http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 3, Pages 833–844 (2024)
DOI:10.31801/cfsuasmas.1434079
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: February 8, 2024; Accepted: May 10, 2024

MATHEMATICAL ANALYSIS AND NUMERICAL SIMULATIONS FOR A NONLINEAR KLEIN GORDON EQUATION IN AN EXTERIOR DOMAIN

Gülnihal MERAL

Department of Mathematics, Ankara Yıldırım Beyazıt University, Ankara, TÜRKİYE

Abstract. In this study, the finite propogation speed properties investigated for a two dimensional exterior problem defined by nonlinear Klein-Gordon equation. Under some assumptions on the initial data and the nonlinearity, the solution is shown to have a finite propogation speed. Furthermore, it is demonstrated that the problem has a unique solution, and accurate numerical solutions have been produced by the use of the dual reciprocity boundary element approach with linear radial basis functions.

1. Introduction

Nonlinear wave equations are used to describe various physical problems, including free surface problems, fields generated at the speed of light, large amplitude problems. The nonlinearity may originate from the material constitutive relations, from the large amplitude of the motion, or from the presence of a free boundary [1, 2]. In most cases, nonlinear exterior wave problems are difficult to analyze theoretically and computationally, since there is an added nonlinearity, the problem is time dependent, the domain is unbounded and periodic waves are not possible.

It is known that solutions of the nonhomogenous linear wave equations have quite different behaviour from solutions of parabolic equations since the energy of a pure wave equation is constant and the initial data are transported with finite velocity [3].

For the linear wave equation as well as for the general class of linear hyperbolic equations in [4] it has been shown that any disturbance originated outside the light

2020 *Mathematics Subject Classification.* 35A01, 35A02, 35A23, 65N38.

Keywords. Nonlinear Klein Gordon equation, exterior problem, finite propogation speed, dual reciprocity boundary element method.

gulnihal.meral@aybu.edu.tr; 0000-0003-0072-0609.

cone with a fixed vertex $(\mathbf{x_0}, \mathbf{t_0})$ has no effect on the solution within the cone and consequently has finite propogation speed. The theoretical and computational analyses of nonlinear wave problems are usually complicated, since there is an added nonlinearity, the problem is time dependent, the domain is unbounded and periodic waves are not possible. The trial equation method has been used to find exact solutions of a nonlinear Klein Gordon equation [5]. An energy decay estimate has been derived in [6] and asymptotic behaviour of the energy for periodic solutions has been studied for a particular semi-linear wave equation,namely damped Klein Gordon equation in [7]. The longtime behaviour of a nonlinear exterior wave problem has been studied in [8]. The existence of a global solution has been studied for exponential type nonlinearity and for a Cauchy problem with small data in [9] and [10], respectively. On the other hand, local energy decay properties have been studied for the dissipative exterior Klein-Gordon equation [11]

It is known that the theoretical solutions are not easy to obtain when the equation is nonlinear and particularly if the problem domain is unbounded. The radial basis functions [12], Taylor matrix method [13], finite difference method(FDM) [14, 15] have been used for the solution of the problems defined by nonlinear Klein Gordon equations. One can find differential quadrature solution for the 2-D IBVP in [16] and artificial boundary method has been applied for the initial value problem (IVP) defined by a coupled nonlinear Klein Gordon equations [17].

Most numerical methods such as the finite element method(FEM), FDM and the differential quadrature method (DQM) have some difficulties for unbounded regions, since they need to discretize the domain itself.

A FEM based method, Dirichlet to Neumann FEM (DNFEM) is a general method for the solution of problems in unbounded domains. DNFEM method constructs an equaivalent problem by introducing an artificial boundary and a map is derived between the original domain and the and the artificial boundary. A detailed review on the method can be found in [18]. Later, another alternative numerical method dual reciprocity boundary element method (DRBEM), which has the advantage of discretizing only the boundary of the region, has been applied to the same problem in [19]. However for the unbounded domains one should be careful with the selection of the approximating radial basis functions (RBF), unless the problem is not guaranteed that the solution vanishes far away from the time-space cylinder.

