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RESEARCH ARTICLE

Effectiveness of a One-fifth Hybrid Block Approach for Second Order Ordinary Differential Equations

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ABSTRACT
In this paper, a class of hybrid block methods for solving second order ordinary differential equations directly was developed. This class was obtained by interpolation and collocation techniques. The methods were analyzed based on the qualitative properties of linear multi-step methods and were found to be zero-stable, consistent and convergent with good region of absolute stability. The proposed methods were analyzed quantitatively and implemented on second order ordinary differential initial value problems. An improved performance of the new methods over existing methods in the literature was shown by solving five numerical examples. The results were presented in tabular form.

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Keywords: Block method, hybrid method, initial value problems, one-fifth step, convergence, stability

1. INTRODUCTION

Issac Newton discovered a large number of differential equations. He was responsible for large number of this type of equations in Physic and Mathematics. He is also responsible for the systematic development of model of motion. He has other discoveries like personal finance, electric circuits, behavior of musical instruments the logistic equation, electric magnetion, quantum chronodynamics oscillation, among others. A differential equation shows relationship between a function that is unknown and its respective derivatives. The order of the equation depends on the highest derivative of the the dependent function. Practically, this work contains extensive qualitative and quantitative analysis of a class of effective numerical methods used in approximating second order class of differential equations.
The solution of Initial Value Problems(IVPs) in Ordinary Differential Equations (ODEs) of the form

$$\begin{aligned} y' &= f(x, y, y' ..., y^{m-1}) \\ y(x_0) &= \eta_0, y'(x_0) = \eta_1, ... y^{m-1}(x_0) = \eta_{m-1}, \end{aligned} \tag{1}$$

with the interval $[x_0, x_1]$ has given rise to the methods of one step and multi-step methods. This is majorly attributed to Linear Multi-step Methods (LMMs). From literature, many scholars look for alternative methods of solving (1) and higher order differential equations without reducing them to systems of first order differential equations. Authors such as

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$$\left.\begin{aligned}
&\alpha_0(z) = 1 - \frac{15}{2}z \\
&\alpha_{\frac{2}{15}}(z) = \frac{15}{2}z \\
&\beta_0(z) = -\frac{14}{675}z + \frac{1}{2}z^2 - \frac{55}{12}z^3 + \frac{75}{4}z^4 - \frac{225}{8}z^5 \\
&\beta_{\frac{1}{15}}(z) = -\frac{11}{225}z + \frac{15}{2}z^3 - \frac{375}{8}z^4 + \frac{675}{8}z^5 \\
&\beta_{\frac{2}{15}}(z) = +\frac{1}{225}z - \frac{15}{4}z^3 + \frac{75}{2}z^4 - \frac{675}{8}z^5 \\
&\beta_{\frac{3}{15}}(z) = -\frac{1}{675}z + \frac{5}{6}z^3 - \frac{75}{8}z^4 + \frac{225}{8}z^5
\end{aligned}\right\} \tag{2}$$

The first derivatives of the equations (2) give

$$\left.\begin{aligned}
&\alpha_0'(z) = -6 \\
&\alpha'_{\frac{2}{15}}(z) = 6 \\
&\beta_0'(z) = z - 11z^2 + 48z^3 - 72z^4 \\
&\beta'_{\frac{1}{12}}(z) = 18z^2 - 120z^3 + 216z^4 \\
&\beta'_{\frac{2}{12}}(z) = -9z^2 + 96z^3 - 216z^4 \\
&\beta'_{\frac{3}{12}}(z) = 2z^2 - 24z^3 + 72z^4
\end{aligned}\right\} \tag{3}$$

