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RESEARCH ARTICLE

ζ(Ric)-vector fields on doubly warped product manifolds

S. Gerdan Aydın[1*], M. Traore[1] and H. M. Taştan[1]

[1]Istanbul University, Faculty of Science, Department of Mathematics, Vezneciler, 34134, Istanbul, Türkiye

ABSTRACT
We investigate ζ(Ric)-vector fields on doubly warped product manifolds. We obtain some results when the vector field is also ζ(Ric) on factor manifolds. We prove that if a vector field is a ζ(Ric)-vector field on a doubly warped product manifold, it is also a ζ(Ric)-vector field on the factor manifolds under certain conditions. Also, we show that a vector field on a doubly warped product manifold can be a ζ(Ric)-vector field with some conditions. Moreover we give two important applications of this concept in the Lorentzian settings, which are the doubly warped product generalized Robertson-Walker space-time and doubly warped product standard static space-time.

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1. INTRODUCTION

There are many special types of smooth vector fields in the literature such as Killing, conformal, concircular, etc. The existence of any special type of vector field can directly influence the geometry of the manifold on which it is defined. For example, any Riemannian manifold with non-zero concircular vector field is a locally warped product (see Chen (2015)). Also, the topological property of a Riemannian manifold can influence the form of a vector field defined in that manifold. For instance, every affine vector field is Killing (see Kobayashi (1995)) on a compact and orientable Riemannian manifold. Moreover, the existence of a vector field and the algebraic topological property of the manifold on which it is defined are closely related.
The notion of ζ(Ric)-vector fields was first defined by Hinterleitner and Kiosak (2008), then many geometers have studied these types of vector fields in several kinds of differentiable structures (see De et al. (2021), Hinterleitner and Kiosak (2009), Kırık and Özen Zengin (2015), Kırık and Özen Zengin (2015), Kırık and Özen Zengin (2019), Özen Zengin and Kırık (2013)).

The concept of warped product manifolds introduced by Bishop and O'Neill Bishop and O'Neill (1969) to investigate Riemannian manifolds with negative sectional curvature. This is the concept that describes the geometry of many significant relativistic space-time, which has a wide range of uses in both differential geometry and mathematical physics (Bishop and O'Neill (1969), O'Neill (1983)).

In the present paper, we consider ζ(Ric)-vector fields on doubly warped product manifolds. We obtain that if a vector field is a ζ(Ric)-vector field on a doubly warped product manifold, it is also a ζ(Ric)-vector field on the factor manifolds under certain conditions. Moreover, we show that a vector field on a doubly warped product manifold can be a ζ(Ric)-vector field with some conditions. Finally, considering ζ(Ric)-vector fields on a doubly warped product generalized Robertson-Walker space-time and doubly warped product standard static space-time, we get some results.

2. DOUBLY WARPED PRODUCT MANIFOLDS WITH ζ(RIC)-VECTOR FIELDS

A *doubly warped product* Ehrlich (1974) ${}_{f_2}M_1 \times_{f_1} M_2$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M = M_1 \times M_2$ and it has the following metric:

$$g = (f_2 \circ \sigma)^2 \sigma_1^*(g_1) + (f_1 \circ \sigma)^2 \sigma_2^*(g_2), \quad (1)$$

Corresponding Author: S. Gerdan Aydın **E-mail:** sibel.gerdan@istanbul.edu.tr
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where (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds and $f_1 \in C^\infty(M_1)$, $f_2 \in C^\infty(M_2)$. σ_1 and σ_2 are defined as canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. For $i = 1, 2$, $\sigma_i^*(g_i)$ is the pullback of g_i via σ_i. We say that f_i is a *warping function* of $({}_{f_2}M_1 \times_{f_1} M_2, g)$. If f_1 or f_2 is constant, then the manifold is a *warped product* Bishop and O'Neill (1969). Also, we get a *direct product manifold* Chen (2017) when both f_1 and f_2 are constant.