In this paper, an IBVP which has significant applications in quantum physics and defined by a Klein Gordon equation (Section 2)) has been considered. In Sections 3 and 4 the IBVP has been shown to have finite propogation speed under some assumptions on the initial data and the solution has also shown to be unique. Unlike [19], one has the advantage of having freedom in the selection of the approximating RBF, since it is guaranteed by the mathematical analysis. For the numerical solution procedure (Section 5), DRBEM is used with linear RBF. The numerical results have been seen to be consistent with the behaviour of the solution and a well agreement with previously given reference solution [20] has been obtained in terms of absolute maximum error.

2. The Problem Definition

In this paper, the finite propogation properties, the uniqueness and the numerical solution of the IBVP

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = \phi(u) \quad \text{in} \quad \Omega \times (0, \infty) \tag{1}$$

$$u = g_1 \quad \text{on} \quad \Gamma_{g_1}, \quad \frac{\partial u}{\partial n} = g_2 \quad \text{on} \quad \Gamma_{g_2} \tag{2}$$

$$u(x, y, 0) = u_0(x, y), \qquad \frac{\partial u}{\partial t}(x, y, 0) = v_0(x, y) \tag{3}$$

are considered. In (1), c is a constant and $\phi\,(u)$ is a given function of the unknown u. In Equations (1)-(3), the infinite exterior problem domain Ω has an inner boundary $\Gamma = \Gamma_{g_1} \cup \Gamma_{g_2}$, n is inward unit normal, $g_1,\ g_2,\ u_0,\ v_0$ are given functions, u_0 and v_0 have compact support.

3. Domain of Dependence

In this section the domain of dependence of solutions to the nonlinear Klein-Gordon equation is examined. In order to prove finite propogation speed a curved 'cone-like' region C is found as in [4]. To this end, the boundary of C is estimated as a level set $\{\mathfrak{p} = 0\}$ where $\mathfrak{p}$ solves the Hamilton-Jacobi equation

$$\mathfrak{p}_t - c^2 \left(\mathfrak{p}_x^2 + \mathfrak{p}_y^2\right)^{1/2} = 0 \qquad \text{in} \qquad \Omega \times (0, \infty)\,. \tag{4}$$

Separating the variables one can write

$$\mathfrak{p}\,(x, y, t) = \mathfrak{q}\,(x, y) + t - t_0 \qquad ((x, y) \in \Omega, 0 \leq t \leq t_0) \tag{5}$$

where $\mathfrak{q}$ solves

$$\begin{cases} c^2 \left((\mathfrak{q}_x)^2 + (\mathfrak{q}_y)^2\right) = 1 \quad \mathfrak{q} > 0 \quad \text{in} \quad \Omega - \{x_0\} \\ \mathfrak{q}\,(x_0, y_0) = 0. \end{cases} \tag{6}$$

for a fixed $(x_0, y_0) \in \Omega$, $t_0 > 0$. Therefore it is assumed that $\mathfrak{q}$ is a smooth solution of (6) on $\Omega - \{(x_0, y_0)\}$. Now one can define C as,

$$C := \{(x, y, t)\,|\mathfrak{p}\,(x, y, t) < 0\} = \{(x, y, t)\,|\mathfrak{q}\,(x, y) < t_0 - t\} \subset \Omega \times (0, \infty)\,.$$

with the cross section of C_t of C for each $t > 0$,

$$C_t := \{x|\mathfrak{q}\,(x, y) < t_0 - t\}\,. \tag{7}$$

Moreover, the cone C is taken far enough such that both boundaries do not touch each other, i.e., $\bar{C} \cap \bar{\Omega^i} \times [0, T] = \emptyset$ where Ω^i is the interior domain bounded by Γ.

Theorem 1 (Finite Propogation Speed)**.** *Assume u is a smooth solution of Equation (1) with $\phi(u) = -mu^n, \quad m > 0, \quad n$ is a positive odd integer. If $u \equiv u_t \equiv 0$ on C_0, then $u \equiv 0$ within the cone C.*

