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Bimultipliers of R-algebroids

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Abstract — Group action is determined by the automorphism group and algebra action is defined by the multiplication algebra. In the study we generalize the multiplication algebra by defining multipliers of an R-algebroid M. Firstly, the set of bimultipliers on an R-algebroid is introduced, it is denoted by $Bim(M)$, then it is proved that this set is an R-algebroid, it is called multiplication R-algebroid. Using this $Bim(M)$, for an R-algebroid morphism $A \longrightarrow Bim(M)$ it is shown that this morphism gives an R-algebroid action. Then we examine some of the properties associated with this action.

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1. Introduction

In the realm of group theory, the interplay between groups and their actions on one another is a subject of profound importance. Central to this discourse is the notion that the action of a group on another group is intricately determined by the automorphism group. This relationship is encapsulated in the form of a homomorphism, mapping the acting group to the automorphism group of the target group. Moreover, any extension of groups also finds its roots in such homomorphisms, further underscoring their significance in understanding the dynamics between groups.

Extending beyond the confines of group theory, similar principles resonate in the domain of algebra, where the action of an algebra on another is closely intertwined with the concept of multiplication algebras. The seminal work of Maclane [1] lays the foundation for this concept, elucidating its pivotal role in algebraic structures. Building upon this framework, Ege and Arvasi [2] introduce actor crossed modules of commutative algebras, leveraging multiplication algebras to generalize aspects from commutative algebras to crossed modules [13], [14].

Within the realm of R-algebroids, a branch of algebraic structures, significant attention has been directed towards their study, notably by Mitchell [3], [4], [5] and Amgott [6]. Mitchell's categorical definition of R-algebroids and Mosa's introduction of crossed modules of R-algeb-
roids serve as pivotal contributions to this field. Notably, the equivalence between crossed modules of R-algebroids and special double algebroids with connections, established by Mosa [7], further enriches our

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understanding of these structures. Subsequent investigations by Akca and Avcioglu [8], [9], [10], [11], [12] delve deeper into crossed modules of R-algebroids, unraveling intricate connections and properties. By means of algebra action, the 2-crossed module structure is defined [15] and the equivalence of 2-crossed modules to simplicial algebras is shown [16]. There are also studies [17], [18], [19], [20], [21] on 2-crossed modules.

In this paper, we embark on a journey to explore the multifaceted landscape of R-algebroids, with a specific focus on their actions and associated properties. Our endeavor begins with the introduction of the set denoted Bim(M), comprising multipliers of an R-algebroid M. Through a rigorous exposition, we establish that this set itself forms an R-algebroid, aptly termed the multiplication R-algebroid, by defining suitable operations. Leveraging this newfound structure, we define an R-algebroid morphism from an arbitrary algebra to Bim(M), thereby elucidating the mechanism through which actions manifest. Finally, we undertake a comprehensive examination of the properties entailed by this action, shedding light on its intricacies and implications.

Throughout our discourse, we maintain R as a fixed commutative ring, anchoring our investigations within a well-defined mathematical framework. As we delve deeper into the intricacies of R-algebroids and their actions, we aim to uncover novel insights and forge connections that resonate across various mathematical domains.

Throughout this paper R will be a fixed commutative ring.

1.1. Preliminaries

Most of the following data can be found in [3–5].

Definition 1.1. An R-category is defined as a category in which each homset possesses an R-module structure, and the composition is R-bilinear. Consequently, a category earns the designation of an R-category only when it satisfies these conditions.

Specifically, a small R-category, termed as an R-algebroid, delineates a more specialized class within this framework. This classification is attributed to a category where homsets exhibit an R-module structure, composition is R-bilinear, and additionally, the category is small in size.

Definition 1.2. An R-linear functor, denoted as an R-functor, denotes a functorial mapping between two R-categories, preserving the R-module structures inherent in their homsets. This functor encapsulates the essence of R-linearity within the categorical framework.

