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Non-Hermitian Dynamics without all-to-all Coupling in Odd-Channel Systems

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Abstract

Here we analytically derive the eigenspectra of non-Hermitian systems with an odd number of channels in a scheme that eliminates the requirement of all-to-all coupling of the channels. The calculated spectrum displays coexisting parity-time broken and parity-time symmetric modes for any choice of coupling and gain/loss values except for the case of N=3 where a certain range of coupling and gain/loss parameters are required for the coexistence of parity-time broken and parity-time symmetric modes.

1. Introduction

Interest in non-Hermitian systems was ignited by the end of the twentieth century following Bender and Boettcher's seminal paper [1], in which they showed that non-Hermitian Hamiltonians can have real spectra if they are invariant under parity-time reversal (PT) transformation. This necessary but not sufficient condition is then classified by Mostafazadeh, revealing that PT-symmetric Hamiltonians fall under the class of pseudo-Hermitian Hamiltonians which possess real spectra [2]. Thus it is understood that for a certain parameter range, PT-symmetric Hamiltonians display real spectra. Then a plethora of applications of PT-symmetry followed, demonstrating various exploitations of PT-symmetry in optical and photonic systems [3-8]. Over the last decade, the interest in non-Hermitian systems continued, in Floquet systems [9,10], waveguides [11] and single-mode lasers [12].

The coexistence of real and complex eigenvalues gives rise to many interesting features for non-Hermitian systems. It is known that the coexistence of PT-symmetric and PT-broken phases requires physical dimensions higher than one, or equivalently at least two input and two output channels [7]. However, it was theoretically shown that it is possible to obtain a mixing of PT-broken and PT-symmetric phases with two channels by introducing other degrees of freedom such as polarization [13] and spin [14]. Furthermore, possible applications of four-channel PT-symmetric systems were investigated [15] and the analytical calculations to obtain the eigenspectra of N-channel PT-symmetric systems for an even number of channels were performed [16]. One drawback of the mentioned N-channel configuration is that it requires all-to-all coupling of the channels. In this manuscript, we will study an alternative case which possesses equal richness in terms of physics without all-to-all coupling. Here we analytically study the odd-channel case and obtain the eigenspectra for N-channel non-Hermitian systems consisting of a neutral channel and (N-1)/2 pairs of gain and loss channels which only couple to the neutral channel. Such platforms could be exploited for realizing PT optoelectronic oscillators [17] with extended features suggested for the all-to-all coupling case [15].

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Therefore, lifting the necessity of all-to-all coupling opens the path for possible experimental demonstrations of the general scheme we suggest here.

2. Methods

Before starting the calculations, it would be necessary to clarify the terminology used throughout the paper. In all cases studied, we start with a non-Hermitian but PT-symmetric matrix. Then we calculate the eigenspectrum. When the eigenvalues are real we say the system is in PT-symmetric phase whereas for the complex eigenvalues, we say the system is in the PT-broken phase.

2.1. Three-Channel Case

It is necessary to clarify that for the neutral channel we use in all cases, we will neglect any possible phase term since this is not the main interest of this paper. The inclusion of such phase terms would be interesting and would be the subject of a separate work by itself.

As a warm-up, we start with the simplest odd-channel case, namely three channels. We study the case with one gain, one loss and one neutral channel as shown in Figure 1. We consider this case (and the other cases that will follow) under the following assumptions: (i) All channels have the same frequency ω. (ii) Gain and loss channels have equal gain/loss values which we set to γ. (iii) Gain and loss channels only couple to the neutral channel with a coupling constant κ.

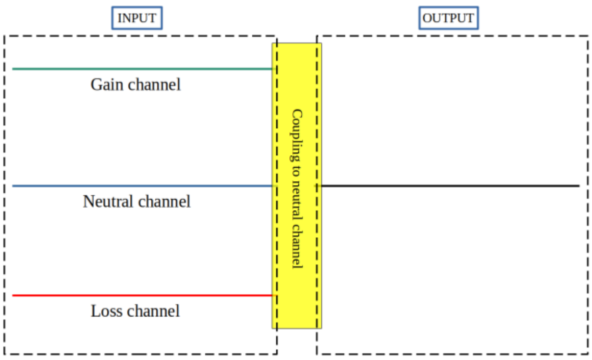

Figure 1. Schematic description of the 3-channel system with one loss, one gain and one neutral channel. Gain and loss channels only couple to the neutral channel