Proof. Defining the energy

$$e(t) := \frac{1}{2}\left\{\int_{C_t}\left(u_t^2 + c^2\left|\nabla u\right|^2\right)dx\right\} - \int_{C_t}\Phi\left(u\right)dx \tag{8}$$

where $\phi(u) = \frac{\partial\Phi}{\partial u}$, $\dot{e}\left(t\right)$ can be computed by making use of the Coarea formula, [4]

$$\begin{aligned}\dot{e}\left(t\right) \quad &= \quad \underbrace{\left[\int_{C_t}\left(u_t u_{tt} + c^2\nabla u.\nabla u_t\right)dx\right]}_{A} \\ &- \quad \underbrace{\left[\frac{1}{2}\int_{\partial C_t}\left(u_t^2 + c^2\left|\nabla u\right|^2\right)\frac{1}{\left|\nabla q\right|}dS + \int_{C_t}\left(\frac{d}{dt}\phi\left(u\right)\right)dx\right]}_{B} - \underbrace{\left[\int_{\partial C_t}\frac{\phi\left(u\right)}{\left|\nabla q\right|}dS\right]}_{C}\end{aligned} \tag{9}$$

Integration by parts in A yields,

$$A \quad = \quad \textstyle\int_{C_t} u_t\left(u_{tt} - c^2\nabla^2 u\right)dx + c^2\int_{\partial C_t} u_t\left(\nabla u.\nu\right)dS \tag{10}$$

where $\nu = \frac{\nabla q}{|\nabla q|}$ is the outer normal to ∂C_t. Using (6), Cauchy-Schwarz and Cauchy inequalities and the fact that u is a solution of (1) one gets,

$$|A| \quad \leq \quad \textstyle\int_{C_t} u_t\phi\left(u\right)dx + c^2\int_{\partial C_t}\left|u_t\right|\left|\nabla u\right|\frac{1}{|\nabla q|}dS \leq \int_{C_t}\frac{d}{dt}\Phi\left(u\right)dx + B \tag{11}$$

and thus

$$\dot{e}\left(t\right) \leq \int_{\partial C_t}\frac{\Phi\left(u\right)}{\left|\nabla q\right|}dS \leq 0 \tag{12}$$

since $\Phi(u) = -m\frac{u^{n+1}}{n+1}$ $(m, n > 0, n$ is an odd integer) Thus $e\left(t\right)$ is a nonincreasing function of t and hence,

$$e(t) \leq e(0) = 0 \quad\quad \forall \;\; 0 \leq t \leq t_0.$$

On the other hand, by its definition (Equation (8)) $e(t)$ is nonnegative and therefore $u_t, \nabla u, u \equiv 0$ within the cone C.

$\square$

4. Uniqueness of the Solution

In this section the aim is to show the uniqueness of the solution for (1-3) for some particular choice of the nonlinear funciton ϕ in equation 1.

Theorem 2 (Uniqueness)**.** *There exists at most one function $u \in L^2(\Omega)$ solving the initial and boundary value problem (1) -(3) with $\phi(u) = -mu^n$, $m > 0$, n is a positive odd integer.*

Proof. To show uniqueness u and $\tilde{u}$ are assumed to be two different solutions of (1-3). If one considers the L^2 inner product of the functions $u_t - \tilde{u}_t$ then obtains $f(u) - f(\tilde{u})$

$$\begin{aligned} \langle u_t - \tilde{u}_t, \phi(u) - \phi(\tilde{u}) \rangle &= \int_\Omega (u_t - \tilde{u}_t)(\phi(u) - \phi(\tilde{u}))\, dx \\ &= \int_\Omega \left(\tfrac{1}{2} \tfrac{d}{dt} \left(u_t^2 - 2u_t.\tilde{u}_t + \tilde{u}_t^2 \right) \right) \\ &+ \int_\Omega \left(\tfrac{1}{2} \tfrac{d}{dt} \left[(\nabla u)^2 - 2(\nabla u.\nabla \tilde{u}) + (\nabla \tilde{u})^2 \right] \right) dx \\ &+ \int_\Gamma \left(\tfrac{\partial u}{\nabla \nu} - \tfrac{\partial \tilde{u}}{\nabla \nu} \right) (\tilde{u}_t - u_t)\, dS \end{aligned} \tag{13}$$

In (13) integration by parts and the fact that u and $\tilde{u}$ are both solutions of the equation (1) are made use of. Also by the boundary conditions the last integral in (13) vanishes, since both solutions satisfy the same boundary condition. Therefore one finally has,

$$2(u_t - \tilde{u}_t, \phi(u) - \phi(\tilde{u})) = \tfrac{d}{dt} \|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{d}{dt} \|\nabla u - \nabla \tilde{u}\|^2_{L^2(\Omega).}$$

