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Unbounded Star Convergence in Lattices

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ABSTRACT

Let L be a vector lattice, (x_α) be a L-valued net, and $x \in L$. If $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in L_+$ then it is said that the net (x_α) unbounded order converges to x and is denoted by $x_\alpha \xrightarrow{uo} x$. This definition of unbounded order convergence has been extensively studied on many structures, including vector lattices, local solid vector lattices, normed lattices and lattice normed spaces. It is not possible to apply this type of convergence to general lattices due to the lack of algebraic structure. Therefore, we will use a type of convergence that is considered to be the motivation for this type of convergence, first defined as independent convergence in semi-ordered linear spaces and later called unbounded order convergence. Namely, L is a lattice, (x_α) is an L-valued net, and $x \in L$. If $(x_\alpha \wedge b) \vee a$ order converges to $(x \wedge b) \vee a$ for every $a, b \in L$ with $a \leqslant b$, then it is said that (x_α) individual converges to x or unbounded order converges to x. This definition can be easily applied to general lattices. In this article, this definition will be understood as unbounded order convergence. Also, even if these two convergences are called by the same name, there is no equivalence between them for general lattices, an example of this is mentioned in this article. Let L be a partially ordered set, (x_α) be an L-valued net and $x \in L$, (x_α) is said to be star convergent to x if every subnet of the net (x_α) has a subnet that is order convergent to x and denoted by $x_\alpha \xrightarrow{s} x$. In this paper, a new type of convergence on lattices is defined by combining unbounded order convergence (individual convergence) and star convergence. Let L be a lattice, (x_α) a net and $x \in L$, (x_α) is said to be unbounded star convergent to x if for every subnet (x_β) of (x_α), there exists a subnet (x_ζ) of (x_β) such that $(x_\zeta \wedge b) \vee a \xrightarrow{o} (x \wedge b) \vee a$ for every $a, b \in L$ with $a \leqslant b$ and it is denoted by $x_\alpha \xrightarrow{us} x$. The differences between the new type of convergence, called unbounded star convergence, and order convergence, star convergence are demonstrated with counterexamples. The meaningfulness of the unbounded star convergence type is analyzed with these counterexamples and the implications presented. In addition, basic questions about unbounded star convergence of a given net on lattices such as convergence of a fixed net, uniqueness of the limit, convergence of the subnet of a convergent net are answered.

