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RESEARCH ARTICLE

A Trapezoid Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces

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ABSTRACT
Generalized trapezoid and trapezoid rules play an important role in approximating the Lebesgue integral in the case of scalar-valued functions defined on a finite interval. Motivated by this reason, in this paper we provided several norm error bounds in approximation the integral of continuous function of the convex combination of some tensorial products in terms of the corresponding tensorial generalized and trapezoid rules. The case of continuously differentiable functions is analysed in detail in the case when the derivative is bounded on a finite interval. Related results for the case when the absolute value of the derivative is convex is also provided. The important particular case for the operator exponential function is also considered and the corresponding norm inequalities revealed.

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1. INTRODUCTION

Assume that the function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then we have the *generalized trapezoid inequality*, see for instance Cerone, P., Dragomir, S. S. (2000)

$$\left|\frac{(b-x) f(b)+(x-a) f(a)}{b-a}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2}\right]\left\|f^{\prime}\right\|_{\infty}(b-a), \tag{1.1}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

For $x = \frac{a+b}{2}$ we get the *trapezoid inequality*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq \frac{1}{4}\left\|f^{\prime}\right\|_{\infty}(b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; where $E_i(\cdot)$ is the spectral measure of A_i for $i = 1, ..., k$; by following Araki, H.,

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Hansen, F. (2000), we define

$$f(A_1,...,A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1,...,\lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction Araki, H., Hansen, F. (2000) extends the definition of Korányi Korányi, A. (1961) for functions of two variables and have the property that

$$f(A_1,...,A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is known that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then (Furuta, T., Mićić Hot, J., Pečarić, J., Seo, Y. 2005, p. 173)

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B)\#(B \otimes A).$$

In 2007, S. Wada Wada, S. (2007) obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A)\end{aligned} \tag{1.5}$$

for $A, B > 0$ and $\alpha \in [0,1]$. For other similar results, see Ando, T. (1979), Aujila, J. S., Vasudeva, H. L. (1995) and Ebadian, A., Nikoufar, I., Gordji, M. E. (2011)-Kitamura, K., Seo, Y. (1998).

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned}&\left\|(1-\lambda) f(A) \otimes 1 + \lambda 1 \otimes f(B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du\right\| \\ &\leq \|f'\|_{I,\infty}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1 \otimes B - A \otimes 1\|\end{aligned}$$

for $\lambda \in [0,1]$. In particular, we have the trapezoid inequality

$$\begin{aligned}&\left\|\frac{f(A) \otimes 1 + 1 \otimes f(B)}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du\right\| \\ &\leq \frac{1}{4}\|f'\|_{I,\infty}\|1 \otimes B - A \otimes 1\|.\end{aligned}$$

2. MAIN RESULTS

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$, the Banach algebra of all bounded linear operators on H.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 2.1. *Assume A and B are selfadjoint operators with* $\operatorname{Sp}(A) \subset I$ *and* $\operatorname{Sp}(B) \subset J$. *Let f, h be continuous on I, g, k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$, then*

$$\begin{aligned} &(f(A) \otimes 1 + 1 \otimes g(B)) \varphi(h(A) \otimes 1 + 1 \otimes k(B)) \\ &= \int_I \int_J (f(t) + g(s)) \varphi(h(t) + k(s)) \, dE_t \otimes dF_s, \end{aligned} \tag{2.6}$$

where A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

For natural number $n \geq 1$ we have

$$\begin{aligned} \mathcal{K} &:= \int_I \int_J (f(t) + g(s))(h(t) + k(s))^n \, dE_t \otimes dF_s \\ &= \int_I \int_J (f(t) + g(s)) \sum_{m=0}^{n} C_n^m [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \int_I \int_J (f(t) + g(s)) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \left[\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \right. \\ &\left. + \int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} \, dE_t \otimes dF_s \right]. \end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned} &\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= f(A) [h(A)]^m \otimes [k(B)]^{n-m} = (f(A) \otimes 1) ([h(A)]^m \otimes [k(B)]^{n-m}) \\ &= (f(A) \otimes 1) ([h(A)]^m \otimes 1) (1 \otimes [k(B)]^{n-m}) \\ &= (f(A) \otimes 1) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m} \end{aligned}$$

and

$$\int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= [h(A)]^m \otimes (g(B) [k(B)]^{n-m}) = (1 \otimes g(B)) ([h(A)]^m \otimes [k(B)]^{n-m})$$
$$= (1 \otimes g(B)) ([h(A)]^m \otimes 1) (1 \otimes [k(B)]^{n-m})$$
$$= (1 \otimes g(B)) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m},$$

with $h(A) \otimes 1$ and $1 \otimes k(B)$ commutative.