Moreover, within the realm of R-algebroids, an R-functor between two such structures assumes the appellation of an R-algebroid morphism. This morphism elucidates the preservation of the algebraic structure, including R-linearity and compositionality, between the respective R-algebroids.

Definition 1.3. Let A be a pre-R-algebroid, and consider the family $I = \{I(x;y) \subseteq A(x;y) : x, y \in A_0\}$ of R-submodules. If $ab, ba' \in I$ for all $b \in I$, $a, a' \in A$ with $ta = sb$, $tb = sa'$, then I is denoted as a two-sided ideal of A.

Definition 1.4. Let A and N be two pre-R-algebroids sharing the same object set A_0. Consider a family of maps defined for all $x, y, z \in A_0$ as follows:

$$\begin{array}{ccc} N(x,y)\times A(y,z) & \longrightarrow & N(x,z) \\ (n,a) & \mapsto & n^a \end{array}$$

is called a right action of A on N if the conditions

$$\begin{array}{llll} 1. & n^{a_1+a_2} = n^{a_1} + n^{a_2} & 4. & (n'n) = n'n^a \\ 2. & (n_1+n_2)^a = n_1^a + n_2^a & 5. & r \centerdot n^a = (r \centerdot n)^a = n^{r \centerdot a} \\ 3. & (n^a)^{a'} = n^{aa'} & & \end{array}$$

and the condition $n^{1_{tn}} = n$, whenever 1_{tn} exists, are satisfied for all $r \in R$, $a, a', a_1, a_2 \in A$, $n, n', n_1, n_2 \in N$ with compatible sources and targets.

In a similar vein, a left action of A on N is established, albeit with a distinction in the side of application. Additionally, if A exhibits both a right and a left action on N, and if the actions conform to the condition $({}^a n)^{a\prime} = {}^a(n^{a'})$ for all $n \in N$, a, $a' \in A$ with $ta = sn$ and $tn = sa'$, where t denotes the target map and s denotes the source map, then A is termed to possess an associative action on N, or to act associatively on N.

Corollary 1.5. Given two pre-R-algebroids A and N with the same object set

i. if A has a left action on N then ${}^{0_{A(x,sn)}} n = 0_{A(x,tn)}$ and ${}^{-a}n = {}^a(-n) = -{}^a n,$

ii. if A has a right action on N then $n^{0_{A(tn,y)}} = 0_{A(sn,y)}$ and $n^{-a'} = (-n)^{a\prime} = -n^{a\prime}$

for all $n \in N, a, a' \in A, x, y \in A_0$ with $ta = sn, tn = sa'$.

Definition 1.6. Let M is an R-Algebroid, for all $m, m', m'' \in M$, with $t(m) = s(m')$ and $t(m'') = s(m)$

$$Ann_M M = \{m \in M : mm' = 0, m''m = 0, m', m'' \in M\}$$

is called Annihilator of M R-Algebroid.

Definition 1.7. [7] For R-algebroids A and M sharing the same object sets and with A exhibiting an associative action on M, an R-algebroid morphism $\eta : M \to A$ is termed a crossed module of R-algebroids if it satisfies the following conditions:

$$\begin{array}{ll} CM1. & \eta({}^a m) = a\eta(m) \\ & \eta(m^{a'}) = \eta(m)a' \\ CM2. & m^{\eta(m')} = mm' = {}^{\eta(m)} m' \end{array}$$

2. Bimultipliers of an R-algebroid

In this section, we commence our exploration by defining the bimultipliers of an R-algebroid M. Subsequently, we embark on a rigorous proof, establishing that the set of bimultipliers of M indeed forms an R-algebroid, which we aptly term the multiplication R-algebroid. This designation arises from the inherent structure and operations defined on this set, which align with the fundamental principles of R-algebroids.