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L bir vektör latis, $(x_\alpha)L$-değerli bir net ve $x \in L$ olmak üzere her $u \in L_+$için $|x_\alpha - x| \wedge u \overset{o}{\rightarrow} 0$ ise x_α neti x elemanına sınırsız sıra yakınsıyor denir ve $x_\alpha \overset{uo}{\rightarrow} x$ ile gösterilir. Sınırsız sıra yakınsaklığın bu tanımı vektör latislerde, yerel solid vektör latislerde, normlu latislerde ve latis normlu uzaylarda başta olmak üzere birçok yapı üzerinde fazlasıyla çalışılmıştır. Bu yakınsaklık tipini genel latislere uygulamak cebirsel bir yapının yokluğundan dolayı mümkün değildir. Bundan dolayı bu yakınsaklık tipinin motivasyonu olarak kabul edilen ve ilk olarak yarı-sıralı lineer uzaylarda bağımsız yakınsama adı ile tanımlanan ve sonrasında sınırsız sıra yakınsama olarak adlandırılan bir yakınsama tipi kullanılacaktır. Şöyle ki L bir latis, $(x_\alpha)L$-değerli bir net ve $x \in L$ olmak üzere $a \leqslant b$ koşulunu sağlayan her $a, b \in L$ için $(x_\alpha \wedge b) \vee a$ neti $(x \wedge b) \vee a$ elemanına sıra yakınsıyor ise $(x_\alpha), x$ *elemanına* bağımsız yakınsıyor ya da sınırsız sıra yakınsıyor denir. Bu tanım genel latislere rahatlıkla uygulanabilir. Bu makalede sınırsız sıra yakınsamadan bu tanım anlaşılacaktır. Ayrıca bu iki yakınsama aynı isimle anılsa dahi aralarında genel latisler için herhangi bir denklik bulunmamaktadır, buna dair bir örnekten makalede bahsedilmiştir. L kısmi sıralı bir küme, $(x_\alpha)L$-değerli bir net ve $x \in L$ olmak üzere (x_α) netinin her alt netinin x elemanına sıra yakınsayan bir alt neti var ise (x_α), x elemanına yıldız yakınsıyor denir ve $x_\alpha \overset{s}{\rightarrow} x$ ile gösterilir. Bu makalede sınırsız sıra yakınsama (bağımsız yakınsama) ile yıldız yakınsama kombine edilerek latisler üzerinde yeni bir yakınsama tipi tanımlanmıştır. Şöyle ki L bir latis, (x_α) bir net ve $x \in L$ olsun. $a \leqslant b$ koşulunu sağlayan her $a, b \in L$ için (x_α)'nın her alt neti (x_β)'nın $(x_\zeta \wedge b) \vee a \overset{o}{\rightarrow} (x \wedge b) \vee a$ olacak biçimde bir (x_ζ) alt neti var ise (x_α), x elemanına sınırsız yıldız yakınsıyor denir ve $x_\alpha \overset{us}{\rightarrow} x$ ile gösterilir. Sınırsız yıldız yakınsama adı verilen yeni tip yakınsamanın sıra yakınsama ve yıldız yakınsama ile farklılıkları ters örnekler ile ortaya konulmuştur. Bu ters örnekler ve sunulan gerektirmeler ile sınırsız yıldız yakınsama tipinin anlamlılığı incelenmiştir. Ayrıca latisler üzerinde verilen bir netin sınırsız yıldız yakınsaklığı ile ilgili sabit netin yakınsaklığı, limitin biricikliği, yakınsak bir netin alt netinin yakınsaklığı gibi temel sorulara cevap verilmiştir.

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1. Introduction

Much as there have been defined different convergence types by using ordering in a lattice L, one of the main and essential convergence is order convergence in a lattice L introduced by Birkhoff (1967) a net $(x_\alpha)_{\alpha \in I}$ is said to be order convergent to $x \in L$ (Briefly; $x_\alpha \overset{o}{\rightarrow} x$) if there exists two monotonic directed nets (t_α) and (u_α) such that $t_\alpha \uparrow x, u_\alpha \downarrow x$ and $t_\alpha \leqslant x_\alpha \leqslant u_\alpha$ for each $\alpha \in I$.

There is also another definition of order convergence in lattices, defined in Birkhoff (1967) and Lowig (1941), named as O_2-convergence since the first definition generally was named as O_1-convergence, defined that; let (x_α) be a net in a lattice and let

$P := \{p\colon \text{there exists } \beta \text{ such that } p \leqslant x_\alpha, \forall \alpha \geqslant \beta\}$

and

$Q \coloneqq \{q\colon \text{there exists } \beta \text{ such that } q \geqslant x_\alpha, \forall \alpha \geqslant \beta\}$.

If the supremum of the set P and infumum of Q exist and equal to $x \in L$, then it is said that $(x_\alpha)O_2$ -converges to x.