Therefore

$$\mathcal{K} = (f(A) \otimes 1 + 1 \otimes g(B)) \sum_{m=0}^{n} C_n^m (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m}$$
$$= (f(A) \otimes 1 + 1 \otimes g(B)) (h(A) \otimes 1 + 1 \otimes k(B))^n,$$

for which the commutativity of $h(A) \otimes 1$ and $1 \otimes k(B)$ has been employed.

We have the following representation result:

Theorem 2.2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$(1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \tag{2.9}$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B) du$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid identity

$$\frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \tag{2.10}$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 \left(u - \frac{1}{2}\right) f'((1-u) A \otimes 1 + u 1 \otimes B) du.$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$\int_a^b (t-x) f'(t) dt = (t-x) f(t)|_a^b - \int_a^b f(t) dt \tag{2.11}$$
$$= (b-x) f(b) + (x-a) f(a) - \int_a^b f(t) dt$$

for $a \le x \le b$ and f absolutely continuous on $[a,b]$.

If we take $x = (1-\lambda) a + \lambda b$, $\lambda \in [0,1]$ and change the variable $t = (1-u) a + ub$, then $dt = (b-a) du$ and by (2.11) we derive

$$(1-\lambda)(b-a) f(b) + \lambda (b-a) f(a) - (b-a) \int_0^1 f((1-u) a + ub) du$$
$$= (b-a)^2 \int_0^1 (u-\lambda) f'((1-u) a + ub) du,$$

namely

$$(1-\lambda) f(b) + \lambda f(a) - \int_0^1 f((1-u) a + ub) du \tag{2.12}$$
$$= (b-a) \int_0^1 (u-\lambda) f'((1-u) a + ub) du,$$

for all $a, b \in I$ and $\lambda \in [0,1]$.

Assume that A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.12) written for $b = s$, $a = t$, then we get

$$\int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.13}$$
$$= \int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u)t + us) \right] dE_t \otimes dF_s .$$

By utilizing Fubini's theorem and Lemma 2.1 we derive

$$\int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.14}$$
$$= (1-\lambda) \int_I \int_I f(s)\, dE_t \otimes dF_s + \lambda \int_I \int_I f(t)\, dE_t \otimes dF_s$$
$$- \int_0^1 \left(\int_I \int_I (f((1-u)t + us))\, dE_t \otimes dF_s \right) du$$
$$= (1-\lambda)\, 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du$$

and

$$\int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.15}$$
$$= \int_0^1 (u-\lambda) \left[\int_I \int_I (s-t) f'((1-u)t + us)\, dE_t \otimes dF_s \right] du$$
$$= \int_0^1 (u-\lambda)(1 \otimes B - A \otimes 1) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du.$$

Therefore, by (2.13)-(2.15) we get the desired identity (2.9).

We have the following generalized trapezoid inequality:

Theorem 2.3. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t \in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\left\| (1-\lambda)\, 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.16}$$
$$\leq \|1 \otimes B - A \otimes 1\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f'\|_{I,\infty}$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid inequality

$$\left\| \frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.17}$$
$$\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\| .$$

Proof. If we take the norm in the identity (2.9) and use the properties of the integral, then we get

$$\left\| (1-\lambda)\, 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.18}$$
$$= \left\| (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\|$$
$$\leq \|1 \otimes B - A \otimes 1\| \left\| \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\|$$
$$\leq \|1 \otimes B - A \otimes 1\| \int_0^1 |u-\lambda| \, \|f'((1-u) A \otimes 1 + u 1 \otimes B)\|\, du$$

for all $\lambda \in [0,1]$.