Definition 2.1. Let M is an R-Algebroid and $f, g : M \to M$ be an R-Linear mappings with identity on object set satisfying the following equations for $m, m' \in M$ with $t(m) = s(m')$,

$$\begin{aligned} f(mm') &= mf(m') \\ g(mm') &= g(m)m' \\ f(m)m' &= mg(m') \end{aligned}$$

The pair (f, g) is called bimultipliers of M. Set of all bimultipliers of M are denoted by $Bim(M)$.

Theorem 2.2. Let $Bim(M)$ be a set of bimultipliers of M. $Bim(M)$ is an R-Algebroid with single object and with the following operations,

$$\begin{aligned} (f,g)+(f',g') &= (f+f',g+g') \\ (f,g)\circ(f',g') &= (f'\circ f, g\circ g') \\ r\cdot(f,g) &= (r\cdot f, r\cdot g) \end{aligned}$$

Proof.

$$\begin{aligned} r\cdot((f,g)+(f',g')) &= r\cdot(f+f',g+g') \\ &= (r\cdot f + r\cdot f', r\cdot g + r\cdot g') \\ &= r\cdot(f,g) + r\cdot(f',g') \end{aligned}$$

$$\begin{aligned} (r_1+r_2)\cdot(f,g) &= ((r_1+r_2)\cdot f, (r_1+r_2)\cdot g) \\ &= (r_1\cdot f + r_2\cdot f, r_1\cdot g + r_2\cdot g) \\ &= (r_1\cdot f, r_1\cdot g) + (r_2\cdot f, r_2\cdot g) \\ &= r_1\cdot(f,g) + r_2\cdot(f,g) \end{aligned}$$

$$\begin{aligned} (r_1r_2)\cdot(f,g) &= (r_1r_2\cdot f, r_1r_2\cdot g) \\ &= r_1(r_2\cdot f, r_2\cdot g) \\ &= r_1\cdot(r_2\cdot(f,g)) \end{aligned}$$

$$\begin{aligned} r\cdot(f,g)\circ(f',g') &= (r\cdot f, r\cdot g)\circ(f',g') \\ &= ((r\cdot f')\circ f, (r\cdot g)\circ g') \\ &= (r\cdot(f'\circ f), r\cdot(g\circ g')) \\ &= r\cdot(f'\circ f, g\circ g') \\ &= r\cdot((f,g)\circ(f',g')) \end{aligned}$$

$$\begin{aligned} (f,g)\circ r\cdot(f',g') &= (f,g)\circ(r\cdot f', r\cdot g') \\ &= ((r\cdot f')\circ f, g\circ(r\cdot g')) \\ &= (r\cdot(f'\circ f), r\cdot(g\circ g')) \\ &= r\cdot(f'\circ f, g\circ g') \\ &= r\cdot((f,g)\circ(f'\circ g')) \end{aligned}$$

In the realm of group theory, the characterization of an action is facilitated by the automorphism group. Specifically, for any group A, its action on itself is delineated by a homomorphism $A \to \mathrm{Aut}(A)$. However, in certain algebraic contexts, the mere structure of automorphisms proves insufficient to define an action. Unlike groups, the set of automorphisms of an algebra typically does not form an algebra itself.

In the study conducted by Arvasi and Ege [2], attention is directed towards the case of commutative algebras, where the limitations of the automorphism structure are explored. Furthermore, MacLane [1] delves into the realm of associative algebras, introducing the notion of the bimultiplication algebra $Bim(M)$ associated with an associative algebra M. This concept serves as an alternative to the automorphism group, effectively fulfilling the role of providing an action within the associative algebraic framework.

Definition 2.3. Let A and M be R-Algebroids with same object we define the set

$$M\ {}_{t}\overset{a}{\times}{}_{s}\ M = \{(m, m') \in M \times M : t(m) = s(a), t(a) = s(m')\}$$

for an $a \in A$.