Using Cauchy's inequality with ϵ one gets,

$$\tfrac{d}{dt} \|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{d}{dt} \|\nabla u - \nabla \tilde{u}\|^2_{L^2(\Omega)} \leq \epsilon \|u_t - \tilde{u}_t\|^2_{L^2(\Omega)} + \tfrac{1}{\epsilon} \|\phi(u) - \phi(\tilde{u})\|^2_{L^2(\Omega).}$$

If one selects $\epsilon > 0$ sufficiently small, uses Poincare inequality and Lipschitz continuity of the polynomial $\phi(u)$ then these lead to

$$\frac{d}{dt} \|u - \tilde{u}\|_{L^2(\Omega)} \leq C \|u - \tilde{u}\|_{L^2(\Omega)}$$

Finally Gronwall's inequality gives

$$\|u - \tilde{u}\|_{L^2(\Omega)} \leq C \|u(0) - \tilde{u}(0)\|_{L^2(\Omega)}$$

which results with $u = \tilde{u}$.

□

5. Numerical Solution

In this section, the IBVP defined by (1)-(3) is solved approximately by using a combination of DRBEM and FDM is used as in [19].The DRBEM has the benefit of discretizing only the region's boundary, and it has been used to solve a variety of issues in a wide range of scientific fields, including fluid dynamics and medicine e.g., [22,23]. However here the nonlinearity function is chosen as given in Theorem 1, so that by the theory given in Section 3 , one is free for the selection of approximating RBF.

To see that, before the application of DRBEM, consider the family of cones

$$\begin{aligned} C_{i,T}\left(x_i\right) &= \left\{(x,t) \in \mathbb{R}^2 \times [0,T] \,|\, |x-x_i| \,\tfrac{1}{c} \leq t, \right. \\ & \left. 0 \leq t \leq T,\, x_i \in \left(supp\left(u_0\right) \cup supp\left(v_0\right)\right)\right\}. \end{aligned}$$

By using these cones one can define the following domain

$$B := \cup C_{i,T}\left(x_i\right)$$

where, $B^C : \mathbb{R}^2 \times [0,T]/B$ and naturally the cones in Section 3 are included in B^C. By Theorem 1 in Section 3 it is obvious that for $t \in [0,T]$ all x with $(x,t) \in B^C$, u vanishes, i.e. $u\left(x,t\right) = 0$, since $x \notin supp\left(u_0\right) \cup supp\left(v_0\right).$

To apply DRBEM, Equation(1) is multiplied by the fundamental solution of the Laplace equation (u^*) and integrated over Ω [21], i.e.,

$$\int_\Omega (c^2 \nabla^2 u) u^* d\Omega = \int_\Omega \left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) u^* d\Omega \tag{14}$$

Then, if for the left hand side of Equation 14 integration by parts is applied, one obtains

$$\begin{aligned} \int_\Omega \left(c^2 \nabla^2 u\right) u^* d\Omega &= \int_{\Omega \cap B^T} \left(c^2 \nabla^2 u\right) u^* d\Omega \\ &= c^2 (d_i u_i + \int_\Gamma (\tfrac{\partial u^*}{\partial n} u - u^* \tfrac{\partial u}{\partial n}) d\Gamma) \end{aligned} \tag{15}$$

where $B^T := \{x \in \Omega |\, (x,T) \in B\}$, i denotes the source (fixed) point, and $d_i = \int_\Omega \Delta(x_i, y_i, x, y) dR$. Here the only integral coming from the boundary is coming from the boundary is the obstacle, namely Γ, because of the fact that the solution u vanishes in the region $\bar{B}^C$ identically.