Observe that, by Lemma 2.1

$$|f'((1-u)A\otimes 1+u1\otimes B)|=\int_I\int_I |f'((1-u)t+us)|\,dE_t\otimes dF_s$$

for $u\in[0,1]$.

Note that

$$|f'((1-u)t+us)|\le \|f'\|_{I,\infty}$$

for $u\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&|f'((1-u)A\otimes 1+u1\otimes B)| \\ &=\int_I\int_I |f'((1-u)t+us)|\,dE_t\otimes dF_s\le \|f'\|_{I,\infty}\int_I\int_I dE_t\otimes dF_s=\|f'\|_{I,\infty}\end{aligned}\tag{2.19}$$

for $u\in[0,1]$. This implies that

$$\|f'((1-u)A\otimes 1+u1\otimes B)\|\le \|f'\|_{I,\infty}$$

for $u\in[0,1]$, which gives

$$\begin{aligned}&\int_0^1 |u-\lambda|\,\|f'((1-u)A\otimes 1+u1\otimes B)\|\,du \\ &\le \|f'\|_{I,\infty}\int_0^1|u-\lambda|\,du=\|f'\|_{I,\infty}\frac{(1-\lambda)^2+\lambda^2}{2} \\ &=\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f'\|_{I,\infty}\end{aligned}$$

for all $\lambda\in[0,1]$, which proves (2.16).

3. RELATED RESULTS

In this section we give some norm trapezoid inequalities under various assumptions of convexity for the absolute value of the derivative $|f'|$ on I.

Theorem 3.1. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with* $\mathrm{Sp}(A),\mathrm{Sp}(B)\subset I$*, then*

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\le \|1\otimes B-A\otimes 1\|\left[p(1-\lambda)\|f'(A)\|+p(\lambda)\|f'(B)\|\right],\end{aligned}\tag{3.1}$$

for $\lambda\in[0,1]$, where

$$p(\lambda):=\frac{1}{6}\left(2\lambda^3-3\lambda+2\right).$$

In particular, we have the trapezoid inequality

$$\begin{aligned}&\left\|\frac{1\otimes f(B)+f(A)\otimes 1}{2}-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\le \frac{1}{8}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right).\end{aligned}\tag{3.2}$$

Proof. Since $|f'|$ is convex on I, then

$$|f'((1-u)t+us)|\le(1-u)|f'(t)|+u|f'(s)|$$

for all $t,s\in I$ and $u\in[0,1]$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&\int_I\int_I |f'((1-u)t+us)|\,dE_t\otimes dF_s \\ &\le (1-u)\int_I\int_I |f'(t)|\,dE_t\otimes dF_s+u\int_I\int_I |f'(s)|\,dE_t\otimes dF_s,\end{aligned}$$

namely

$$\left|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right|\leq(1-u)\left|f'(A)\right|\otimes 1+u\left|f'(B)\right|\otimes 1 \tag{3.3}$$

for all $u\in[0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned}\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\| &\leq\left\|(1-u)\left|f'(A)\right|\otimes 1+u\left|f'(B)\right|\otimes 1\right\| \\ &\leq(1-u)\left\|f'(A)\right\|+u\left\|f'(B)\right\|\end{aligned} \tag{3.4}$$

for all $u\in[0,1]$.

By (2.18) and (3.4) we derive

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ &\leq\|1\otimes B-A\otimes 1\|\int_0^1|u-\lambda|\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\|du \\ &\leq\|1\otimes B-A\otimes 1\| \\ &\times\left[\left\|f'(A)\right\|\int_0^1|u-\lambda|(1-u)du+\left\|f'(B)\right\|\int_0^1 u|u-\lambda|du\right],\end{aligned} \tag{3.5}$$

for $\lambda\in[0,1]$.

Observe that, for $\lambda\in[0,1]$,

$$\int_0^1 u|u-\lambda|du=p(\lambda)\ \text{ and }\ \int_0^1(1-u)|u-\lambda|du=p(1-\lambda).$$

By utilizing (3.5) we derive (3.1).