Let $({}_{f_2}M_1 \times_{f_1} M_2, g)$ be a doubly warped product manifold. In this study, the same notation will be used for a vector field and for its lift. It is also true for a metric and its pullback. Because each σ_i is a (positive) homothety, the connection is preserved. Also, we can use the same notation for a connection on M_i and for its pullback via σ_i. For $({}_{f_2}M_1 \times_{f_1} M_2, g)$, the covariant derivative formulas Ehrlich (1974) are obtained as follows:

$$\nabla_Z T = \nabla^1_Z T - g(Z,T)\nabla(\ln(f_2 \circ \pi_2)), \tag{2}$$

$$\nabla_Z W = \nabla_W Z = W(\ln(f_2 \circ \pi_2))Z + Z(\ln(f_1 \circ \pi_1))W, \tag{3}$$

$$\nabla_V W = \nabla^2_V W - g(V,W)\nabla(\ln(f_1 \circ \pi_1)), \tag{4}$$

for $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$. Here ∇ and ∇^i are the Levi-Civita connections of ${}_{f_2}M_1 \times_{f_1} M_2$ and M_i respectively, for $i \in \{1, 2\}$. Also, we use the notation $\mathfrak{L}(M_i)$ for the set of lifts of vector fields on M_i. On the other hand, we obtain $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds and their mean curvature vector fields are closed in ${}_{f_2}M_1 \times_{f_1} M_2$ Gutierrez and Olea (2012). Here $p_1 \in M_1$ and $p_2 \in M_2$.

Remark 2.1. Here, $l = \ln f_2$ (resp. $k = \ln f_1$) and for the function l (resp. k) and its pullback $l \circ \sigma_2$ (resp. $k \circ \sigma_1$), the same symbol is used from now on.

Let $\mathcal{S}$, $\mathcal{S}^1$ and $\mathcal{S}^2$ be the lifts of Ricci curvature tensors of $({}_{f_2}M_1 \times_{f_1} M_2, g)$, (M_1, g_1) and (M_2, g_2) respectively. Then, the followings are hold:

Lemma 2.2. *Blaga and Taştan (2022) Let $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$. Then, we have*

$$\mathcal{S}(Z,T) = \mathcal{S}^1(Z,T) - \tfrac{m_2}{f_1} h_1^{f_1}(Z,T) - g(Z,T)\Delta l, \tag{5}$$

$$\mathcal{S}(Z,V) = (m_1 + m_2 - 2)Z(k)V(l), \tag{6}$$

$$\mathcal{S}(V,W) = \mathcal{S}^2(V,W) - \tfrac{m_1}{f_2} h_2^{f_2}(V,W) - g(V,W)\Delta k, \tag{7}$$

where Δ is the Laplacian operator on $({}_{f_2}M_1 \times_{f_1} M_2, g)$, $m_i = dim(M_i)$ for $i \in \{1, 2\}$ and $h_1^{f_1}(Z,T) = ZT(f_1) - (\nabla^1_Z T)(f_1)$ and $h_2^{f_2}(V,W) = VW(f_2) - (\nabla^2_V W)(f_2)$.

Now, we recall the definition of $\zeta(\text{Ric})$ vector field defined by Hinterleitner and Kiosak (2008).

Definition 2.3. A vector field ζ is called $\zeta(\text{Ric})$ if for any vector field X on a Riemannian manifold (M^m, g) the equation

$$\nabla_X \zeta = \mu Q X \tag{8}$$

holds, where ∇ is the Levi-Civita connection of the metric g, Q is the Ricci operator of the Ricci tensor $\mathcal{S}$ of M and μ is a constant.

For a doubly warped product manifold, we give the main theorem about $\zeta(\text{Ric})$-vector fields as follows:

Theorem 2.4. *Let the vector field $\zeta = \zeta_1 + \zeta_2$ be $\zeta(\text{Ric})$ on $(M = {}_{f_2} M_1 \times_{f_1} M_2, g)$ for $i = 1, 2$, $\zeta_i \in \mathfrak{L}(M_i)$. Then, we have*

(i) The vector field ζ_1 is $\zeta_1(\text{Ric})$ on $M_1 \Leftrightarrow$

$$\tfrac{\mu m_2}{f_1} h_1^{f_1}(Z,T) + \Big\{\mu\Delta l + \zeta_2(l)\Big\} g(Z,T) = 0, \tag{9}$$

(ii) The vector field ζ_2 is $\zeta_2(\text{Ric})$ on $M_2 \Leftrightarrow$

$$\tfrac{\mu m_1}{f_2} h_2^{f_2}(V,W) + \Big\{\mu\Delta k + \zeta_1(k)\Big\} g(V,W) = 0, \tag{10}$$

where $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$.