The choice of approximating functions is not restricted, since for $u \in C^2\left(\Omega \times (0,T)\right)$ the integrand of the right hand side,i.e. , $\left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right)$ vanishes outside B by Theorem 1 in Section 3 and the right hand side of (14) can be approximated using linear RBF $f = 1 + r$ with r being the distance function and considered as the modulus of a vector r_{kj} where for each boundary point k on the obstacle, j represents each of the other boundary and internal nodes. The approximation would be as follows:

$$\left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) \approx \sum_{j=1}^{N+L} \alpha_j(t) f_j \tag{16}$$

with N and L being the number of discretization points on the inner boundary and inside the exterior domian, respectively. Choosing the RBF f_j s to be related to the other distance functions $\hat{u}_j(x, y)$ through the relation $\nabla^2 \hat{u}_j = f_j$ with the condition $\sum_{j=1}^{N+L} \alpha_j(t)\, \hat{u}_j \equiv 0$ in B_C^T one obtains,

$$\begin{aligned}
\int_\Omega \left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) u^* d\Omega &= \int_{B^T \cap \Omega} \left(\frac{\partial^2 u}{\partial t^2} - \phi(u)\right) u^* d\Omega \\
&= \int_{B^T \cap \Omega} \nabla^2 u^* \sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j d\Omega \\
&+ \int_\Gamma \left(\frac{\partial}{\partial n} u^*\right) \sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j d\Gamma \\
&- \int_\Gamma u^* \frac{\partial}{\partial n} \left(\sum_{j=1}^{N+L} \alpha_j(t) \hat{u}_j\right) d\Gamma
\end{aligned} \tag{17}$$

Here there is no boundary integral coming from the boundary $B_T \cap \Omega$, since $\sum_{j=1}^{N+L} \alpha_j(t)\, \hat{u}_j \equiv 0$ on the boundary of $B^T \cap \Omega$ by the assumption.

Combining (15) and (17) yields,

$$\begin{aligned}
&c^2 \left(c_i u_i + \int_\Gamma \left(\frac{\partial u^*}{\partial n} u - u^* q\right) d\Gamma\right) = \\
&\qquad \sum_{j=1}^{N+L} \alpha_j(t) \left[c_i \hat{u}_{ij} + \int_\Gamma \left(\frac{\partial u^*}{\partial n} \hat{u}_j - u^* \hat{q}_j\right) d\Gamma\right] \quad .
\end{aligned} \tag{18}$$

where $\hat{q}_j = \frac{\partial \hat{u}_j}{\partial n}$.

Finally, rewriting Equation (18) in matrix vector form, one gets a system of ordinary differential equations of size $(N + L) \times (N + L)$ as;

$$c^2 (\mathbf{Hu} - \mathbf{Gq}) \quad = \quad \mathbf{S}(\ddot{\mathbf{u}} - \boldsymbol{\phi}) \tag{19}$$

where $\mathbf{S} = \left(\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{Q}}\right) \mathbf{F^{-1}}$ with G and H being the matrices with entries containing the fundamental solution of Laplace equation and its normal derivative, respectively. Furthermore, each coloumn of the matrices F , $\hat{U}$ and $\hat{Q}$ consists of the vectors of approximating functions f_j, particular solutions $\hat{u}_j$ and $\hat{q}_j$, respectively. Note that, $\ddot{\mathbf{u}}$ is the $(N+L)\times 1$ vector containing second order time derivative at discretization points.

In order to approximate the solution at the discretization points at a time T, the time interval $[0, T]$ is divided into K with an equal time step size of Δt. The second order time derivative in (19) is discretized by using the central difference scheme having $O(\Delta t^2)$, i.e.,

$$\ddot{\mathbf{u}} = \frac{1}{\Delta t^2}\left(\mathbf{u}^{k+1} - 2\mathbf{u}^k + \mathbf{u}^{k-1}\right) \quad \text{for} \quad k = 0, 1, 2, \cdots N \tag{20}$$

with k denoting the time level. In order to obtain the DRBEM solution for the first time level, the initial conditions (3) are made use of. To this end, the first order derivative in (3) is discretized by using the $O(\Delta t)$ backward difference scheme which gives $\mathbf{u}^{-1}$ at the discretization points as,

$$\mathbf{u}^{-1} = \mathbf{u}^0 - \Delta t \mathbf{v}^0 = \mathbf{u}_0 - \Delta t \mathbf{v}_0 \tag{21}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ contains the values of the initial conditions in (3) at the discretization points.