We recall that the function $g:I\to\mathbb{R}$ is *quasi-convex*, if

$$g((1-\lambda)t+\lambda s)\leq\max\{g(t),g(s)\}=\frac{1}{2}(g(t)+g(s)+|g(t)-g(s)|)$$

for all $t,s\in I$ and $\lambda\in[0,1]$.

Theorem 3.2. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, then*

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ &\leq\frac{1}{2}\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right] \\ &\times\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right).\end{aligned} \tag{3.6}$$

In particular,

$$\begin{aligned}&\left\|\frac{1\otimes f(B)+f(A)\otimes 1}{2}-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ &\leq\frac{1}{8}\|1\otimes B-A\otimes 1\| \\ &\times\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right).\end{aligned} \tag{3.7}$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$\left|f'((1-u)t+us)\right|\leq\frac{1}{2}\left(\left|f'(t)\right|+\left|f'(s)\right|+\left|\left|f'(t)\right|-\left|f'(s)\right|\right|\right)$$

for all for $u\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&\int_I\int_I\left|f'((1-u)t+us)\right|dE_t\otimes dF_s \\ &\leq\frac{1}{2}\int_I\int_I\left(\left|f'(t)\right|+\left|f'(s)\right|+\left|\left|f'(t)\right|-\left|f'(s)\right|\right|\right)dE_t\otimes dF_s\end{aligned}$$

namely

$$\begin{aligned}&|f'((1-u)A\otimes 1+u1\otimes B)|\\&\leq \frac{1}{2}(|f'(A)|\otimes 1+1\otimes |f'(B)|+||f'(A)|\otimes 1-1\otimes |f'(B)||)\end{aligned}$$

for all for $u\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned}&\|f'((1-u)A\otimes 1+u1\otimes B)\|\\&\leq \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)\end{aligned}\tag{3.8}$$

for all for $u\in[0,1]$.

By (2.18) and (3.8)

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\&\leq \|1\otimes B-A\otimes 1\|\int_0^1|u-\lambda|\,\|f'((1-u)A\otimes 1+u1\otimes B)\|\,du\\&\leq \frac{1}{2}\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\\&\times(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)\end{aligned}$$

for all $\lambda\in[0,1]$ and the inequality (3.6) is proved.

4. EXAMPLES

It is known that if U and V are commuting, i.e. $UV=VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V)=\exp(V)\exp(U)=\exp(U+V).$$

Also, if U is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b\exp(tU)\,dt=U^{-1}[\exp(bU)-\exp(aU)].$$

Moreover, if U and V are commuting and $V-U$ is invertible, then

$$\begin{aligned}\int_0^1\exp((1-s)U+sV)\,ds&=\int_0^1\exp(s(V-U))\exp(U)\,ds\\&=\left(\int_0^1\exp(s(V-U))\,ds\right)\exp(U)\\&=(V-U)^{-1}[\exp(V)-\exp(U)].\end{aligned}$$

Since the operators $U=A\otimes 1$ and $V=1\otimes B$ are commutative and if $1\otimes B-A\otimes 1$ is invertible, then

$$\begin{aligned}&\int_0^1\exp((1-u)A\otimes 1+u1\otimes B)\,du\\&=(1\otimes B-A\otimes 1)^{-1}[\exp(1\otimes B)-\exp(A\otimes 1)].\end{aligned}$$

If A, B are selfadjoint operators with $\mathrm{Sp}(A),\mathrm{Sp}(B)\subset[m,M]$, with $m<M$ real numbers and $1\otimes B-A\otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}&\|(1-\lambda)\exp(A)\otimes 1+\lambda 1\otimes\exp(B)\\&-(1\otimes B-A\otimes 1)^{-1}[\exp(1\otimes B)-\exp(A\otimes 1)]\|\\&\leq\exp(M)\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|,\end{aligned}\tag{4.1}$$

for $\lambda\in[0,1]$.

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 3.1 we get

$$\begin{aligned} &\|(1-\lambda)\exp(A)\otimes 1 + \lambda 1\otimes \exp(B) \\ &-(1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)]\| \\ &\le \frac{1}{2}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\| \\ &\times(\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|) \end{aligned} \tag{4.2}$$

for $\lambda \in [0, 1]$.

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