Proof. Let the vector field ζ be ζ(Ric) on M. Then, we get $\mu\mathcal{S}(Z,T) = g(\nabla_Z\zeta, T)$ for all $Z, T \in \mathfrak{L}(M_1)$. From (5), we get

$$\mu\mathcal{S}(Z,T) = \mu\Big\{\mathcal{S}^1(Z,T) - \tfrac{m_2}{f_1}h_1^{f_1}(Z,T) - g(Z,T)\Delta l\Big\}.$$

Hence, from (2), we obtain

$$\mu\,\mathcal{S}^1(Z,T) - \mu\tfrac{m_2}{f_1}h_1^{f_1}(Z,T) - \mu g(Z,T)\Delta l$$

$$= g(\nabla^1_Z\zeta_1 - g(Z,\zeta_1)\nabla l, T) + g(Z(k)\zeta_2 + \zeta_2(l)Z, T).$$

Thus, we have

$$\begin{aligned}\mu\,\mathcal{S}^1(Z,T) = \;& g(\nabla^1_Z\zeta_1, T) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(Z,T) + \mu g(Z,T)\Delta l\\ & -g(Z,\zeta_1)g(\nabla l, T) + Z(k)g(\zeta_2, T) + \zeta_2(l)g(Z,T)\\ = \;& g(\nabla^1_Z\zeta_1, T) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(Z,T) + \Big\{\mu\Delta l + \zeta_2(l)\Big\}g(Z,T).\end{aligned}$$

This concludes the first assertion.

Regarding the second assertion, we have $\mu\mathcal{S}(V,W) = g(\nabla_V\zeta, W)$ for all $V, W \in \mathfrak{L}(M_2)$, since the vector field ζ is ζ(Ric) on M. Using (7), we get

$$\mu\mathcal{S}(V,W) = \mu\Big\{\mathcal{S}^2(V,W) - \tfrac{m_1}{f_2}h_2^{f_2}(V,W) - g(V,W)\Delta k\Big\}.$$

Hence, using (4) we obtain

$$\mu\,\mathcal{S}^2(V,W) - \mu\tfrac{m_1}{f_2}h_2^{f_2}(V,W) - \mu g(V,W)\Delta k$$

$$= g(\nabla^2_V\zeta_2 - g(V,\zeta_2)\nabla k, W) + g(V(l)\zeta_1 + \zeta_1(k)V, W).$$

After some calculations, we obtain

$$\begin{aligned}\mu\,\mathcal{S}^2(V,W) \;=\;& g(\nabla^2_V\zeta_2, W) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(V,W) + \mu g(V,W)\Delta k\\ & -g(V,\zeta_2)g(\nabla k, W) + V(l)g(\zeta_1, W) + \zeta_1(k)g(V,W)\\ =\;& g(\nabla^2_V\zeta_2, W) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(V,W) + \Big\{\mu\Delta k + \zeta_1(k)\Big\}g(V,W).\end{aligned}$$

Thus the assertion is hold.

Theorem 2.5. *Let the vector field $\zeta = \zeta_1 + \zeta_2$ be defined on a doubly warped product $(M = {}_{f_2}M_1 \times_{f_1} M_2, g)$, where $\zeta_i \in \mathfrak{L}(M_i)$, for $i = 1, 2$. If*