By evaluating the first derivative equation (2) together with (3) at points $p = 0, \frac{1}{15}, \frac{2}{15}$ and $\frac{3}{15}$, we obtain

$$hy'_n = -\frac{5}{2}y_n + \frac{15}{2}y_{n+\frac{2}{5}} + h(-\frac{14}{675}f_n - \frac{11}{225}f_{n+\frac{1}{15}} - \frac{1}{1225}f_{n+\frac{2}{15}} + \frac{1}{675}f_{n+\frac{3}{15}}) \tag{4}$$

$$hy'_{n+\frac{1}{15}} = -\frac{5}{2}y_n + \frac{15}{2}y_{n+\frac{2}{5}} + h^2(\frac{23}{5400}f_n - \frac{7}{1800}f_{n+\frac{1}{15}} - \frac{17}{1800}f_{n+\frac{2}{15}} + \frac{7}{5400}f_{n+\frac{3}{15}}) \tag{5}$$

$$hy'_{n+\frac{2}{15}} = -\frac{5}{2}y_n + \frac{15}{2}y_{n+\frac{2}{5}} + h^2(\frac{1}{675}f_n - \frac{1}{25}f_{n+\frac{1}{15}} + \frac{2}{75}f_{n+\frac{2}{15}} - \frac{1}{675}f_{n+\frac{3}{15}}) \tag{6}$$

$$hy'_{n+\frac{2}{15}} = -\frac{5}{2}y_n + \frac{15}{2}y_{n+\frac{2}{5}} + h^2(\frac{23}{5400}f_n + \frac{7}{1800}f_{n+\frac{1}{15}} + \frac{143}{1800}f_{n+\frac{2}{15}} + \frac{127}{5400}f_{n+\frac{3}{15}}) \tag{7}$$

Next, we derive the block for a new one-fifth step hybrid method
In order to get the blocks for derivation of the block methods and to test for the zero stability, we combine equations (5), (6) and (7) and use their coefficients in the block form

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{15}{2} & 0 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{15}{2} \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{15}} \\ y_{n-\frac{2}{15}} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y'_{n-\frac{1}{15}} \\ y'_{n-\frac{2}{15}} \\ y'_n \end{bmatrix}$$

$$+h^2 \begin{bmatrix} 0 & 0 & -\frac{1}{5400} \\ 0 & 0 & \frac{1}{5400} \\ 0 & 0 & -\frac{14}{675} \end{bmatrix} \begin{bmatrix} f_{n-\frac{1}{15}} \\ f_{n-\frac{2}{15}} \\ f_n \end{bmatrix} + h^2 \begin{bmatrix} -\frac{1}{540} & -\frac{1}{5400} & 0 \\ -\frac{1}{450} & -\frac{7}{1800} & \frac{1}{2700} \\ -\frac{11}{225} & \frac{1}{225} & -\frac{1}{675} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{15}} \\ f_{n+\frac{2}{15}} \\ f_{n+\frac{3}{15}} \end{bmatrix} \tag{8}$$

After normalizing the equation (8) we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{15}} \\ y_{n-\frac{2}{15}} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & 0 & n+\frac{2}{15} \\ 0 & 0 & n+\frac{1}{5} \end{bmatrix} \begin{bmatrix} y'_{n-\frac{1}{15}} \\ y'_{n-\frac{2}{15}} \\ y'_n \end{bmatrix}$$

$$+h^2 \begin{bmatrix} 0 & 0 & -\frac{13}{3000} \\ 0 & 0 & \frac{28}{10125} \\ 0 & 0 & -\frac{97}{81000} \end{bmatrix} \begin{bmatrix} f_{n-\frac{1}{15}} \\ f_{n-\frac{2}{15}} \\ f_n \end{bmatrix} + h^2 \begin{bmatrix} -\frac{3}{250} & -\frac{3}{1000} & -\frac{1}{1500} \\ \frac{22}{3375} & -\frac{2}{3375} & \frac{2}{10125} \\ \frac{19}{13500} & -\frac{13}{2700} & \frac{1}{10125} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{15}} \\ f_{n+\frac{2}{15}} \\ f_{n+\frac{3}{15}} \end{bmatrix} \tag{9}$$

