Tedford (2019) [5] and Vieira (2020) [7].

The Perrin sequence is closely related to the Padovan sequence. In a similar way as with the Fibonacci and Lucas sequence. With this, it is worth highlighting the work of Benjamin and Quinn (2003) [1] in which they carried out a study around the combinatorial model of Fibonacci and Lucas, investigating Lucas bracelets. So, the n-bracelet is defined as being a cover of a circular n-board. Lucas sequence has its combinatorial interpretation by means of bracelets, as l_n being the number of ways to tile a circular board composed of n cells marked with squares and 1 x 2

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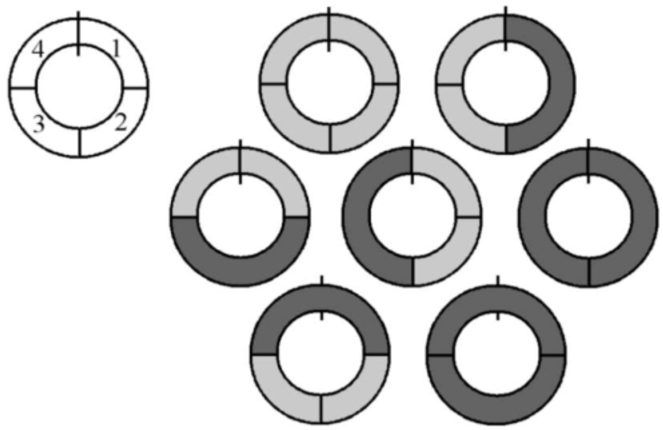
Figure 1. Size 4 Lucas bracelets. Source: Benjamin and Quinn[1]



dominoes. figure:lucas the number of tiles on Lucas bracelet of size 4, that is, l_4, obtaining a total of 7 ways to tile the bracelet.

Based on this, the work of Tedford (2019) [5] is presented, in which Padovan's combinatorial model is defined, based on a construction rule with the pieces: blue dominoes of size 1 x 2, gray triminoes of size 1 x 3 and green tetramino of size 1 x 4, all with weight 1. The particular rules mentioned are defined for the theorem concerning Padovan tiling [5].

In Figure 2, on the left side, some examples are provided in order to fill in the n-board corresponding to the Padovan sequence. On the right side are the terms corresponding to the Padovan numbers. With this, it is possible to perceive the term p_n as being the amount of tile shapes on the n-board, following the aforementioned rules, determines the relationship: $p_n = P_n, n \geqslant 0$.

In view of this, the bracelets of Perrin, Tetrarrin and their polynomial forms will be defined in a primordial way in this research, introducing the combinatorial model of Perrin, Tetrarrin, polynomial of Perrin and polynomial of Tetrarrin.

2. The Perrin sequence and its polynomial form

The Perrin sequence is a third-order, numerically recurring linear recurrent sequence given by the recurrence: $R_n = R_{n-2} + R_{n-3}, R_0 = 3, R_1 = 0, R_2 = 2, n \geq 3$ [9]. These numbers have a close relationship with the Padovan sequence, $\{P_n\}$, differing in their initial values, which are given by: $P_0 = P_1 = P_2 = 1$ [7]. The Tridovan sequence ($\{T_n\}$), for its part, was defined by Vieira (2020) [7], as being a fourth order sequence, derived from the Padovan sequence with recurrence $T_n = T_{n-2} + T_{n-3} + T_{n-4}$ and initial values given by $T_0 = 1, T_1 = 0, T_2 = T_3 = 1$. Thus, in this work an extension of the Perrin numbers is carried out, naming the Tetrarrin sequence, Te_n (fourth order) and its polynomial form, which will be discussed later.

FIGURE 2. Padovan tiling. Source: Adapted from [5].