$$\begin{aligned}\mu\,\mathcal{S}^1(X_1,Y_1) \;=\;& g(\nabla^1_{X_1}\zeta_1, Y_1) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(X_1,Y_1) + \mu g(X_1,Y_1)\Delta l\\ & -g(X_1,\zeta_1)Y_2(l) + \zeta_2(l)g(X_1,Y_1) + X_2(l)g(\zeta_1,Y_1)\\ & -\mu(m_1+m_2-2)X_1(k)Y_2(l)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mu\,\mathcal{S}^2(X_2,Y_2) \;=\;& g(\nabla^2_{X_2}\zeta_2, Y_2) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(X_2,Y_2) + \mu g(X_2,Y_2)\Delta k\\ & -g(X_2,\zeta_2)Y_1(k) + \zeta_1(k)g(X_2,Y_2) + X_1(k)g(\zeta_2,Y_2)\\ & -\mu(m_1+m_2-2)X_2(l)Y_1(k),\end{aligned} \tag{12}$$

hold, then the vector field ζ is ζ(Ric) with scalar μ, where $X_1, Y_1 \in \mathfrak{L}(M_1)$ and $X_2, Y_2 \in \mathfrak{L}(M_2)$.

Proof. Let $T, W \in \mathfrak{X}(M)$, where $T = X_1 + X_2$ and $W = Y_1 + Y_2$. Suppose that the vector field ζ is ζ(Ric) on M with scalar μ. Then, $\mu\mathcal{S}(T,W) = g(\nabla_T\zeta, W)$. Using (5) and (7), we have

$$\mu\mathcal{S}(X_1+X_2, Y_1+Y_2) = g(\nabla_{X_1+X_2}(\zeta_1+\zeta_2), Y_1+Y_2).$$

Then, we have

$$\mu\Big\{\mathcal{S}(X_1,Y_1)+\mathcal{S}(X_1,Y_2)+\mathcal{S}(X_2,Y_1)+\mathcal{S}(X_2,Y_2)\Big\}$$
$$=g(\nabla^1_{X_1}\zeta_1-g(X_1,\zeta_1)\nabla l+X_1(k)\zeta_2+\zeta_2(l)X_1+\zeta_1(k)X_2+X_2(l)\zeta_1$$
$$+\nabla^2_{X_2}\zeta_2-g(X_2,\zeta_2)\nabla k,Y_1+Y_2).$$

Hence, we obtain

$$\begin{aligned}&\mu\Big\{\mathcal{S}^1(X_1,Y_1)-\tfrac{m_2}{f_1}h_1^{f_1}(X_1,Y_1)-g(X_1,Y_1)\Delta l\\&+(m_1+m_2-2)X_1(k)Y_2(l)+(m_1+m_2-2)X_2(l)Y_1(k)\\&+\mathcal{S}^2(X_2,Y_2)-\frac{m_1}{f_2}h_2^{f_2}(X_2,Y_2)-g(X_2,Y_2)\Delta k\Big\}\\&=g(\nabla^1_{X_1}\zeta_1,Y_1)-g(X_1,\zeta_1)g(\nabla l,Y_2)+X_1(k)g(\zeta_2,Y_2)\\&+\zeta_2(l)g(X_1,Y_1)+\zeta_1(k)g(X_2,Y_2)+X_2(l)g(\zeta_1,Y_1)\\&+g(\nabla^2_{X_2}\zeta_2,Y_2)-g(X_2,\zeta_2)g(\nabla k,Y_1).\end{aligned} \tag{13}$$

If the equations (11) and (12) hold, the assertion is hold from (13), which completes the proof.

In the remaining part, we give the definitions of a standard static space-time (SSS-T) and a generalized Robertson-Walker space-time (GRW). Let $(M_2,\ g_2)$ be an m_2-dimensional Riemannian manifold and J is an open connected interval of $\mathbb{R}$. If a $(m_2+1)-$ dimensional doubly warped product $\bar{M}={}_{f_2}J\times_{f_1}M_2$ has the metric tensor

$$\bar{g}=-(f_2^2)dt^2\oplus(f_1^2)g_2,$$

then it is called a *doubly warped product generalized Robertson-Walker space-time*. Here, $f_1\in C^\infty(J)$ and $f_2\in C^\infty(M_2)$, respectively and dt^2 is defined as the usual Euclidean metric tensor on J. For more details, see Flores and Sánchez (1974), Sánchez (1999), Sánchez (1998).