By rewriting equation (9) explicitly, we get:

$$\left.\begin{aligned} y_{n+\frac{1}{15}} &= y_n + \frac{1}{15}hy'_n + h^2(\frac{97}{81000}f_n + \frac{19}{3500}f_{n+\frac{1}{15}} + \frac{13}{27000}f_{n+\frac{2}{15}} + \frac{1}{10125}f_{n+\frac{3}{15}} \\ y_{n+\frac{2}{5}} &= y_n + \frac{2}{15}hy'_n + h^2(\frac{28}{10125}f_n + \frac{22}{3375}f_{n+\frac{1}{15}} - \frac{2}{3375}f_{n+\frac{2}{15}} + \frac{2}{10125}f_{n+\frac{3}{15}} \\ y_{n+\frac{1}{5}} &= y_n + \frac{3}{15}hy'_n + h^2(\frac{13}{3000}f_n + \frac{3}{250}f_{n+\frac{1}{15}} + \frac{13}{3000}f_{n+\frac{2}{15}} + \frac{1}{1500}f_{n+\frac{3}{15}}) \end{aligned}\right\} \tag{10}$$

Substituting $y_{n+\frac{2}{15}}$ of equation (10) into the equations (5), (6) and (7) gives

$$\left.\begin{aligned} y'_{n+\frac{1}{15}} &= y_n + h^2(\frac{1}{40}f_n + \frac{19}{360}f_{n+\frac{1}{15}} - \frac{1}{72}f_{n+\frac{2}{15}} + \frac{1}{360}f_{n+\frac{3}{15}}) \\ y'_{n+\frac{2}{15}} &= y_n + h^2(\frac{1}{45}f_n + \frac{4}{45}f_{n+\frac{1}{15}} - \frac{1}{45}f_{n+\frac{2}{15}}) \\ y'_{n+\frac{1}{5}} &= y_n + h^2(\frac{1}{4}f_n + \frac{3}{40}f_{n+\frac{1}{15}} + \frac{3}{40}f_{n+\frac{2}{15}} + \frac{1}{40}f_{n+\frac{3}{15}}) \end{aligned}\right\} \tag{11}$$

2. ANALYSIS OF THE METHODS

The methods have some basic properties which establish their validity. The properties: order error constant, consistency and zero stability reveal the nature of convergence of the methods.

2.1. Order and Error Constant

We define the truncation error associated with equation (10) by the difference operator

$$\mathcal{L}(y(x,h)) = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - \alpha_{vj} y(x_n + vjh) - h^2\beta_j y''(x_n + jh) - h^2\beta_{vj} y''(x_n + jh)\right], \tag{12}$$

where $y(x)$ is an arbitrary test function which is continuously differentiable in the interval expanding (x) in Taylor series about x_n and collecting like terms in h and y gives

$$\mathcal{L}(y(x)) = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + C_3 h^3 y'(x) + \ldots + C_{p+3} 3h^{p+3} y^{p+3}(x), \tag{13}$$

where the coefficient $C_q, q = 0, 1, 2, \ldots$ are given as

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

$$C_2 = \frac{1}{2}\sum_{j=2}^{k} j^2\alpha_j,$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)\sum_{j=1}^{k} j^q \beta_j j^{q-3}\right].$$

According to Henrici (1962) method (13) has order p if $C_0 = C_1 = C_2 = ...C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$
The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x)$ is the principal local truncation error at the point x_n. Using Taylor series expansion on equations (10) and (11) we get the order of the new proposed block methods respectively as $(4, 4, 4, 4, 4, 4)$ with error constants as

$$\left(\frac{-31}{1366875000}, \frac{16}{170859375}, \frac{7}{16875000}, \frac{-1}{20250}, \frac{-4}{10125}, \frac{-1}{750}\right).$$