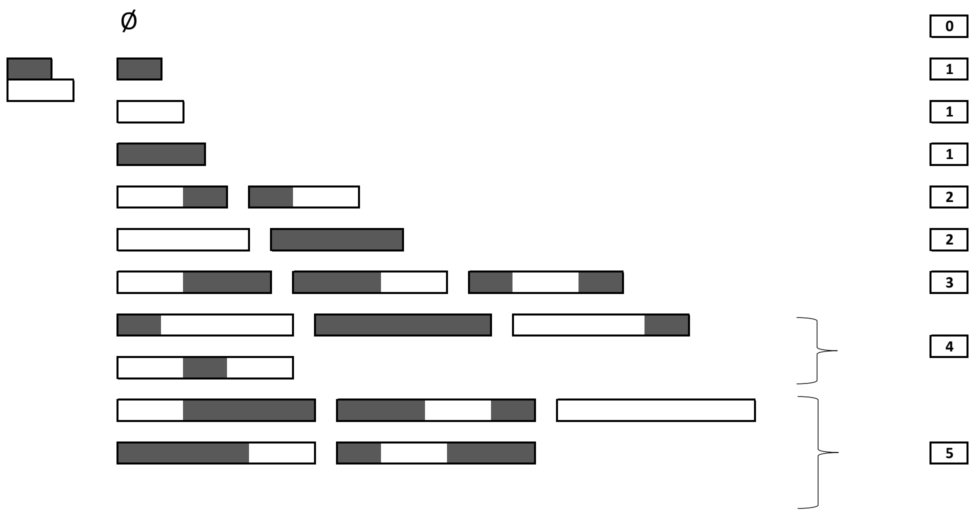


TABLE 1. First terms of the Perrin sequence. Source: Prepared by the authors.

R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
3	0	2	3	2	5	5	7	10	12	17

The Perrin polynomial numbers represent the polynomial form of the Perrin sequence.

Definition 2.1. For $x \in \mathbb{Z}$, the Perrin polynomial is defined, with $n \geqslant 3, n \in \mathbb{N}$, by recurrence:

$$R_n(x) = xR_{n-2}(x) + R_{n-3}(x),$$

with $R_0(x) = 3, R_1(x) = 0, R_2(x) = 2x$.

Thus, the first terms presented in Table 2.

TABLE 2. Perrin's first ten polynomial terms. Source: Prepared by the authors.

n	$R_n(x)$
0	3
1	0
2	$2x$
3	3
4	$2x^2$
5	$5x$
6	$2x^3+3$
7	$7x^2$
8	$2x^4+8x$
9	$9x^3+3x$

Yilmaz and Taskara (2013) [12] enabled an arithmetic relationship between the Padovan and Perrin sequences, through the equation: $R_n = 3P_{n-5} + 2P_{n-4}$.

3. The Tetrarrin sequence and its polynomial form

Based on the study by Vieira (2020) [7], who carried out an extension of the Padovan sequence, expanding the order of this sequence and defining new sequences arising from the Padovan numbers, we have the study for the Perrin numbers. With this, an extension of the Perrin sequence is performed, defining the Tetrarrin sequence.

The Tetrarrin sequence is a linear and recurrent sequence of the fourth order, primarily studied in this research.

Definition 3.1. The Tetrarrin sequence, represented by $Te_{(n)}$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:

$$Te_{(n)} = Te_{(n-2)} + Te_{(n-3)} + Te_{(n-4)},$$

with the following initial values: $Te_{(0)} = 3$, $Te_{(1)} = 0$, $Te_{(2)} = 2$ and $Te_{(3)} = 3$.

Thus, we have the first terms of this sequence as being in Table 3.

TABLE 3. First terms of the Tetrarrin sequence. Source: Prepared by the authors.

Te_0	Te_1	Te_2	Te_3	Te_4	Te_5	Te_6	Te_7	Te_8	Te_9	Te_{10}
3	0	2	3	5	5	10	13	20	28	43

Based on Yilmaz and Taskara (2013) [12], in which they presented a relationship between the Padovan sequence and Perrin, we then sought to obtain a linear combination of the terms of the Tetrarrin sequence ($Te_{(n)}$) and Tridovan ($T_{(n)}$). Taking as a premise that this linear combination is possible, the following system of equations was modeled: $Ax = y$, presenting the following definitions:

$A = \begin{bmatrix} T_{(0)} & T_{(1)} & T_{(2)} & T_{(3)} \\ T_{(1)} & T_{(2)} & T_{(3)} & T_{(4)} \\ T_{(2)} & T_{(3)} & T_{(4)} & T_{(5)} \end{bmatrix}$, $y = \begin{bmatrix} Te_{(4)} \\ Te_{(5)} \\ Te_{(6)} \\ Te_{(7)} \end{bmatrix}$ and x is a vector of coefficients satisfying the system. Thus, it was possible to obtain the relation:

$$Te_{(n)} = 3T_{(n-4)} + 3T_{(n-3)} + 4T_{(n-2)}. \tag{3.1}$$

Since other identities can be obtained, arising from arithmetic operations on the mathematical relation presented in Equation 3.1, we have: $Te_{(n)} = 2T_{(n-2)} + 3T_{(n-3)} + 3T_{(n-4)}$. From there, Tetrarrin's bracelet, $te_{(n)}$, will be defined in the next section.