Theorem 2.4. Let A and M be R-Algebroids with same object set and $Ann(M) = 0$ or $M^2 = M$. For the maps

$$\begin{aligned} f_a : \quad & M \to M \\ & m \mapsto f_a(m) = m^a \end{aligned}$$

and

$$\begin{aligned} g_a : \quad & M \to M \\ & m' \mapsto g_a(m') = {}^a m' \end{aligned}$$

for an $a \in A$ with $(m, m') \in M\ {}_{t}\overset{a}{\times}{}_{s}\ M$, let $(f_a, g_a) \in Bim(M)$. Then the R-Algebroid morphism

$$\begin{aligned} \phi : \quad & A \to Bim(M) \\ & a \mapsto \phi(a) = \phi_a = (f_a, g_a) \end{aligned}$$

gives an R-Algebroid action of A on M.

Proof.

(i) Since ϕ is an R-algebroid homomorphism, then

$$r \cdot \phi(a) = \phi(r \cdot a) \Rightarrow r \cdot \phi(a) = \phi(r \cdot a)$$

for $a \in A$ and

$$\begin{aligned} r \cdot \phi_a(m, m') &= r \cdot (f_a, g_a)(m, m') \\ &= r \cdot (f_a(m), g_a(m')) \\ &= (r \cdot f_a(m), r \cdot g_a(m')) \end{aligned}$$

$$\begin{aligned} \phi_{r \cdot a}(m, m') &= (f_{r \cdot a}, g_{r \cdot a})(m, m') \\ &= (f_{r \cdot a}(m), g_{r \cdot a}(m')) \end{aligned}$$

for $(m, m') \in M\ {}_{t}\overset{a}{\times}{}_{s}\ M$. Therefore we get

$$f_{r \cdot a}(m) = r \cdot f_a(m) \Rightarrow m^{r \cdot a} = r \cdot m^a$$

$$g_{r \cdot a}(m') = r \cdot g_a(m') \Rightarrow m'^{(r \cdot a)} = r \cdot (m'^a) = {}^{r \cdot a} m' = r \cdot {}^a m'.$$

(ii) Since ϕ is an R-Algebroid homomorphism, then

$$\phi(a_1+a_2)=\phi(a_1)+\phi(a_2)\Rightarrow \phi_{a_1+a_2}=\phi_{a_1}+\phi_{a_2}$$

for $a_1,a_2\in A$ with $s(a_1)=s(a_2), t(a_1)=t(a_2)$ and

$$\phi_{a_1+a_2}(m,m')=(f_{(a_1+a_2)},g_{(a_1+a_2)})(m,m')$$

$$\begin{aligned}\phi_{a_1}(m,m')+\phi_{a_2}(m,m') &= (f_{a_1}(m),g_{a_1}(m'))+(f_{a_2}(m),g_{a_2}(m'))\\ &= (f_{a_1}(m)+f_{a_2}(m),g_{a_1}(m')+g_{a_2}(m'))\end{aligned}$$

for $(m,m')\in M\ {}_t\overset{a}{\times}_s M$.

Therefore we get

$$f_{a_1+a_2}(m)=f_{a_1}(m)+f_{a_2}(m)\Rightarrow m^{a_1+a_2}=m^{a_1}+m^{a_2}$$

$$g_{a_1+a_2}(m')=g_{a_1}(m')+g_{a_2}(m')\Rightarrow {}^{a_1+a_2}m'={}^{a_1}m'+{}^{a_2}m'$$

.