2.2. Zero stability of One-fifth Step-length For Second Order Differential Equation

In order to test for zero stability of the block method (10), we consider the matrix difference equation of the form

$$p^0 Y_{m+1} = p^{'} y_m + h^2[Q^0 F_{m+1} + Q^{'} F_m + hR^{'}], \tag{14}$$

where

$$Y_{m+1} = [y_{n+\frac{1}{15}}, ...y_{n+\frac{1}{5}}]^T, Y_m = [y_{n-\frac{1}{15}}, ...y_n]^T F_{m+1} = [f_{n+\frac{1}{15}}, ...f_{n+\frac{1}{5}}]^T, F_m = [f_{n-\frac{1}{15}}, ...f_n]^T. \tag{15}$$

The matrices $p^0, p^{'}, Q^0, Q^{'}$ and $R^0$are the coefficients of equation (10) which defined as follows

$$p^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

$$p^{'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$Q^0 = \begin{bmatrix} \frac{19}{3500} & \frac{13}{27000} & \frac{1}{10125} \\ \frac{22}{3375} & -\frac{2}{3375} & \frac{2}{10125} \\ \frac{3}{250} & \frac{13}{3000} & \frac{1}{1500} \end{bmatrix}, \tag{18}$$

$$Q^{'} = \begin{bmatrix} 0 & 0 & \frac{97}{81000} \\ 0 & 0 & \frac{28}{10125} \\ 0 & 0 & \frac{13}{3000} \end{bmatrix}, \tag{19}$$

$$R^{'} = \begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & 0 & \frac{2}{15} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}. \tag{20}$$

A block method is said to be zero stable if the roots

$$\left|[\lambda p^0 - p^1]\right| = 0$$

are sample with maximum modulus 1.
Now

$$\left|[\lambda p^0 - p^{'}]\right| = \left|\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0 \tag{21}$$

implies that $\lambda^3 - \lambda^2 = 0$, This gives $\lambda = 0, 0, 1$.
Since the root have modulus less than or equal to one and are simple, the method in zero-stable

2.3. Consistency of One-fifth Step Length For Second Order Differential Equation.

The first and second characteristic polynomials of method (10) are given by Remark
condition (i) is the a sufficient condition for the associated block method to be consistent.
Consistency of the main method (10) The first and second characteristic polynomials of method (10) are given by

$$\rho(z) = z^{\frac{1}{5}} + \frac{1}{2}z^0 - \frac{3}{2}z^{\frac{2}{15}} \tag{22}$$

$$\sigma(z) = \frac{1}{5400}z^0 + \frac{1}{450}z^{\frac{1}{15}} + \frac{7}{1800}z^{\frac{2}{15}} + \frac{1}{2700}z^{\frac{3}{15}} \tag{23}$$

The method (10) is consistent if it satisfies the condition

a The order of the method is $\rho = 4 \geq 1$.

b

$$\alpha_0 = \frac{1}{2},$$

$$\alpha_{\frac{2}{15}} = \frac{-3}{2}$$

and

$$\alpha_{\frac{1}{5}} = 1.$$

Thus,

$$\sum_j \alpha_j = \frac{1}{2} - \frac{3}{2} + 1 = 0. \;\; j = 0, \frac{1}{15}, \frac{2}{15}$$

c

$$\rho(1) = \frac{1}{2} - \frac{3}{2} + 1 = 0$$

$$\rho'(z) = \frac{1}{5}z^{\frac{4}{5}} - \frac{1}{5}z^{\frac{13}{15}} = 0$$

for $z = 1$

$$\rho'(1) = \frac{1}{5}(1)^{\frac{4}{5}} - \frac{1}{5}(1)^{\frac{13}{15}} = 0$$

$$\rho(1) = \rho'(1) = 0.$$

Hence, this condition is satisfied

d

$$\rho''(z) = \frac{4}{25}z^{\frac{9}{5}} + \frac{13}{75}z^{\frac{28}{15}} = 0$$

$$\rho''(1) = \frac{4}{25}(1)^{\frac{9}{5}} + \frac{13}{75}(1)^{\frac{28}{15}} = 0$$

$$\sigma(1) = \frac{1}{5400}(1)^0 + \frac{1}{450}(1)^{\frac{1}{15}} + \frac{7}{1800}(1)^{\frac{2}{15}} + \frac{1}{2700}(1)^{\frac{1}{5}} = 150$$

and

$$2!\sigma(1) = 2\frac{1}{150}$$

$$\rho''(1) = 2!\sigma(1) = \frac{1}{75}$$

Hence the method is consistent.