So, based on the extension of the polynomial Padovan sequence, which is called the polynomial Tridovan, based on the definitions established by [10, 6, 11], thus defining the polynomial sequence of Tetrarrin.

Definition 3.2. The Tetrarrin polynomial sequence, $Te_{(n)}(x)$, satisfies the following recurrence formula, for $n \in \mathbb{N}$ and $n \geqslant 4$.

$$Te_{(n)}(x) = x^2 Te_{(n-2)}(x) + xTe_{(n-3)}(x) + Te_{(n-4)}(x),$$

with the initial terms: $Te_{(0)}(x) = 3, Te_{(1)}(x) = 0, Te_{(2)}(x) = 2x^2, Te_{(3)}(x) = 3x$.

Thus, we have the Table 4 with the first terms of the Tetrarrin polynomial sequence.

TABLE 4. First ten polynomial terms of Tetrarrin. Source: Prepared by the authors.

n	$Te_{(n)}(x)$
0	3
1	0
2	$2x^2$
3	$3x$
4	$2x^4 + 3$
5	$5x^3$
6	$2x^5 + 8x^2$
7	$7x^5 + 6x$
8	$2x^7 + 15x^4 + 3$
9	$7x^7 + 2x^6 + 13x^3 + 6x^2$

With this, the relationship between the polynomial sequences of Tridovan and Tetrarrin is investigated, through the resolution of linear systems, obtaining:

$$Te_{(n+2)}(x) = 2T_{(n-2)}(x) + 3T_{(n-3)}(x) + 3T_{(n-4)}(x) \tag{3.2}$$

Given this, the study of Tetrarrin's polynomial combinatorial model can be established.

4. The combinatorial approach

In view of the definitions and discussions of the sequences, a study of their combinatorial interpretations is carried out.

Let r_n be the amount of coverage of a circular board with n positions labeled clockwise, using blue curved dominoes and gray curved triminoes. It is called a n-bracelet, a covering of a circular n-board. It should be noted that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. Note that this definition can be extended in the case of a trimino in positions $(n-1, n, 1)$ or $(n, 1, 2)$.

Theorem 4.1. *For $n \geqslant 2$, the possible Perrin bracelets of size 1 x n with blue curved dominoes and gray curved triminoes, all weighing 1 is given by: $r_n = R_n$, where r_n is the number of $n-$Perrin bracelets and R_n is the nth term of the Perrin sequence.*

Proof. A simple count shows that for $n = 5$ we have exactly 5 3-bracelets (*in phase* and *out of phase*). The last piece of a $n-$ bracelet is defined by the one that occupies, even partially, the n position. Note that this tile ends with a domino in position $(n-3, n-2)$ or position $(n-2, 1)$, or it could be a trimino in position $(1, 2, 3)$, or position $(n-2, 1, 2)$ or at position $(n-3, n-2, n)$.

In the first case, there are $n-4$ positions left that must be covered in r_{n-2} ways. In case the last piece is a trimino, there are $n-5$ positions left that must be covered in r_{n-3} ways. So $r_n = r_{n-2} + r_{n-3}$. Or equivalently, $R_n = R_{n-2} + R_{n-3}$,

Combinatorial Identity Proof: $R_n = 3P_{n-5} + 2P_{n-4}$. Consider $n-$bracelets. These can be of two types, *in phase* and *out of phase*. A bracelet *in phase* can be stretched into a n cover, so there are $P_{n-2} = P_{n-4} + P_{n-5}$ n bracelets *in phase*. For the case of a bracelet out of phase, it is known that either there is a blue domino in position $(n-2, 1)$ or a gray trimino among positions $(n-3, n-2, 1)$ $(n-2, 1, 2)$. In the first case, the n bracelet can be stretched into a $(n-4)$ cover. In the second case, the n bracelet can be stretched into a $(n-5)-$ cover. Therefore, the number of bracelets out of phase is equal to $P_{n-4} + 2P_{n-5}$. The result follows, since every bracelet is either *in phase* or *out of phase*, soon: $R_n = P_{n-4} + P_{n-5} + P_{n-4} + 2P_{n-5} = 3P_{n-5} + 2P_{n-4}$. □

To exemplify the Theorem 4.1, we have Figure 3 with cases from r_2 to r_5.