(iii) Since $\phi_a=(f_a,g_a)\in Bim(M)$ for $a\in A$, then,

$$\phi_a((m_1,m_1')+(m_2,m_2'))=\phi_a(m_1,m_1')+\phi_a(m_2,m_2')$$

and

$$\begin{aligned}\phi_a((m_1,m_1')+(m_2,m_2')) &= \phi_a(m_1+m_2,m_1'+m_2')\\ &= (f_a(m_1+m_2),g_a(m_1'+m_2'))\\ &= ((m_1+m_2)^a,{}^a(m_1'+m_2')),\end{aligned}$$

$$\begin{aligned}\phi_a(m_1,m_1')+\phi_a(m_2,m_2') &= (f_a(m_1),g_a(m_1'))+(f_a(m_2),g_a(m_2'))\\ &= (m_1^a,{}^am_1')+(m_2^a,{}^am_2')\\ &= (m_1^a+m_2^a,{}^am_1'+{}^am_2')\end{aligned}$$

for $(m_1,m_1'),(m_2,m_2')\in M\ {}_t\overset{a}{\times}_s M$,$(s(m_1)=s(m_2))$ and $(t(m_1')=t(m_2'))$ therefore we get

$$(m_1+m_2)^a=m_1^a+m_2^a$$

and

$${}^a(m_1'+m_2')={}^am_1'+{}^am_2'.$$

(iv) Since $\phi_a=(f_a,g_a)\in Bim(M)$ for $a\in A$, then

$$\begin{aligned}\phi_a(m_1m_2,m_1'm_2') &= (f_a,g_a)(m_1m_2,m_1'm_2')\\ &= (f_a(m_1m_2),g_a(m_1'm_2'))\\ &= ((m_1m_2)^a,{}^a(m_1'm_2'))\end{aligned}$$

and

$$\begin{aligned}(f_a(m_1m_2),g_a(m_1'm_2')) &= (m_1f_a(m_2),g_a(m_1')m_2')\\ &= (m_1(m_2^a),(({}^am_1')m_2'))\end{aligned}$$

for $(m_1m_2, m_1'm_2') \in M\ {}_t\overset{a}{\times}_s M$ and $t(m_1) = s(m_2)$, $t(m_1') = s(m_2')$ therefore we get

$$m_1m_2^a = m_1(m_2)^a$$

and

$$^a m_1'm_2' = (^a m_1'm_2').$$

(v) Since ϕ is an R-Algebroid homomorphism, then

$$\phi_{aa'} = \phi_a \circ \phi_{a'}$$

$$\phi_{aa'} = (f_{aa'}, g_{aa'})$$

$$\begin{aligned} \phi_a \circ \phi_{a'} &= (f_a, g_a) \circ (f_{a'}, g_{a'}) \\ &= (f_{a'} \circ f_a, g_a \circ g_{a'}) \end{aligned}$$

for $a, a' \in A$ with $t(a) = s(a')$ and

$$\begin{aligned} \phi_{aa'}(m, m') &= (f_{aa'}, g_{aa'})(m, m') \\ &= (f_{aa'}(m), g_{aa'}(m')) \\ &= (m^{aa'}, {}^{aa'}m') \end{aligned}$$

$$\begin{aligned} (\phi_a \circ \phi_{a'})(m, m') &= (f_{a'} \circ f_a, g_a \circ g_{a'})(m, m') \\ &= ((f_{a'} \circ f_a)(m), (g_a \circ g_{a'})(m')) \\ &= (f_{a'}(f_a(m)), g_a(g_{a'}(m'))) \\ &= (f_{a'}(m^a), g_a({}^{a'}m')) \\ &= ((m^a)^{a'}, {}^a({}^{a'}m')) \end{aligned}$$

for $(m, m') \in M\ {}_t\overset{aa'}{\times}_s M$, therefore we get $m^{aa'} = (m^a)^{a'}$ and ${}^{aa'}m' = {}^a({}^{a'}m')$.

Thus, $\phi : A \to Bim(M)$ R-Algebroid morphism induces an R-Algebroid action of A on M.