If the lattice L is a complete lattice, then the notion of order in a vector lattice convergence is given equivalently as (x_α) order converges to x if and only if

$$\inf_\beta \left(\sup_{\alpha \geqslant \beta} x_\alpha \right) = \mathrm{limsup} x_\alpha = x = \lim\inf x_\alpha = \sup_\beta \left(\inf_{\alpha \geqslant \beta} x_\alpha \right).$$

The other convergence type is star convergence presented and studied in (Birkhoff,1967) by Birkhoff which was introduced by independently in Uryshon (1926), Kantorovich (1937) and Von Neumann (1935), respectively. Here is the definition of star convergence; A net (x_α) in a lattice $(L, \leqslant)$ is star convergent to $x \in L$ (Briefly; $x_\alpha \overset{s}{\rightarrow} x$) if every subnet (x_β) of (x_α) has a subnet (x_γ) such that $x_\gamma \overset{o}{\rightarrow} x$

Another type of convergence, individual convergence, which we call unbounded order convergence after the work of DeMarr (1964), was defined by Nakano (1948) for semi-ordered linear spaces and later studied by Kaplan (1997) for vector lattices. A net $(x_\alpha)_{\alpha \in I}$ in a vector lattice is said to be unbounded order convergent to $x \in L$ (Briefly; $x_\alpha \overset{uo}{\rightarrow} x$) if for every pair $a, b \in L$ with $a \leqslant b$ the net $(x_\alpha \wedge b) \vee a$ order converges to $(x \wedge b) \vee a$.

Unbounded star convergence, defined above, could be understood in the sense of convergence type, defined by Wickstead (1977) in vector lattices. Namely in a vector lattice, say L, a net (x_α) is called unbounded order convergent to x, if the net $(| x_\alpha - x | \wedge u)$ is order convergent to zero for any vector u in L^+, briefly; $x_\alpha \overset{uo}{\rightarrow} x$. Even though two different convergences have the same name, the individual convergence is suitable for lattice structures because it does not require an algebraic system. Moreover, even in vector lattices these two types of convergence do not coincide, for which a counterexample is given in this paper.

The definition of the new convergence type to be taken as basis in this study is as follows:

Definition 1.1

Let $(L, \leqslant)$ be a lattice, (x_α) be a net in L and $x \in L$ be an element. (x_α) is unbounded star convergence to $x \in L$ (Briefly; $x_\alpha \overset{us}{\rightarrow} x$) if every subnet (x_β) of (x_α) has a subnet (x_γ) such that (x_γ) unbounded order convergent to x.

2. Materials and Method

2.1. Meaningfulness of the Definition of Unbounded Star Convergence

In this section, we will examine the meaningfulness of the definition of unbounded order convergence in relation to order convergence, star convergence and unbounded order convergence in general lattices with the help of counterexamples and implications.

Let $(L, \leqslant)$ be a lattice, $(x_\alpha)_{\alpha \in I}$ be a net in L and $x \in L$. If (x_α) order converges to x, so there exists two monotonic directed nets (t_α) and (u_α) such that $t_\alpha \uparrow x, u_\alpha \downarrow x$ and $t_\alpha \leqslant x_\alpha \leqslant u_\alpha$ for each $\alpha \in I$. Since the subnet of any subnet of net (x_α) will satisfy the above monotonicity and squeezing condition for a cofinal and increasing index $\alpha_{\beta\xi}$ with (t_α), (u_α) the net (x_α) star converges to x. Let the net (x_α) order converges to an element $x \in L$, from the isotoneness of the join and meet operations

$(t_\alpha \wedge a) \vee b \leqslant (x_\alpha \wedge a) \vee b \leqslant (u_\alpha \wedge a) \vee b$ for each $\alpha \in I$

and also

$(t_\alpha \wedge a) \vee b \uparrow (x \wedge a) \vee b, (u_\alpha \wedge a) \vee b \downarrow (x \wedge a) \vee b.$

Hence every order convergent net is also unbounded order convergent. Since star convergence implies unbounded star convergence, order convergence will also imply unbounded star convergence.

Also, if the net $(x_\alpha \wedge b) \vee a$ order converges to $(x \wedge b) \vee a$, then for any subnet (x_β) of x_α, $(x_\beta \wedge b) \vee a$ order converges to $(x \wedge b) \vee a$, so unbounded order convergent implies unbounded star convergent.