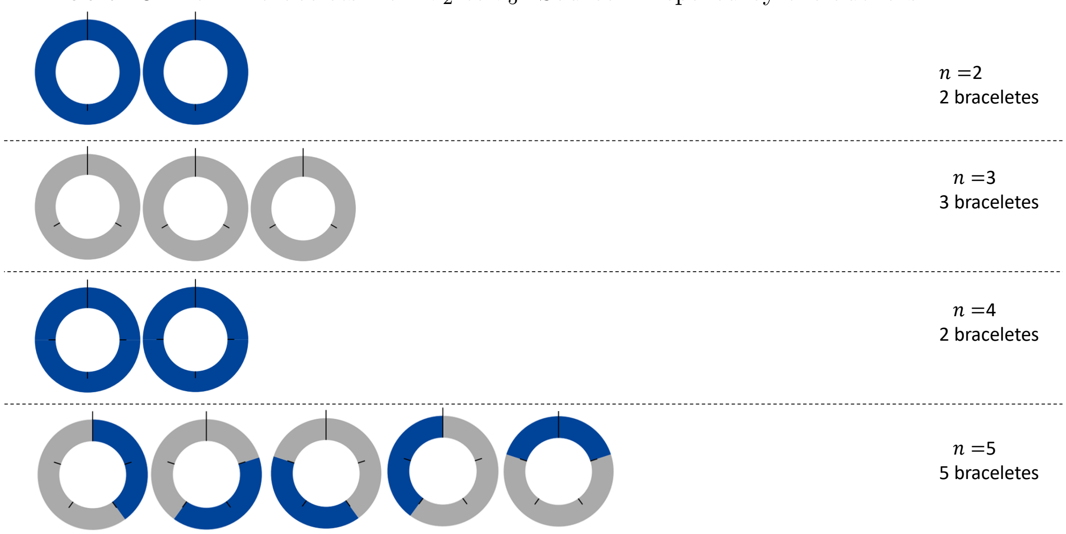
Note that for the initial case r_2, we have 2 bracelets, representing the R_2 term of the Perrin sequence. For r_3, there are 3 bracelets rotated, representing the R_3 of the sequence. For r_4, we have the amount of 2 bracelets, representing the R_4 of the sequence. For r_5, we have the amount of 5 bracelets, representing the R_5 of the sequence. The other cases can be made for the reader to follow the line of reasoning of the research.

In general, a combinatorial approach determines the size of a collection of objects in two different ways. This is done during the discussion of the theorem studied, referring to Perrin's combinatorial interpretation, conditioning the last element of each side by side, with the bias of demonstrating the recursive relationship

Next, there is Perrin's polynomial combinatorial interpretation.

Define $r_n(x)$ the covering amount of a circular polynomial Perrin board with n clockwise labeled positions, using blue curved dominoes and gray curved triminoes. It is called a n-bracelet, a covering of a circular n-board. It should be noted that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise

FIGURE 3. Perrin bracelets from r_2 to r_5. Source: Prepared by the authors.



it is said to be *in phase*. Note that this definition can be extended in the case of a trimino in positions $(n-1, n, 1)$ or $(n, 1, 2)$.

Theorem 4.2. *For $n \geqslant 2$, the possible Perrin polynomial bracelets of size 1 x n with blue curved dominoes of weight x and gray curved triminoes of weight 1 is given by: $r_n(x) = R_n(x)$, where $r_n(x)$ is the number of $n-$Perrin polynomial bracelets and $R_n(x)$ is the nth term of the Perrin polynomial sequence.*

Proof. The proof is analogous to Theorem. 4.1. □

To exemplify, there is Figure 4 with cases from $r_2(x)$ to $r_5(x)$.