Definition 2.5. Let A be an R-Algebroid. For an R-Algebroid morphism

$$\begin{aligned} \phi : A &\to Bim(A) \\ a &\mapsto \phi(a) = (f_a, g_a) \end{aligned}$$

the pair $(f_a, g_a)(a', a'') = (f_a(a'), g_a(a'')) = (a'a, aa'')$ is called inner bimultipliers of A for $(a', a'') \in A\ {}_t\overset{a}{\times}_s A$. Set of all bimultipliers of A are denoted by $I(A)$ and $I(A) = Im(\phi)$.

Theorem 2.6. Let M be an R-Algebroid. The kernel of homomorphism

$$\begin{aligned} \phi : \quad & M \to Bim(M) \\ & m \mapsto \phi(m) = (f_m, g_m) \end{aligned}$$

is Annihilator of M.

Proof.

The annihilator of M is

$$Ann_M(M) = \{m \in M : f_m(m') = m'm = 0, g_m(m'') = m''m = 0, m', m'' \in M\}.$$

$$\begin{aligned} f_{m_1m_2}(m') &= m'(m_1m_2) \\ &= (m'm_1)m_2 \\ &= f_{m_2}(m'm_1) \\ &= f_{m_2}(f_{m_1}(m')) \\ &= (f_{m_2} \circ f_{m_1})(m') \end{aligned}$$

$$\begin{aligned} g_{m_1m_2}(m'') &= (m_1m_2)(m'') \\ &= m_1(m_2m'') \\ &= g_{m_1}(m_2m'') \\ &= g_{m_1}(g_{m_2}(m'')) \\ &= (g_{m_1} \circ g_{m_2})(m'') \end{aligned}$$

and

$$\begin{aligned} \phi_{m_1m_2} &= (f_{m_1m_2}, g_{m_1m_2}) \\ &= (f_{m_2} \circ f_{m_1}, g_{m_1} \circ g_{m_2}) \\ &= (f_{m_1}, g_{m_1}) \circ (f_{m_2}, g_{m_2}) \\ &= (\phi_{m_1}\phi_{m_2}) \end{aligned}$$

for $(m', m'') \in M \; {}_t\overset{m_1m_2}{\times}{}_s \; M$. Also

$$m \in Ker\phi \quad \Leftrightarrow \phi_m = (f_m, g_m) = (\mathbf{0}, \mathbf{0})$$

and

$$f_m(m') = \mathbf{0}, g_m(m'') = \mathbf{0} \Leftrightarrow m'm = 0, mm'' = 0 \Leftrightarrow m \in Ann_M(M)$$

for $(m', m'') \in M \; {}_t\overset{m}{\times}{}_s \; M$. Thus $Ker\phi = Ann_M(M)$.

Theorem 2.7. Let $I(M)$ be image of $\phi : M \to Bim(M)$ algebroid homomorphism. $I(M)$ is ideal of $Bim(M)$.

Proof.
For $(f_m, g_m) \in I(M)$ and $(f', g') \in Bim(M)$ and $(m', m'') \in M \; {}_t\overset{m}{\times}{}_s \; M$.

$$\begin{aligned} I(M) \times Bim(M) &\to I(M) \\ ((f_m, g_m), (f', g')) &\mapsto ((f_m, g_m) \circ (f', g')) = ((f' \circ f_m), (g_m \circ g')) \end{aligned}$$

$$\begin{aligned} f'f_m(m') &= f'(m'm) \\ &= m'f'(m) \\ &= f_{f'(m)}(m') \end{aligned}$$

$$\begin{aligned} g_mg'(m'') &= mg'(m'') \\ &= f_{g'(m'')}(m) \end{aligned}$$

and

$$\begin{aligned} Bim(M)\times I(M) &\to I(M)\\ ((f',g'),(f_m,g_m)) &\mapsto ((f',g')\circ(f_m,g_m)) = ((f_m\circ f'),(g'\circ g_m)) \end{aligned}$$

$$\begin{aligned} f_m f'(m') &= f'(m')m\\ &= g_{f'(m')}(m) \end{aligned}$$

$$\begin{aligned} g' g_m(m'') &= g'(mm'')\\ &= g'(m)m''\\ &= g_{g'(m)}(m'') \end{aligned}$$

Thus $I(M)$ is ideal of $Bim(M)$.