The coupled mode equations for three channels can be expressed as $i\dot{\vec{a}} = \mathbb{M}_3\vec{a}$ where dot denotes time derivative, $\vec{a} = [a_1, a_2, a_3]^T$ are the amplitudes in respective channels and $\mathbb{M}_3$ is the dynamical matrix given by:

$$\mathbb{M}_3 = \begin{pmatrix} \omega + i\gamma & \kappa & 0 \\ \kappa & \omega & \kappa \\ 0 & \kappa & \omega - i\gamma \end{pmatrix}, \quad (1)$$

from which the eigenfrequencies can be calculated by solving $\det[\mathbb{M}_3 - \mathbb{1}\,\lambda] = 0$, (where $\mathbb{1}$ 3×3 is unit matrix) that yields:

$$\lambda_1 = \omega,$$
$$\lambda_{2,3} = \omega \pm \sqrt{2\kappa^2 - \gamma^2}. \quad (2)$$

As it can be seen from Equation (2), the system is in PT-symmetric phase for $2\kappa^2 > \gamma^2$, whereas for $2\kappa^2 < \gamma^2$ the it is in mixed phase, with one eigenvalue being real and other two being complex.

2.2. Five-Channel Case

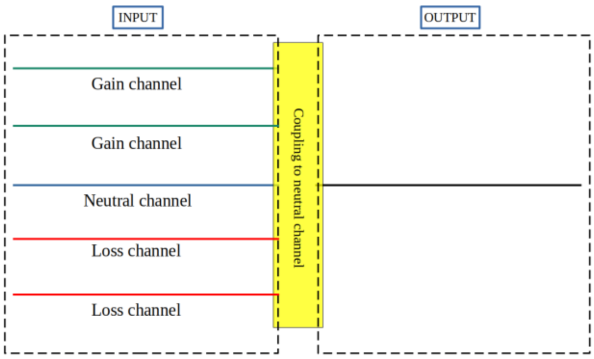

Figure 2. Schematic description of the five-channel system with two loss channels, two gain channels and one neutral channel. Gain and loss channels only couple to the neutral channel

Next, we study the five-channel case. The pictorial description is given in Figure 2. Coupled mode equations for the five-channel case is given by $i\dot{\vec{a}} = \mathbb{M}_5\vec{a}$ with $\vec{a} = [a_1, a_2, a_3, a_4, a_5]^T$ and the dynamical matrix given by:

$$\mathbb{M}_5 = \begin{pmatrix} \omega + i\gamma & 0 & \kappa & 0 & 0 \\ 0 & \omega - i\gamma & \kappa & 0 & 0 \\ \kappa & \kappa & \omega & \kappa & \kappa \\ 0 & 0 & \kappa & \omega + i\gamma & 0 \\ 0 & 0 & \kappa & 0 & \omega - i\gamma \end{pmatrix}$$

(3)

The eigenspectrum can be deduced by solving $\det[\mathbb{M}_5 - \mathbb{1}\ \lambda] = 0$ where $\mathbb{1}$ this time is five by five:

$$\lambda_1 = \omega \quad (4a)$$

$$\lambda_{2,3} = \omega \pm i\gamma \quad (4b)$$

$$\lambda_{4,5} = \omega \pm \sqrt{4\kappa^2 - \gamma^2} \quad (4c)$$

There is a crucial difference between the three-channel case and the present case that now the system cannot be purely in the PT-symmetric phase, with one eigenvalue always being real, two of them being complex and the other two being real for $4\kappa^2 > \gamma^2$ and complex for $4\kappa^2 < \gamma^2$.

2.3. N-Channel Case

We finally derive the eigenspectrum for the most general case, i.e. N-odd channels. The schematic view is given in Figure 3. The coupled mode equations have the form $i\dot{\vec{a}} = \mathbb{M}_N \vec{a}$ with $\vec{a} = [a_1, a_2, \ldots, a_N]^T$ and

the dynamical matrix given by:

$$\mathbb{M}_N = \mathbb{D}_N + \mathbb{C}_N \quad (5)$$

$$\mathbb{D}_N = \begin{pmatrix} \omega + i\gamma & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \omega - i\gamma & 0 & \ldots & \ldots & \ldots & \vdots \\ \vdots & 0 & \ddots & 0 & \ldots & \ldots & \vdots \\ \vdots & \ldots & 0 & \omega & 0 & \ldots & \vdots \\ \vdots & \ldots & \ldots & 0 & \ddots & 0 & \vdots \\ \vdots & \ldots & \ldots & \ldots & 0 & \omega + i\gamma & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & \omega - i\gamma \end{pmatrix} \quad (6a)$$