Now let's investigate the opposite directions of the implications in the diagram. Let's consider the lattice L of finite and co-finite subsets of real numbers with respect to ordering of inclusion and the sequence (x_n) of single points defined by $x_n = \{n\}$ for each $n \in \mathbb{N}$. In Rennie (1950), it is stated that (x_n) does not order converge to empty set. But for any $x, y \in L$ with $x \leqslant y$ the equalities

$(x_n \vee y) \wedge x = x$ and $(\emptyset \vee y) \wedge x = x$

implies that the sequence (x_n) unbounded order converges to empty set since constant sequences always order converge to its constant value. This example shows that unbounded order convergence does not require order convergence. We can use the same example to show that unbounded star convergence does not require star convergence.

Let's consider the set of all real-valued continuous functions on $\mathbb{R}$, denoted by $C(\mathbb{R})$ with the order relation

$f \leqslant g \Leftrightarrow f(x) \leqslant g(x) \forall x \in \mathbb{R}$.

Let's enumarate the rational numbers which belong to $[0,1]$ as $r_1, r_2, r_3, \dots$ and consider the sequence of functions

$f_n(x) = g_n(x - r_n) \ \forall x \in \mathbb{R}$

where

$$g_n(x) = \begin{cases} 1 - n \cdot |x| & , x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ 0 & , \text{otherwise} \end{cases}.$$

Let (f_{n_k}) be any subsequence of (f_n) since (f_{n_k}) is a sequence in $[0,1]$ then there exists convergent subsequence $(r_{n_{k_i}})$ which converges to any $r \in [0,1]$. Consider the function

$$h_i(x)=\begin{cases}0 & ,x\leqslant r-t_i-\frac{1}{i}\\ 1+i(x-r+t_i) & ,x\in\left(r-t_i-\frac{1}{i},r-t_i\right)\\ 1 & ,x\in[r-t_i,r+t_i]\\ 1-i.(x-r-t_i) & ,x\in\left(r+t_i,r+t_i+\frac{1}{i}\right)\\ 0 & ,x\geqslant r+t_i+\frac{1}{i}\end{cases}$$

where $t_i=\max\left\{\left|r_{n_{k_j}}-r\right|:j\geqslant i\right\}$. Then the inequality below holds for all $i\in\mathbb{N}$ and $x\in\mathbb{R}$

$0\leqslant f_{n_{k_i}}(x)\leqslant h_i(x)$

which implies the order convergent conditions.

The sequence (f_n) does not converge to a constant zero function. Suppose it converges to a constant zero function, then there must be such a sequence of the function (u_n) such that

$f_n(x)\leqslant u_n(x)\downarrow 0$.

Given $n_0\in\mathbb{N}$, the inequality $u_n(r_j)\geqslant u_j(r_j)\geqslant f_j(q_j)=1$ holds. With the density of $\{r_j:j\geqslant n_0\}$ and the continuity of u_n we obtain

$u_n(x)\geqslant 1, x\in[0,1], n\in\mathbb{N}$.

Therefore the infimum of u_n cannot be a constant zero function.

Now, we give the proof that sub-subsequence $\left(f_{n_{k_i}}\right)$ order converges to constant zero function.

In Anguelov and Walt (2005), this example shows that sequence convergence is not a topological convergence in the space $C(X)$. The reason why it is not topological is that sequence convergence does not have the Urysohn property. Also in Birkhoff (1967), it is shown that star convergence always has Urysohn property.

Let us consider the lattice of Borel sets of unit interval and the sequence.

$x_n=(n\cdot 2^{-m}-1,(n+1)\cdot 2^{-m}-1)$

where $m\in\mathbb{Z}, 1\leq n.2^{-m}<2$. (x_n) does not order converge to $\emptyset$ but star converges to $\emptyset$ (Rennie, 1950). Also (x_n) does not unbounded order converges to $\emptyset$. Hence star convergence does not imply order convergence and unbounded star convergence does not imply unbounded order convergence.