2.4. Convergence

The convergence of the continuous implicit hybrid block method base on the basic properties discussed above with the fundamental theorem of Dahlquist for the linear multi-step methods. The theorem is stated below without proof.
Theorem 1
The necessary and sufficient condition for a linear multi-step method to be convergent is for it to be consistent and zero stable.

2.5. Region of Absolute Stability of the Block Method

The stability matrix for the method is defined as follows:

$$M(z) = V + zB(M - zA)U \tag{24}$$

and the stability function

$$p(\eta, z) = det(\eta I - M(z)). \tag{25}$$

Then, we represent the block method in form of

$$\begin{bmatrix} y_n \\ --- \\ y_{n+\frac{3}{15}} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ --- \\ y_{i-1} \end{bmatrix} \tag{26}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{97}{81000} & \frac{19}{13500} & \frac{-13}{27000} & \frac{1}{10125} \\ \frac{28}{10125} & \frac{22}{3375} & \frac{-2}{3375} & \frac{2}{10120} \\ \frac{13}{3000} & \frac{3}{250} & \frac{3}{1000} & \frac{1}{1500} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{97}{81000} & \frac{19}{13500} & \frac{-13}{27000} & \frac{1}{10125} \\ \frac{28}{10125} & \frac{22}{3375} & \frac{-2}{3375} & \frac{2}{10120} \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & I \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & I \end{bmatrix}, f(y) = \begin{bmatrix} y_n \\ y_{n+\frac{1}{15}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix}, Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{15}} y_{n+\frac{3}{15}} \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.$$

This gives the stability polynomial of the one-fifth step method which was plotted below

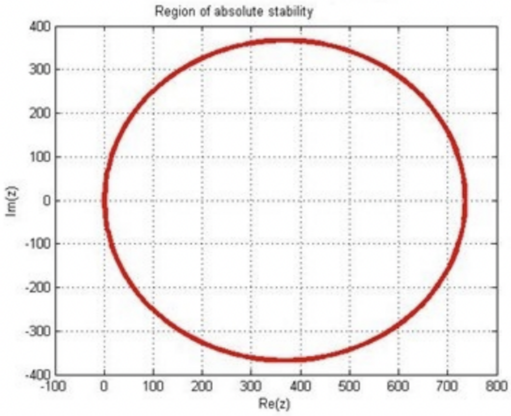

Figure 1: Region of Absolute stability of $\frac{1}{5}$ HBMS

3. NUMERICAL EXAMPLES

In this section, some numerical examples of second order ordinary differential equations are solved. The methods are implemented directly without using any starting value and with use of Matlab and Maple. The table below shows some notations that were use to present the numerical and graphical results obtained for some test problems by application of the proposed schemes.

Problem 1: Consider a Linear non-homogeneous test problem

$$y'' = 3y' + 8e^{2x},$$

$$y(0) = 1,$$

$$y'(0) = 1, h = 0.05$$

Exact solution:

$$y(x) = -4e^{2x} + 3e^{3x} + 2.$$

Problem 2: Consider a specially oscillatory test problem

$$y'' = -\lambda^2 y,$$

$$y(0) = 1, \; y'(0) = 2, \; h = 0.01, \; \lambda = 2$$

Exact solution:

$$y(x) = \cos 2x + \sin 2x.$$

Problem 3: Consider a singular non-homogeneous test problem

$$y'' = \frac{2y'}{x} + xe^x - y(1 + \frac{2}{x^2})$$

$$y(\frac{\pi}{2}) = 4 - \pi + \frac{1}{4}(e^{\frac{\pi}{2}})(\pi + 2)$$

$$y'(\frac{\pi}{2}) = \frac{\pi}{2}(8 + e^{\frac{\pi}{2}})$$

$$h = 0.003125$$

Exact solution:

$$y(x) = 2x\cos x + 4x\sin x + \frac{1}{2}xe^x.$$

Problem 4: We consider a linear homogeneous test problem

$$y'' = y',$$

$$y(0) = 0,\ y'(0) = -1,\ h = 0.1$$

Exact solution :

$$y(x) = 1 - e^x$$

Problem 5: We consider a non-linear non-homogeneous test problem

$$y'' = x(y')^2$$

$$y(0) = 1,\ y'(0) = \frac{1}{2},\ h = 0.1$$

Exact solution:

$$y(x) = 1 + \frac{1}{2}\ln(\frac{2+x}{2-x}).$$

3.1. Tabular Presentation of Numerical Results

Here we present numerical and errors results for $\frac{1}{5}HBM$ in tabular form below

Table 1: Numerical Results for Problem 1

x	Exact	Computed	$Errorinourmethod$	Error in $Areo$(2016)
0.0100	1.01979867335991	1.01979867335989	$1.8874E-14$	$8.4599E-14$
0.0200	1.03918944084761	1.03918944084754	$7.5939E-13$	$3.4861E-13$
0.0300	1.05816454641465	1.05816454641448	$1.7231E-13$	$7.8870E-13$
0.0400	1.07671640027179	1.07671640027149	$3.0509E-13$	$1.4004E-12$
0.0500	1.09483758192485	1.09483758192438	$4.7384E-13$	$2.1791E-12$
0.0600	1.11252084314279	1.11252084314211	$6.7768E-13$	$3.1208E-12$
0.0700	1.12975911085687	1.12975911085596	$9.1704E-13$	$4.2211E-12$
0.0800	1.14654548998987	1.14654548998868	$1.1884E-12$	$0.23146E-11$
0.0900	1.16287326621395	1.16287326621245	$1.4924E-12$	$6.8801E-12$
0.1000	1.17873590863630	1.17873590863448	$1.8268E-12$	$8.4293E-12$

Table 2: Numerical Results for Problem 2

x	Exact	Computed	Error in our method	Error in $Areo$(2016)
0.0050	1.00513852551049	1.00513852547870	$3.1790E-11$	$1.2349E-09$
0.0100	1.01055824175353	1.01055824162003	$8.4730E-11$	$2.6905E-09$
0.0150	1.01626544391208	1.01626544360366	$3.0842E-10$	$4.3738E-09$
0.0200	1.02226654286653	1.02226654230753	$5.5900E-10$	$6.2921E-09$
0.0250	1.02856806714980	1.02856806626307	$8.8673E-10$	$8.9697E-09$
0.0300	1.03517666493419	1.03517666364109	$1.2931E-09$	$1.0863E-08$
0.0350	1.04209910605025	1.04209910426619	$1.7841E-09$	$1.6463E-08$
0.0400	1.04934228403829	1.04934228168012	$2.3581E-09$	

Table 3: Numerical Results for Problem 3

x	Exact	Computed	Error in our method	Error in $Areo(2016)$
1.7000	10.95785118097658	10.95785118071406	$2.6252E-10$	
1.8000	11.63820762976944	11.63820762869096	$1.0785-09$	$4.0964E-09$
1.9000	12.31472912025427	12.31472911749325	$2.7610E-09$	$1.6840E-08$
2.0000	12.99859200531184	12.99859199968641	$5.6254E-09$	$0.43121E-08$
2.1000	13.70481572693030	13.70481571693856	$9.9917E-09$	$8.7867E-08$
2.2000	14.45259109075646	14.45259107457567	$1.6181E-08$	$1.5609E-07$
2.3000	15.26561176327774	15.26561173877142	$2.4506E-08$	$3.8289E-07$
2.4000	16.17241142639307	16.17241139112289	$3.5270E-08$	$5.5108E-07$
2.5000	17.20670978769302	17.20670973893582	$4.8757E-08$	$7.6182E-07$
2.6000	18.40777146832744	18.40777140309677	$6.5231E-08$	$1.0192E-06$