In order to overcome the stability problems a relaxation procedure is applied for the unknown u as

$$\mathbf{u} = (1-\beta)\mathbf{u}^k + \beta \mathbf{u}^{k+1} \tag{22}$$

positioning the values of $\mathbf{u}$ between the time levels k and $k+1$. Together with the relaxation procedure (22) the unknown $\mathbf{u}$ is obtained at the discretization points by solving the system of linear algebraic equations

$$\mathbf{A_1 u^1} \quad = \quad \mathbf{A_2 u^0} + \boldsymbol{\Delta}\mathbf{t^2}\boldsymbol{\phi}\left(\mathbf{u^0}\right) + \mathbf{A_3 q^1} + \boldsymbol{\Delta}\mathbf{t v^0} \tag{23}$$

$$\mathbf{B_1 u^{k+1}} = \mathbf{B_2 u^k} + \boldsymbol{\Delta}\mathbf{t^2}\boldsymbol{\phi}\left(\mathbf{u^k}\right) + \mathbf{A_3 q^{k+1}} - \mathbf{u^{k-1}} \tag{24}$$

at the first and $(k+1)$-st time levels, respectively. Note that the matrices in equations (23) and (24) are given as

$$\begin{gathered}
\mathbf{A_1} = \left((1-\beta)\,\mathbf{I} + \Delta t^2 \beta \bar{\mathbf{H}}\right), \\
\mathbf{A_2} = \left((1-\beta)\,I - \Delta t^2\,(1-\beta)\,\bar{\mathbf{H}}\right), \quad \mathbf{A_3} = \Delta t^2 \bar{\mathbf{G}} \\
\mathbf{B_1} = (1-2\beta)\,I + \Delta t^2 \beta \bar{\mathbf{H}}, \\
\mathbf{B_2} = 2\,(1-\beta)\,\mathbf{I} - \Delta t^2\,(1-\beta)\,\bar{\mathbf{H}}
\end{gathered}$$

with

$$\bar{\mathbf{H}} = -c^2 \mathbf{S}^{-1}\mathbf{H}, \quad \bar{\mathbf{G}} = -c^2 \mathbf{S}^{-1}\mathbf{G}.$$

Example: Computations have been carried out for the initial and boundary value problem (1)-(3) with $\phi(u) = -mu^3$ and $c = 200$. The region is taken as the exterior of a circle with radius $a = 0.25$. The initial conditions u_0 and v_0 are taken to be 0 and the solution is assumed to be 1 on the interior boundary. A reference solution [20] is used to compare the numerical results with the technique described here. In the calculations, $N = 120$ constant boundary elements and $L = 100$ interior points are used, and the time step is taken as 0.01. In order to compare the results with the reference solution (denoted by u_{ref}) in [20], the interior points are taken along a portion (between $a = 0.25$ and $R = 0.5$) of a straight line radiating from the origin of the circle.

In Tables 1 and 2, both of the solutions, obtained here(u_{DRBEM}) and the reference solution [20] (u_{ref}), are presented, for different times at the point $R = 0.5$ with $\phi(u) = -mu^3$ for $m = 0.0$, $m = 10000$, respectively. In the third row of the table the reference and DRBEM solutions are compared in terms of absolute relative error τ which is calculated by

$$\tau = \left| \frac{u_{ref} - u_{DRBEM}}{u_{ref}} \right|$$

One can observe that the DRBEM solution with linear RBF are accurate almost with 4 significant digits for both linear and nonlinear cases.

In the Figures 1 and 2, the reference solution and the DRBEM solution are illustrated at different times for the linear and nonlinear ($m = 10000$) cases; respectively. One can see that DRBEM solution agrees well with the reference solution.

Table 1. $R = 0.5, m = 0$

t	0.02	0.05	0.08	0.1
u_{ref}	0.9441	0.8174	0.8673	0.8598
u_{DRBEM}	0.9448	0.8176	0.8677	0.8593
τ	7×10^{-4}	2×10^{-4}	4×10^{-4}	5×10^{-4}

Table 2. $R = 0.5, m = 10000$

t	0.02	0.05	0.08	0.1
u_{ref}	0.9164	0.7509	0.7867	0.7736
u_{DRBEM}	0.9165	0.7511	0.7861	0.7742
τ	1×10^{-4}	2×10^{-4}	6×10^{-4}	6×10^{-4}