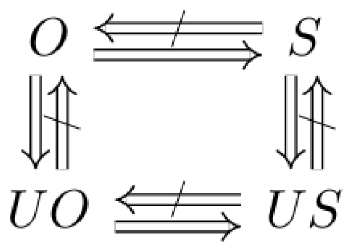

Figure 1. Relationship Diagram

3. Main Results and Discussion

3.1. Elementary Theorems on Unbounded Star Convergence

In this section, we will give some properties of unbounded star convergence on general lattices. Let us start with a theorem.

Theorem 3.1.1. Every constant sequence is unbounded star convergent to its constant value.

Proof. Let (x_n) be a constant sequence defined by $x_n := x$ for some $x \in L$. For any pair $b, c \in L$ with $b \leq c$, $(x_n \wedge c) \vee b = (x \wedge c) \vee b$ for each $n \in \mathbb{N}$. Let's take $u_n = t_n = (x_n \wedge c) \vee b$ then conditions of order convergence hold; $t_n \uparrow (x \wedge c) \vee b$, $u_n \downarrow (x \wedge c) \vee b$ and $t_n \leq (x_n \wedge c) \vee b \leq u_n$. It completes the proof.

Theorem 3.1.2. If a net is unbounded star convergent, then the order limit point is unique.

Proof. Let (x_α) be a net in L and $x, y \in L$. If $x_\alpha \overset{us}{\to} x$ and $x_\alpha \overset{us}{\to} y$ then for any pair $b, c \in L$ with $b \leqslant c$ any subnet of a subnet of (x_α), say $\left(x_{\alpha_{\beta_\gamma}}\right)$ order converges to x and also y, that is,

$$(x_\alpha \wedge c) \vee b \overset{o}{\to} (x \wedge c) \vee b$$

And

$$(x_\alpha \wedge c) \vee b \overset{o}{\to} (y \wedge c) \vee b.$$

Then for

$b \coloneqq \inf\{x, y\}$ and $c \coloneqq \sup\{x, y\}$

it follows that

$$\begin{aligned}(x \wedge c) \vee b &= (x \vee b) \wedge (c \vee b)\\ &= (x \vee \inf\{x, y\}) \wedge (\sup\{x, y\} \vee \inf\{x, y\})\\ &= x \wedge (\sup\{x, y\})\\ &= x\end{aligned}$$

and similarly $(y \wedge c) \vee b = y$. Hence $(x_\alpha \wedge c) \vee b$ order converges to x and y but order convergence has Hausdorff property, therefore $x = y$.

Theorem 3.1.3. If a net is unbounded star converges to a point x then any subnet of the net is also unbounded star converges to x.

Proof. Let $(x_\alpha)_{\alpha \in I}$ be a net in a lattice L, $(x_\beta)_{\beta \in J}$ be a subnet of (x_α) and $x \in L$. For any pair $b, c \in L$ with $b \leq c$. If $(x_\alpha \wedge c) \vee b \overset{o}{\to} (x \wedge c) \vee b$, then there exists two monotonic directed nets (u_α) and (t_α) such that $u_\alpha \downarrow (x \wedge c) \vee b, t_\alpha \uparrow (x \wedge c) \vee b$ and $t_\alpha \leq (x_\alpha \wedge c) \vee b \leq u_\alpha$ for each $\alpha \in I$. There exists a cofinal and incereasing map σ from J into I since (x_β) *is a subnet of* (x_α). Let's take $u_\beta \coloneqq u_{\sigma(\beta)}$ and $t_\beta \coloneqq t_{\sigma(\beta)}$ which satisfy the convergence conditions. It completes the proof.

Theorem 3.1.4. Let x_α and y_α unbounded star converge to x. If there is a net z_α such that $x_\alpha \leqslant z_\alpha \leqslant y_\alpha$ then $z_\alpha \overset{us}{\to} x$.