$$\mathbb{C}_N = \begin{pmatrix} 0 & \ldots & 0 & \kappa & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & 0 & \kappa & 0 & \ldots & 0 \\ \kappa & \ldots & \kappa & 0 & \kappa & \ldots & \kappa \\ 0 & \ldots & 0 & \kappa & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & 0 & \kappa & 0 & \ldots & 0 \end{pmatrix} \quad (6b)$$

To obtain the eigenvalues we solve $\det[\mathbb{M}_N - \mathbb{1}\ \lambda] = 0$ ($\mathbb{1}$ is N by N):

$$\lambda_1 = \omega \quad (7a)$$

$$\lambda_{2 \ldots N-2} = \omega \pm i\gamma \quad (7b)$$

$$\lambda_{N-1,N} = \omega \pm \sqrt{(N-1)\kappa^2 - \gamma^2} \quad (7c)$$

For the case of N-channels, as a generalization of the five-channel case studied above, PT-broken and PT-symmetric phases coexist for any value of κ and γ. One of the eigenvalues is always real, whereas two pairs of eigenvalues with (N-3)/2-degeneracy are complex. The final two eigenvalues are real for $(N-1)\kappa^2 > \gamma^2$ and complex for $(N-1)\kappa^2 < \gamma^2$.

2.4. Symmetries of the Dynamical Matrix

It is noteworthy to show the relevant symmetries of the dynamical matrix for the N-odd channel case. The dynamical matrices we studied here are all non-Hermitian. Further, they are PT-symmetric, which requires their invariance under PT transformation, or in other words, PT operator commutes $\mathbb{M}_N$, which is satisfied as shown below:

$$(\mathcal{PT})\mathbb{M}_N(\mathcal{PT})^{-1} = \mathbb{M}_N. \quad (8)$$

Here $\mathbb{M}_N$ is given by Equations (5,6) and parity and time reversal transformations are given by:

$$\mathcal{P} = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ \vdots & ⋰ & ⋰ & 0 \\ 0 & ⋰ & ⋰ & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \qquad \mathcal{T} = \hat{K}. \quad (9)$$

where $\mathcal{P}$ is N by N and $\hat{K}$ is complex conjugation.

3. Results and Discussion

Different than the all-to-all coupling case for N-channels (N even) [16], our results showed that for the case without all-to-all coupling in the odd-channel case, one always ends up with mixing of PT-broken and PT-symmetric phases (except for the N=3 case discussed below). Equations 7(a-c) show that for every value of κ and γ coexistence of PT-broken and PT-symmetric phases is obtained. The advantage of the suggested scheme is due to the lifting of the necessity of all-to-all coupling which is experimentally challenging to satisfy.

The scheme studied here yields at least one real eigenvalue (system being in PT-symmetric phase), in contrast with the all-to-all coupling case in which real eigenvalues are only obtained for a certain range of κ and γ. Therefore, compared with the all-to-all coupling scheme, the odd-channel scheme studied here is more favorable for obtaining coexisting PT-symmetric and PT-broken phases. The only exception is the special case of the suggested scheme for N=3, in which for $2\kappa^2 > \gamma^2$ the system is purely in the PT-symmetric phase and coexistence of PT-broken and PT-symmetric phases only arises when $2\kappa^2 < \gamma^2$ as seen from Equations (2a-b). However, starting with the N=5 case, the system always displays the coexistence of PT-broken and PT-symmetric phases.

4. Conclusions

In conclusion, we analytically derived the eigenspectra for the (odd) N-channel non-Hermitian systems that do not rely on all-to-all coupling of its channels. We showed that except for the case of N=3, all other configurations always yield a mixing of PT-broken and PT-symmetric phases independent of the values of κ and γ. Thus the scheme suggested here as compared to all-to-all coupling schemes has the advantage of yielding coexisting PT-symmetric and PT-broken phases without the need of tuning κ and γ. Moreover, in terms of experimental implementation, the suggested scheme here is more favorable, since it eliminates the requirement of all-to-all coupling.

Declaration of Ethical Standards

The author of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Conflict of Interest

The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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