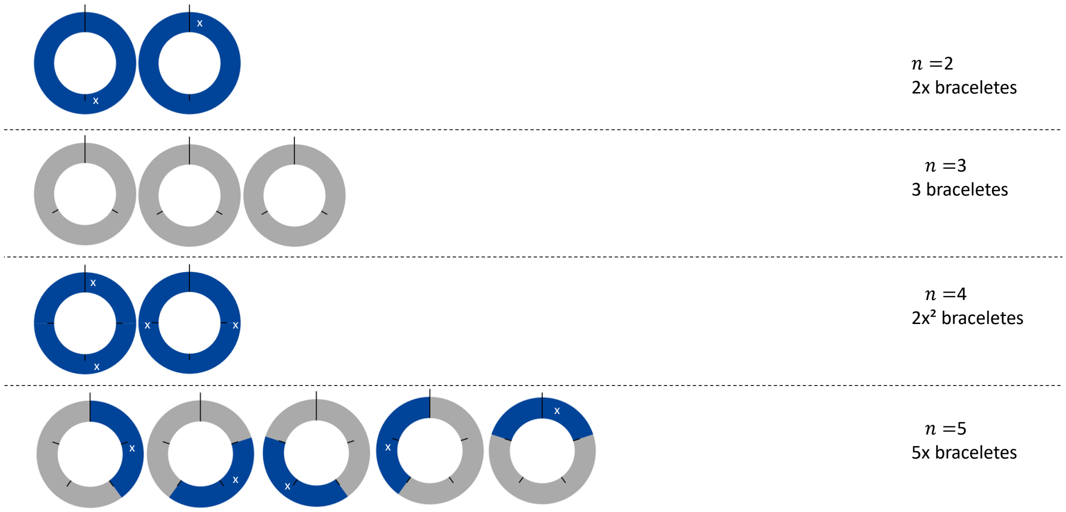
For the case $r_2(x)$, there are 2 bracelets of weight x, resulting in $2x$, representing the term $R_2(x)$. For the case $r_3(x)$, we have 3 bracelets weighing 1, resulting in 3 and representing the term $R_3(x)$. For the case $r_4(x)$, we have 2 bracelets of weight x^2, resulting in $2x^2$ and representing the term $R_4(x)$. For the case $r_5(x)$, we have 5 bracelets of weight x, resulting in $5x$ and representing the term $R_5(x)$.

Next, there is the combinatorial interpretation of Tetrarrin.

Set te_n the amount of coverage of a circular board with n positions labeled clockwise, using the pieces: blue curved dominoes, gray curved triminoes and green curved tetraminos. It is called a n-bracelet, a covering of a circular n-board. Note that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. The present definition is also valid for the cases of gray curved tetraminoes in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminoes in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$. The blue curved domino rotates only twice. The gray curved trimino and the green curved tetramino rotate only three times.

Theorem 4.3. *For $n \geqslant 2$, the possible bracelets of size 1 x n with blue curved dominoes, gray curved triminoes and green curved tetraminos, all with weight 1 is*

FIGURE 4. Polynomial Perrin bracelets from $r_2(x)$ to $r_5(x)$. Source: Prepared by the authors.



given by: $te_n = Te_n$, *where* te_n *is the number of* $n-$ *Tetrarrin bracelets and* Te_n *is the nth term of the Tetrarrin sequence.*

Proof. The proof is analogous to Theorem 4.1. □

To exemplify, there is Figure 5 with cases from te_2 to te_5.

For the case te_2, there are 2 bracelets, representing the term Te_2. For the case te_3, 3 bracelets are accounted for, representing the term Te_3. For the case te_4, 5 bracelets are accounted for, representing the term Te_4. For the case te_5, 5 bracelets are counted, representing the term Te_5.

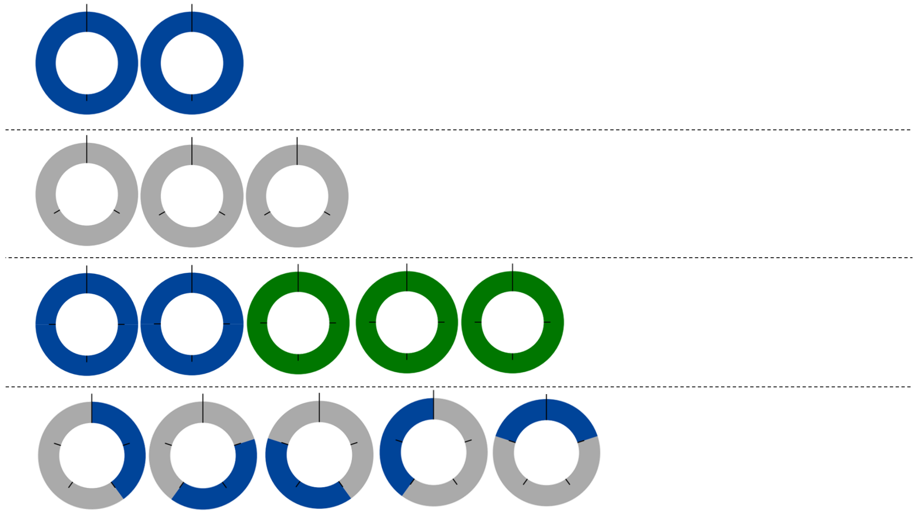
Next, there is the polynomial combinatorial interpretation of Tetrarrin.