Proof. Consider the inequalities $x_\alpha \geqslant t_\alpha \uparrow x$ and $y_\alpha \leqslant u_\alpha \downarrow x$ in the definition of unbounded star convergence for (x_α) and (y_α). By the hypothesis, the inequalities

$t_\alpha \leqslant x_\alpha \leqslant z_\alpha \leqslant y_\alpha \leqslant u_\alpha$ for each α

hold. $t_\alpha \leqslant z_\alpha \leqslant u_\alpha$ and $t_\alpha \uparrow x, u_\alpha \downarrow x$ shows that $z_\alpha \overset{us}{\rightarrow} x$.

Theorem 3.1.5. If (x_α) and (y_α) unbounded star converge to x and y, respectively with $x_\alpha \leqslant y_\alpha$ for each $\alpha \in I$ then $y \nless x$.

Proof. Let $\left(x_{\alpha_{\beta_\gamma}}\right)$ be a subnet of the subnet of (x_α) unbounded star converging to x, i.e. for any pair $b, c \in L$ with $b \leqslant c$

$$\left(x_{\alpha_{\beta_\gamma}} \wedge c\right) \vee b \overset{o}{\rightarrow} (x \wedge c) \vee b$$

similarly,

$$\left(y_{\alpha_{\beta_\gamma}} \wedge c\right) \vee b \overset{o}{\rightarrow} (y \wedge c) \vee b$$

If we take $b := \inf\{x, y\}$ and $c := \sup\{x, y\}$ and write in place of the above convergences then

$$x_{\alpha_{\beta_\gamma}} \vee sup\{x, y\} \overset{\circ}{\rightarrow} x$$
$$y_{\alpha_{\beta_\gamma}} \vee sup\{x, y\} \overset{\circ}{\rightarrow} y.$$

Let us assume that $y < x$. Then by using the nets $t_{\alpha_{\beta_\gamma}} \uparrow x$ and $u_{\alpha_{\beta_\gamma}} \downarrow y$ we can write the inequality

$$t_{\alpha_{\beta_\gamma}} \vee x \leqslant x_{\alpha_{\beta_\gamma}} \vee x \leqslant y_{\alpha_{\beta_\gamma}} \vee x \leqslant u_{\alpha_{\beta_\gamma}} \vee x$$

for a given γ_0,

$$x \leqslant x_{\alpha_{\beta_{\gamma_0}}} \vee x \leqslant y_{\alpha_{\beta_{\gamma_0}}} \vee x \leqslant x$$

so by above theorem we obtain $x = y$ which leads to contradiction.

Theorem 3.1.6. If an order bounded net (x_α) unbounded star converges to an element x, then the (x_α) star converges to x.

Proof. Let $\left(x_\gamma\right)$ be subnet of the subnet of $\left(x_\beta\right)$ of (x_α), by hypothesis for any b, c with $b \leq c$

$$\left(x_\gamma \wedge b\right) \vee a \rightarrow (x \wedge b) \vee a.$$

If (x_α) is order bounded then there exist a_1, a_2 such that $a_1 \leq x_\alpha \leq a_2, \forall \alpha$. Then taking

$b = a_2, c = a_1$ ends the proof.

4. Conclusion

A new type of convergence, unbounded star convergence, is defined and the meaningfulness of this definition is studied. The basic order properties of unbounded star convergence are studied. When O_2 order convergence is taken instead of O_1 convergence, a new convergence type, may named as Star_2 convergence, could be defined.

Let's consider the lattice of all sequences converging to zero, denoted by c_0 and sequence of unit elements (e_n), defined as for each $n \in N$; (e_n)'s n_{th} - term is one and rest of it is zero. It is known that (e_n) does not order converge to zero sequence and unbounded order converges to zero sequence in the sense of the definition of Wickstead but it does not unbounded order converge to zero sequence in the sense of individual convergence.

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Conflict of interest

The author declares no conflict of interest.

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