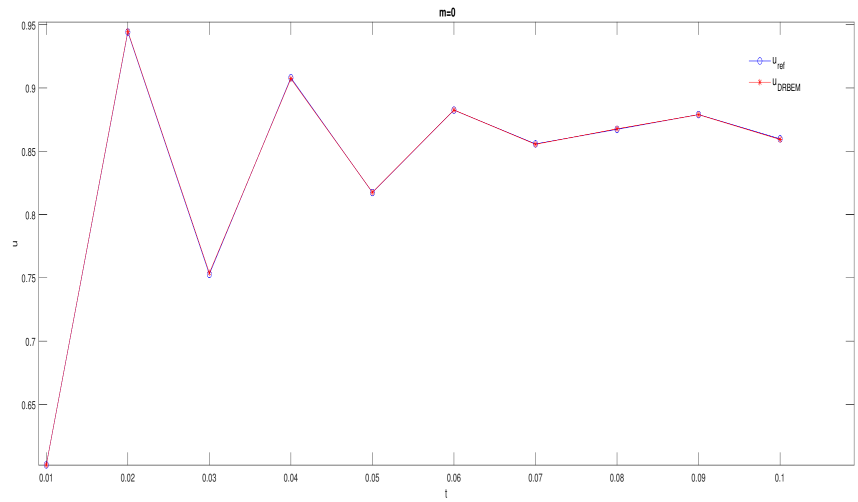

FIGURE 1. Reference and DRBEM Solutions for $m = 0$ at different times

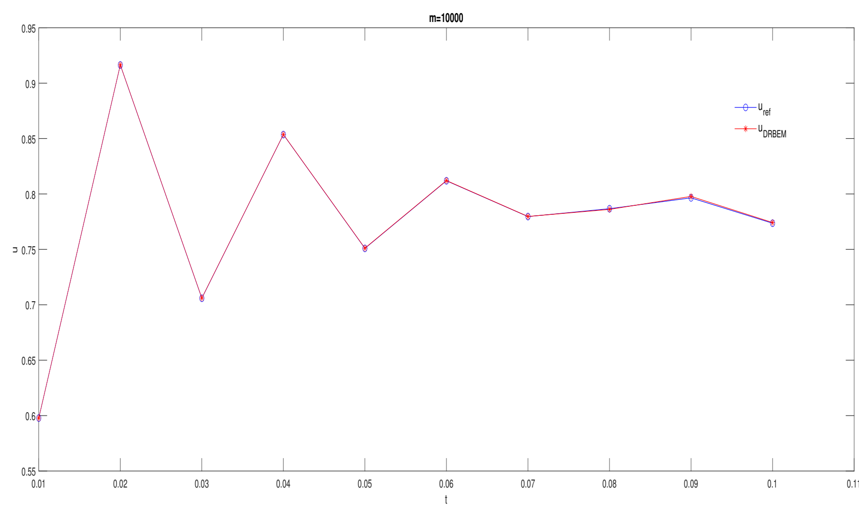

FIGURE 2. Reference and DRBEM Solutions for $m = 10000$ at different times

6. Conclusion

In this paper, finite propogation speed properties are shown for a nonlinear two-dimensional exterior Klein Gordon problem. The solution of the problem is shown to be unique. For the numerical solution of the corresponding problem DRBEM is used and the nonhomogenity is approximated with the help of linear RBF which is known to have some difficulties when the domain is an exterior one. However, the theory in Section 3 shows that under certain conditions on the nonlinearity and initial conditions the solution is vanishing far away from the time-space cyclinder and thus the approximation by the RBF is taken only within the finite region of integration. The numerical results show good agreement with a previously obtained reference solution in terms of absolute relative error.

Declaration of Competing Interests This work does not have any conflict of interest.