Table 4: Numerical Results for Problem 4

x	Exact	Computed	Error in our method	Error in Ramos et al(2016)
0.1000000	−0.105170918075647710	−0.105170918075646940	$7.77156117E-15$	−0.105170918075645880
0.2000000	−0.221402758160170080	−0.221402758160166880	$3.19189120E-14$	$5.441E-07$
0.3000000	−0.349858807576003410	−0.349858807575995860	$7.54951657E-14$	$9.114E-07$
0.4000	−0.491824697641270790	−0.491824697641256470	$1.43218770E-13$	$1.329E-06$
0.5000	−0.648721270700128640	−0.648721270700105430	$2.32036612E-13$	$1.447E-06$
0.6000	−0.822118800390509770	−0.822118800390473690	$3.60822483E-13$	$2.435E-06$
0.7000	−1.013752707470477500	−1.013752707470424700	$5.28466160E-13$	$3.153E-6$
0.8000	−1.225540928492468800	−1.225540928492394400	$7.43849426E-13$	$4.965E-06$
0.9000	−1.459603111156951200	−1.459603111156850200	$1.01030295E-12$	$4.948E-06$

Table 5: Numerical Results for Problem 5

x	Exact	Computed	Error in our method	Error in Ramos et al(2016)
0.1000000	1.050041729278491400	1.050041729278491800	$0.444089210E-15$	$1.18393E-10$
0.2000000	1.100335347731074900	1.100335347731074900	$0.000000E-00$	$2.3749E-10$
0.3000000	1.151140435936465000	1.151140435936462300	$0.26645E-14$	$4.2485E-10$
0.4000	1.202732554054079200	1.202732554054065400	$0.13766E-13$	$6.1628E-10$
0.5000	1.255412811882991000	1.255412811882949100	$0.41966E-13$	$1.0233E-09$
0.6000	1.309519604203106100	1.309519604203005100	$0.10103E-12$	$1.4483E-09$
0.7000	1.365443754271389100	1.365443754271173500	$2.1560E-013$	$2.5449E-09$
0.8000	1.423648930193593300	1.423648930193167200	$0.4261E-12$	$3.7221E-09$
0.9000	1.484700278594041300	1.484700278593238400	$0.80291E-12$	$7.3287E-08$

4. DISCUSSION OF RESULTS AND CONCLUSION

In this paper, we developed one-fifth order initial value problems directly with out reducing the system of first order differential equation. the method that was develop was test by using it to solve numerical examples which are linear,non linear and stiff initial value problems of second order ordinary differential equation. The table of results of our method is show below comparing the proposed method with the exalt and the existing method. In the table of result,the first,second and third example solved was compared with Areo & Rufai (2016) were the fourth and fifth example was compared with Ramos *et al.* (2016) the newly develop method performed better. Overall, in this paper, a class of numerical schemes are developed in which fractions was used as the step-lengths for second order ordinary differential equations. The resulting methods are consistent and zero stable, therefore it convergences. The methods have good region of absolute stability. The results of the problem show that the method is effective and accurate compared with Areo & Rufai (2016) and Ramos *et al.* (2016) methods.

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REFERENCES

Adeniyi, R. B. and Alabi, M. O. (2007). Continuous formulation of a class accurate implicit linear multi-step methods with Chebyshev basis function in a collocation technique. *Journal of Mathematical Association of Nigeria (ABACUS)* , **24**(2A), 58-77.

Adesanya, A. O., Anake, T. A. & Oghonyon, G. J. (2009). ontinuous implicit method for the solution of general second order ordinary differential Equations *Journal of Nigerian Association of Mathematical Physics*, **15**, 71-78.