The following lemma is the direct consequences of (2)~(4), see also El-Sayied et al. (2020), pp. 3775.

Lemma 2.6. *Let $(\bar{M}={}_{f_2}J\times_{f_1}M_2,\ \bar{g})$ be a doubly warped product generalized Robertson-Walker space-time and $U,V\in\mathfrak{L}(M_2)$. Then we have*

$$\nabla_{\partial t}\partial t=f_2^2\nabla l, \tag{14}$$

$$\nabla_V\partial t=\nabla_{\partial t}V=V(l)\partial t+k'V, \tag{15}$$

$$\nabla_UV=\nabla^2_UV-\bar{g}(U,V)\nabla k, \tag{16}$$

for the components of the Levi-Civita connection of $\bar{M}$.

From Lemma 2.2, we get the following result directly, see also El-Sayied et al. (2020), pp. 3775.

Lemma 2.7. *Let $(\bar{M}={}_{f_2}J\times_{f_1}M_2,\ \bar{g})$ be a doubly warped product generalized Robertson-Walker space-time. Then we have*

$$\mathcal{S}(\partial t,\partial t)=(-k''+(k')^2)m_2+f_2^2\Delta l-\bar{g}(\nabla l,\nabla l), \tag{17}$$

$$\mathcal{S}(\partial t,U)=k'U(l)(m_2-1), \tag{18}$$

$$\mathcal{S}(U,V)=f_1^2\mathcal{S}^2(U,V), \tag{19}$$

for the non-zero components of the Ricci tensor of $\bar{M}$, where $U,V\in\mathfrak{L}(M_2)$.

Remark 2.8. The vector field $h\partial t$ is a ζ_1(Ric)-vector field on $(J,-dt^2)$ such that $h\in C^\infty(J)\Leftrightarrow h'=0$ on J. Here, " $'$ " is the derivative with respect to "t" on J.

Theorem 2.9. *Let the vector field $\bar{\zeta} = h\partial t + \zeta_2$ be $\bar{\zeta}(\mathrm{Ric})$ on a doubly warped product GRW space-time of the form $(\bar{M} =_{f_2} J \times_{f_1} M_2, \bar{g})$ with scalar μ and $U, V \in \mathfrak{L}(M_2)$. Then, the following conditions hold:*

$$\mu\Big\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\Big\} = (h' + \zeta_2(l))\bar{g}(\partial t, \partial t) \tag{20}$$

and

$$\mu f_1^2 \mathcal{S}^2(U, V) = hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) \tag{21}$$

or ζ_2 is a $\zeta_2(\mathrm{Ric})$-vector field if and only if $hk' = 0$.

Proof. Let the vector field $\bar{\zeta}$ be $\bar{\zeta}(\mathrm{Ric})$ on $\bar{M}$. Then, for all $T, W \in \mathfrak{X}(\bar{M})$, $\mu\mathcal{S}(T, W) = \bar{g}(\nabla_T\bar{\zeta}, W)$. Hence, we get $\mu\mathcal{S}(\partial t, \partial t) = \bar{g}(\nabla_{\partial t}\bar{\zeta}, \partial t)$ for $T = \partial t$, $W = \partial t$. Using (17), we obtain

$$\mu\Big\{(k'' + (k')^2)m_2 - k'\partial t(l) + f_2^2\Delta l - \bar{g}(\nabla l, \nabla l)\Big\} = \bar{g}(\nabla_{\partial t}(h\partial t) + \nabla_{\partial t}\zeta_2, \partial t).$$

Since $\partial t(l) = 0$, we obtain

$$\begin{aligned}\mu\Big\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\Big\} &= \bar{g}(\partial(h)\partial t + h\nabla_{\partial t}\partial t + \nabla_{\partial t}\zeta_2, \partial t)\\ &= \bar{g}(h'\partial t + hf_2^2\nabla l + \zeta_2(l)\partial t + k'\zeta_2, \partial t)\\ &= h'\bar{g}(\partial t, \partial t) + \zeta_2(l)\bar{g}(\partial t, \partial t).\end{aligned}$$