To do this, define $te_n(x)$ the amount of coverage of a circular board with n positions labeled clockwise, using blue curved dominoes of weight x^2, gray curved triminoes of weight x and tetraminos weight green curves 1.

In this way, the previously mentioned denomination referring to the n-bracelet follows, bearing in mind that it has a covering of a circular n-tray. Similarly, one can say that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. It is noteworthy that this definition can be extended to the cases of gray curved tetraminos in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminos in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$. The blue curved domino rotates only twice. The gray curved trimino and the green curved tetramino rotate only three times.

Theorem 4.4. *For* $n \geqslant 2$, *the possible bracelets of size 1 x n with curved blue dominoes of weight* x^2, *curved gray triminoes of weight* x *and curved green tetraminoes*

FIGURE 5. Tetrarrin bracelets from te_2 to te_5. Source: Prepared by the authors.



of weight 1, is given by: $te_n(x) = Te_n(x)$, *where* $te_n(x)$ *is the number of* $n-$ *Tetrarrin polynomial bracelets and* $Te_n(x)$ *is the nth term of the polynomial sequence of Tetrarin.*

Proof. The proof follows analogous to the validation of the Theorem 4.3. □

An example of the model is Figure 6 for cases from $n = 2$ to $n = 5$.

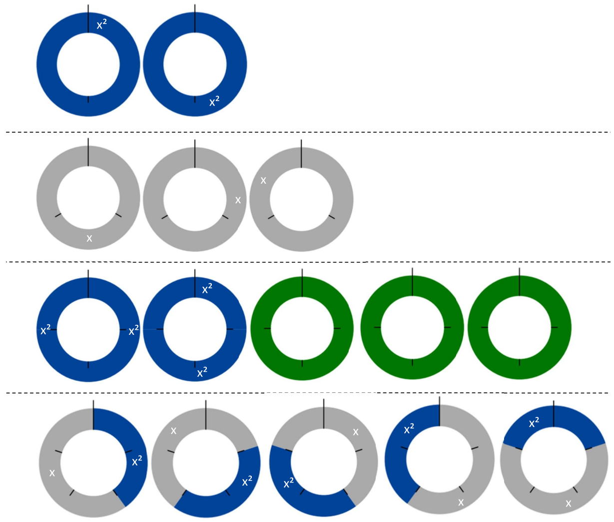
For the case $te_2(x)$, there are 2 bracelets of weights x^2, resulting in $2x^2$ and representing the term $Te_2(x)$. For the case $te_3(x)$, 3 weight bracelets x are accounted for, resulting in $3x$ and representing the term $Te_3(x)$. For the case $te_4(x)$, 5 bracelets are accounted for, resulting in $3x^4+3$ and representing the term $Te_4(x)$. For the case $te_5(x)$, 5 bracelets are accounted for, resulting in $5x^3$ and representing the term $Te_5(x)$.

5. Conclusions

From the relationship between the Padovan and Perrin sequences, it was possible to deepen Padovan's combinatorial approach in order to introduce Perrin's combinatorial interpretation. Thus, the present study introduced Perrin's combinatorial approaches and its Tetrarrin extension. Furthermore, the combinatorial approaches of Perrin and Tetrarrin polynomial sequences were obtained, contributing to the study of linear and recurrent sequences.

In fact, combinatorial models make it possible to integrate the study of sequences in the area of combinatorics, allowing a visualization of the numbers of these investigated sequences.

FIGURE 6. Tetrarrin polynomial bracelets from $te_2(x)$ to $te_5(x)$.
Source: Prepared by the authors.



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The author(s) declared that no conflict of interest or common interest

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that

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