Definition 2.8. Let $I(M)$ be ideal of $Bim(M)$ algebroid,

$$O(M) = Bim(M)/I(M)$$

division algebroid is called the outer multiplication of M algebroid and denoted by $O(M)$.

Theorem 2.9. Let M be an R-Algebroid such that $Ann(M) = 0$ or $M^2 = M$ and

$$\begin{aligned} \eta : M &\to Bim(M)\\ m &\mapsto \eta(m) = (f_m, g_m) \end{aligned}$$

be an R-Algebroid morphism with the pair $(f_m, g_m)(m', m'') = (f_m(m'), g_m(m'')) = (m'm, mm'')$ for $(m', m'') \in M\ {}_t\overset{m}{\times}_s\ M$. Then $(M, Bim(M), \eta)$ is a crossed module.

Proof.
$Bim(M)$ acts on M by

$$\begin{aligned} Bim(M)\times M &\to M\\ ((f',g'),m') &\mapsto (f',g')\cdot m' = g'(m') \end{aligned}$$

and

$$\begin{aligned} M\times Bim(M) &\to M\\ (m'',(f',g')) &\mapsto m''\cdot(f',g') = f'(m'') \end{aligned}$$

for $(m', m'') \in M\ {}_t\overset{m}{\times}_s\ M$ and

$$\begin{aligned} f'_m : M &\to M\\ m' &\mapsto f'_m(m') = m'm \end{aligned}$$

and

$$\begin{aligned} g'_m : M &\to M\\ m'' &\mapsto g'_m(m'') = mm'' \end{aligned}$$

such that

$$\begin{aligned} \eta : M &\to Bim(M)\\ m &\mapsto \eta(m) = (f'_m, g'_m). \end{aligned}$$

CM1.

$$\begin{aligned}
\eta((f',g')\cdot m)(m',m'') &= \eta(g'(m))(m',m'') \\
&= (f'_{g'(m)}, g'_{g'(m)})(m',m'') \\
&= (m'g'(m), g'(m)m'') \\
&= (f'(m')m, g'(mm'')) \\
&= (f'_m(f'(m')), g'(g'_m(m''))) \\
&= (f'_m f', g' g'_m)(m',m'') \\
&= (f',g') \circ (f'_m, g'_m)(m',m'')
\end{aligned}$$

then

$$\begin{aligned}
\eta((f',g')\cdot m) &= (f',g') \circ (f'_m, g'_m) \\
&= (f',g') \circ \eta(m)
\end{aligned}$$

$$\begin{aligned}
\eta(m\cdot(f',g'))(m',m'') &= \eta(f'(m))(m',m'') \\
&= (f'_{f'(m)}, g'_{f'(m)})(m',m'') \\
&= (m'f'(m), f'(m)m'') \\
&= (f'(m'm), mg'(m'')) \\
&= (f'(f'_m(m')), g'_m(g'(m''))) \\
&= (f' f'_m, g'_m g')(m',m'') \\
&= (f'_m, g'_m) \circ (f',g')(m',m'')
\end{aligned}$$

then

$$\begin{aligned}
\eta(m\cdot(f',g')) &= (f'_m, g'_m) \circ (f',g') \\
&= \eta(m) \circ (f',g')
\end{aligned}$$

CM2.

$$\begin{aligned}
\eta(m')\cdot m &= (f'_{m'}, g'_{m'}) \\
&= g'_{m'}(m) \\
&= m'm
\end{aligned}$$

$$\begin{aligned}
m'\cdot\eta(m) &= m'\cdot(f'_m, g'_m) \\
&= f'_m(m') \\
&= m'm
\end{aligned}$$

Thus $(M, Bim(M), \eta)$ is a crossed module.

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