Adesanya, A. O., Odekunle, M. R and Alkali, M. A. (2012). Order six block predictor corrector for the solution of $y'' = f(x, y, y')$, *Canadian J. on Science and Engineering Mathematics*, **3**(4), 180-185.

Adeyefa, E. O. (2013). Collocation approach for continuous hybrid block methods for second order ordinary differential equations with Chebyshev basis function. *Unpublished doctoral thesis, University of Ilorin*, Ilorin, Nigeria.

Adeyeye, O. & Omar, Z. (2016). Maximal Order Block Method For The Solution Of Second Order Ordinary Differential Equations, *IAENG International Journal of Applied Mathematics* **46**(4).

Anake, T. A. (2011). Continuous implicit hybrid one-step Methods for solution of initial value problems of general second ordinary differential equation *Ph.D. Thesis,(Unpublished) Covenant University*, Ota Nigeria.

Areo, E. A., Ademiluyi R. A. and Babalola, P. O.(2008). Accurate collocation multi-step methods for integration of first order ordinary differential equations. *International Journal Computer Mathematics* **2**(1), 15-27.

Areo E. A. and Rufai M. A. (2016). Develop a new uniform fourth order one-third step continuous block method for the direct solution. *British journal of Mathematics and computer science* **15**(4), 1-12.

Areo, E. and Omojola, M. T. (2017). One-twelveth step continuous block method for the solution of $y''' = f(x, y, y', y'')$ *International journal of pure and applied mathematics*.

Awoyemi, D. O. (1992). On some continuous linear multi-step methods for initial value problems *Unpublished doctoral thesis, University of Ilorin*, Ilorin Nigeria.

Butcher, J. C. (1965). A modified multi-step method for numerical integration of ordinary differential equations *Journal of the ACM*,**12**, 124-135.

Dahlquist, G. (1959). Stability and error bound in the numerical integration of ODEs *Transcript/130/, Royal Institute of Technology,* Stockholom.

Dahlquist, G. (1963). A special stability problem for linear multi-step methods *BIT* **3**, 27-43.

Ehigie, J. O., Okunuga S. A. and Sofoluwe, A. B. (2011). A class of 2-step continuous hybrid implicit linear multi-step methods for IVP's *Journal of Nigerian Mathematical Society* **30**, 145-161.

Fotta, A. U., Alabi, T. J., & Abdulqadir, B. (2015). Block method with one hybrid point for the solution of first order initial value problems of ordinary differential equations. *International journal of pure and applied mathematics*.

Mohammed, U. and Adeniyi, R. B. (2014). A Three Step Implicit Hybrid Linear Multistep Method for the Solution of Third Order Ordinary Differential Equations. *Gen. Math. Notes,* **25**(1), 62-74.

Muhammed, U., Ma'ali A. I., and Badeggi, A. Y. (2014). Derivation of block hybrid method for the solution of first order initial value problems in ODEs" *Pacific Journal of Science and Technology* **15**(1), 93-98.

Olabode, B.T. (2013). Block Multistep Method for Direct Solution of Third Order Ordinary Differential Equations. *FUTA Journal of Research in Science* **2**, 194-200.

Rajabi, M. Ismail, F., Senu, N. (2016). Linear 3 and 5-step methods using Taylor series expansion for solving special 3rd order ODEs.

Ramos H., Mehta S. and Vigo-Aguiter J. (2016). A unified approach for k-step block Falkner methods for solving general second-order problems in ODEs. *Journal of Computation and Applied Mathematics*

Rufai M. K., Duramola M. K. and Ganiyu.A. A. (2016). Derivation of one-sixth hybrid block method for solving general first order ordinary differential equations *IOSR Journal of Mathematics***12**(5), 20-27.

Sahi, R. K., Jator, S. N. and Khan, N. A. (2013). Continuous Fourth Derivative Method for Third Order Boundary Value Problems. International Journal of Pure and Applied Mathematics, *International Journal of Pure and Applied Mathematics*, **85**(5), 907-923.