Hence, we get

$$\mu\Big\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\Big\} = (h' + \zeta_2(l))\bar{g}(\partial t, \partial t), \tag{22}$$

which proves (20). Since $\mathcal{S}(U, V) = \bar{g}(\nabla_U\bar{\zeta}, V)$ for $U, V \in \mathfrak{L}(M_2)$, using (19), we get

$$\begin{aligned}\mu f_1^2 \mathcal{S}^2(U, V) = \;& \bar{g}(\nabla_U(h\partial t) + \nabla_U\zeta_2, V)\\ &= \bar{g}(U(h)\partial t + h\nabla_U\partial t + \nabla_U\zeta_2, V)\\ &= \bar{g}(h(U(l)\partial t + k'U) + \nabla_U^2\zeta_2 - \bar{g}(U, \zeta_2)\nabla k, V)\\ &= hU(l)\bar{g}(\partial t, V) + hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) - \bar{g}(U, \zeta_2)\bar{g}(\nabla k, V)\\ &= hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V).\end{aligned} \tag{23}$$

Thus, we have (21) from (23). On the other hand, using (23), we get

$$\mu\mathcal{S}^2(U, V) = hk'g_2(U, V) + g_2(\nabla_U^2\zeta_2, V). \tag{24}$$

Then, the vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow$ the condition $hk' = 0$ is satisfied in (24), i.e. $k' = 0$ or $h = 0$. Hence, $\bar{M}$ is a GRW space-time or $\bar{\zeta} = \zeta_2$, where ζ_2 is also $\zeta_2(\mathrm{Ric})$-vector field on $\bar{M}$. The proof is completed.

If a $(m_2 + 1)-$dimensional doubly warped product $\bar{M} =_{f_1} J \times_{f_2} M_2$ has a metric tensor

$$\bar{g} = -(f_1^2)dt^2 \oplus (f_2^2)g_2,$$

then it is called a *doubly warped product SSS-T*, where (M_2, g_2) be an m_2-dimensional Riemannian manifold, here $f_1 \in C^\infty(M_2)$ and $f_2 \in C^\infty(J)$. Also dt^2 is defined as the usual Euclidean metric tensor on J, where J is an open connected interval of $\mathbb{R}$. For more details about standard static space-times, see Allison (1988)-Besse (2007)). From (2)~(4), we have:

Lemma 2.10. *Let $(\bar{M} =_{f_1} J \times_{f_2} M_2, \bar{g})$ be a doubly warped product SSS-T. Then we have*

$$\nabla_{\partial t}\partial t = 2\dot{k}\partial t + f_1^2\nabla k, \tag{25}$$

$$\nabla_V\partial t = \nabla_{\partial t}V = V(k)\partial t + \partial t(l)V, \tag{26}$$

$$\nabla_U V = \nabla_U^2 V - \bar{g}(U, V)\nabla l, \tag{27}$$

for the components of Levi-Civita connection of $\bar{M}$, where $U, V \in \mathfrak{L}(M_2)$. Here, "$\cdot$" is the derivative with respect to ∇^2.

From Lemma 2.2, we get the following result directly.

Lemma 2.11. *Let* $(\bar{M} ={}_{f_1} J \times_{f_2} M_2, \bar{g})$ *be a doubly warped product SSS-T. Then we have*

$$\mathcal{S}(\partial t, \partial t) = -m_2(-l' + (l')^2 - 2l'\dot{k}) + f_1^2\bar{g}(\nabla k, \nabla k) + f_1^2\Delta k, \tag{28}$$

$$\mathcal{S}(\partial t, U) = U(k)(1 - l' m_2), \tag{29}$$

$$\mathcal{S}(U, V) = f_2^2\mathcal{S}^2(U, V), \tag{30}$$

for the non-zero components of the Ricci tensor of $\bar{M}$, *where* $U, V \in \mathfrak{L}(M_2)$.