References

[1] Segel, L. A., Mathematics Applied to Continuum Mechanics, Macmillan Publication, New York, 1977.
[2] Whitham, G. B., Linear and Nonlinear Waves, Wiley Interscience Publication, New York, 1974.
[3] Axelsson, O., Finite Difference Methods, Encyclopedia of Computational Mechanics, Stein E.,de Borst R.,Hughes T.,eds., Vol. 1, chap. 2, John Wiley & Sons. Ltd., West Sussex, 2004.
[4] Evans, L. C., Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, United States of America, 1998.
[5] Mácias-Díaz, J. E., Medina-Guavera, M. G., Vargas-Rodríguez, H., Exact solutions of non-linear Klein-Gordon equation with non-constant coefficients through the trial equation method, *J. Math. Chem.*, 59 (2021) 827-839. https://doi.org/10.1007/s10910-021-01220-y
[6] Nakao, M., Energy decay for a nonlinear generalized Klein-Gordon equation in exterior domains with a nonlinear localized dissipative term, *J. Math. Soc. Japan.*, 64(3), (2012) 851-883. https://doi.org/10.2969/jmsj/06430851
[7] Mohamad, H., Energy asymptotics for the strongly damped Klein-Gordon equation, *Partial Differ. Equ.*, 3(71) 2022, 1-12. https://doi.org/10.1007/s42985-022-00207-x
[8] Datti, P. S., Nonlinear wave equations in exterior domains, *Nonlinear Anal. Theory Methods Appl.*, 15(4), (1990) 312-331. https://doi.org/10.1016/0362-546X(90)90140-C
[9] Taskesen, H., Global existence and nonexistence of solutions for a Klein-Gordon equation with exponential type nonlinear term, *TWMS J. App. and Eng. Math.*, 10(3), (2020) 669-676.
[10] Hörmander, L., Remarks on the Klein-Gordon equation, *J. Équations aux Dérivées Partielles*, (1987), 1-9.
[11] Malloug, M., Local energy decay for the damped Klein-Gordon equation in exterior domain, *Appl. Anal.*, 96(2)(2017), 349-362. https://doi.org/10.1080/00036811.2015.1136821
[12] Dehghan, M., Shokri, A., Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions, *J. Comput. Appl. Math.*, 230 (2009), 400-410. https://doi.org/10.1016/j.cam.2008.12.011
[13] Bülbül, B., Sezer, M., A new approach to numerical solution of nonlinear Klein-Gordon equation, *Math. Probl. Eng.*, 2013(2013), 1-7. http://dx.doi.org/10.1155/2013/869749

[14] Mácias-Díaz, J. E., Puri, A., A numerical method for computing radially symmetric solutions of a dissipative nonlinear modified Klein Gordon equation, *Numer. Methods Partial Differ. Equ.*, 21(5), (2005),998-1015. https://doi.org/10.1002/num.20094

[15] Mácias-Díaz, J. E., On the bifurcation of energy in media governed by (2+1)-dimensional modified Klein-Gordon equations, *Appl. Math. Comput.*, 206, (2008), 221-235. https://doi.org/10.1016/j.amc.2008.09.013

[16] Pekmen, B., Tezer-Sezgin, M., Differential quadrature solution of nonlinear Klein-Gordon and sine-Gordon equations, *Comput. Phys. Commun.*, 183 (2012), 1702-1713. https://doi.org/10.1016/j.cpc.2012.03.010

[17] Tai, Y., Zhou, Z., Jiang, Z., Numerical solution of coupled nonlinear Klein-Gordon equations on unbounded domains, *Phys. Rev. E.*, 106(2) (2022), 025317-1-10. https://doi.org/10.1103/PhysRevE.106.025317

[18] Givoli, D., Recent Advances in the DtN FE Method, *Arch. Comput. Methods Eng.*, 6(2), 71-116, 1999. https://doi.org/10.1007/BF02736182

[19] Meral, G., Tezer-Sezgin, M., DRBEM solution of exterior wave problem using FDM and LSM time integrations. *Eng. Anal. Bound. Elem.*, 34 (2010) 574-580. https://doi.org/10.1016/j.enganabound.2010.01.006

[20] Givoli, D., Patlashenko, I., Finite element solution of nonlinear time dependent exterior wave problems, *Jour. of Comput. Phys.*, 143 (1998) 241-258. https://doi.org/10.1006/jcph.1998.9999

[21] Brebbia, C. A., Dominguez, J., Boundary Elements an Introductory Course, 2nd edn. Comput. Mech. Publications, Southampton, Boston, 1992.

[22] Senel, P., Comparison study on the numerical stability of dual reciprocity boundary element method for the MHD slip flow problem, *Eng. Anal. Bound. Elem.*, 151(2023) 370-386. https://doi.org/10.1016/j.enganabound.2023.03.010

[23] Meral, G., DRBEM-FDM solution of a chemotaxis–haptotaxis model for cancer invasion, *J. Comput. Appl. Math.*, 354 (2019) 299-309. https://doi.org/10.1016/j.cam.2018.04.047