Theorem 2.12. *Let the vector field* $\bar{\zeta} = h\partial t + \zeta_2$ *be* $\bar{\zeta}(\mathrm{Ric})$ *on a doubly warped product SSS-T of the form* $(\bar{M} ={}_{f_1} J \times_{f_2} M_2, \bar{g})$ *with scalar* μ. *Then, we have*

$$\begin{aligned}\mu\Big\{ - m_2\Big(- l' + (l')^2 - 2l'\dot{k}\Big) + f_1^2\Big(|\nabla k|^2 + \Delta k\Big)\Big\}\\ = \Big\{\dot{h} + 2h\dot{k} + \zeta_2(k)\Big\}|\partial t|^2,\end{aligned} \tag{31}$$

and

$$\mu f_2^2\mathcal{S}^2(U, V) = \partial t(l)\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) \tag{32}$$

or the vector field ζ_2 *is* $\zeta_2(\mathrm{Ric})$ *on* $M_2 \Leftrightarrow \partial t(l) = 0$, *namely* $\bar{M}$ *is a SSS-T.*

Proof. Let the vector field $\bar{\zeta}$ be $\bar{\zeta}(\mathrm{Ric})$ on $\bar{M}$. Then, $\mu\mathcal{S}(T, W) = \bar{g}(\nabla_T\bar{\zeta}, W)$ for all $T, W \in \mathfrak{X}(\bar{M})$. It follows that $\mu\mathcal{S}(\partial t, \partial t) = \bar{g}(\nabla_{\partial t}\bar{\zeta}, \partial t)$. Hence, using (28) we get

$$\begin{aligned}&\mu\Big\{ - m_2(-l' + (l')^2 - 2l'\dot{k}) + f_1^2\bar{g}(\nabla k, \nabla k) + f_1^2\Delta k\Big\}\\ &= \bar{g}(\nabla_{\partial t}(h\partial t) + \nabla_{\partial t}\zeta_2, \partial t)\\ &= \bar{g}(\partial t(h)\partial t + h\nabla_{\partial t}\partial t + \nabla_{\partial t}\zeta_2, \partial t)\\ &= h'\bar{g}(\partial t, \partial t) + h\bar{g}(2\dot{k}\partial t + f_1^2\nabla k, \partial t) + \bar{g}(\zeta_2(k)\partial t + \partial t(l)\zeta_2, \partial t)\\ &= h'\bar{g}(\partial t, \partial t) + 2\dot{k}h\bar{g}(\partial t, \partial t) + hf_1^2\bar{g}(\nabla k, \partial t) + \zeta_2(k)\bar{g}(\partial t, \partial t) + \partial(l)\bar{g}(\zeta_2, \partial t)\\ &= \Big\{h' + 2h\dot{k} + \zeta_2(k)\Big\}|\partial t|^2.\end{aligned}$$

Hence, we get (31). Since $\mu\mathcal{S}(U, V) = \bar{g}(\nabla_U\bar{\zeta}, V)$, using (30), we get

$$\begin{aligned}\mu f_2^2\,\mathcal{S}^2(U, V) &= \bar{g}(\nabla_U(h\partial t) + \nabla_U\zeta_2, V)\\ &= \bar{g}(h(U(l)\partial t + \dot{k}U) + \nabla_U\zeta_2 - \bar{g}(U, \zeta_2)\nabla k, V)\\ &= \bar{g}(hU(l)\partial t + \partial t(l)U, V) + \bar{g}(\nabla_U^2\zeta_2 - \bar{g}(U, \zeta_2)\nabla l, V)\\ &= \partial t(l)\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V),\end{aligned} \tag{33}$$

for $U, V \in \mathfrak{L}(M_2)$. Thus, we have (32) from (33). Then, using (33), we obtain

$$\mu\mathcal{S}^2(U, V) = \partial t(l)g_2(U, V) + g_2(\nabla_U^2\zeta_2, V). \tag{34}$$

Thus, the vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow$ the condition $\partial t(l) = 0$ is satisfied in (34), i.e. l is constant. It follows that $\bar{M}$ is a SSS-T and hence, the proof is completed.

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LIST OF AUTHOR ORCIDS

S. Gerdan Aydın	https://orcid.org/0000-0001-5278-6066
M. Traore	https://orcid.org/0000-0003-2132-789X
H. M. Taştan	https://orcid.org/0000-0002-0773-9305

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