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RESEARCH ARTICLE

Lichnerowicz Type Estimate for the p-Laplacian Under Weighted Integral Curvature Bounds

Shoo Seto*

(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this short note, we prove a quantitative lower bound in terms of the dimension and curvature, known as a Lichnerowicz-type estimate, for the first eigenvalue of the p-Laplacian on Riemannian manifolds with a bound on the integral norm of the Bakry-Émery curvature.

Keywords: *p-Laplacian, Bakry-Émery curvature, integral curvature.*
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1. Introduction

Geometric estimates under the assumption of lower Ricci curvature bounds have been extensively studied and have produced many fundamental results. In particular, studying the spectrum of the Laplacian on Riemannian manifolds with lower Ricci curvature has received a considerable amount of attention due to its applications in the study of PDE's involving the Laplacian, i.e., Poisson equation, heat equation, wave equation, etc.

Let (M^n, g) be an n-dimensional closed Riemannian manifold with $\mathrm{Ric} \geq (n-1)K$, $K \in \mathbb{R}$. The Laplacian is defined by the formula

$$\Delta u := \mathrm{div}(\nabla u),$$

and its corresponding eigenvalue equation

$$\Delta u + \lambda u = 0.$$

The eigenvalues can be given an extremal characterization as the minimizer of the normalized L^2 energy and of special interest is the smallest or first nonzero eigenvalue

$$\lambda_1 := \min_{u \in C^2(M)} \left\{ \frac{\|\nabla u\|_{L^2}}{\|u\|_{L^2}} \mid \int_M u dV = 0 \right\}.$$

The classical result of Lichnerowicz gives a quantitative lower bound of λ_1 in terms of the dimension and curvature, with the rigidity result by Obata.

Theorem 1.1. *Let M^n be a closed manifold with $\mathrm{Ric} \geq (n-1)K > 0$. Then*

$$\lambda_1(M^n) \geq nK = \lambda_1(\mathbb{M}^n_K), \tag{1.1}$$

where $\mathbb{M}^n_K$ is the space form of curvature K. Moreover, equality holds if and only if M is isometric to $\mathbb{M}^n_K$.

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*** Corresponding author***

Remark 1.1. The estimate is proved using the Bochner formula which essentially relates the Laplacian with the Ricci curvature.

In this paper, we generalize the Lichnerowicz estimate in two broad directions: the curvature assumption and the Laplacian operator itself.

1.1. *Integral Bakry-Émery Curvature*

Let $(M, g, e^{-f}dV_g)$ be an n-dimensional weighted Riemannian manifold with weight function f. The conformal factor in the volume form gives rise to the weighted L^2 inner product given by

$$\langle f, g\rangle_{L^2,f} = \int_M fge^{-f}dV_g.$$

Such a framework has been extensively used in geometric analysis, for example, polarized Kähler manifolds locally can be viewed as a weighted $\mathbb{C}^n$ space with weight function $f = |z|^2$. Under the weighted L^2 inner product, the Laplacian must be changed appropriately to maintain self-adjointness. We define the weighted Laplacian (or the Witten Laplacian) by

$$\Delta_f u = \operatorname{div}(e^{-f}\nabla u)e^f = \Delta u - \langle\nabla u, \nabla f\rangle.$$

The appropriate curvature tensor to study is the N-Bakry-Émery Ricci Tensor

$$\operatorname{Ric}_f^N = \operatorname{Ric} + \operatorname{Hess} f - \frac{df \otimes df}{N}.$$

When $N = \infty$, i.e. $\operatorname{Ric}_f := \operatorname{Ric} + \operatorname{Hess} f$, Bakry and Emery [2] studied (and generalized) this tensor in the context of probability and diffusion processes. Wei and Wylie [11] showed that many of the geometric comparison results under a lower Ricci curvature can be carried over to an analogous comparison result under bounds on the N-Bakry-Émery tensor.

For $x \in M$, $N > 0$, and $K \in \mathbb{R}$, let $\lambda(x)$ be the smallest eigenvalue of $\operatorname{Ric}_f^N : T_xM \to T_xM$. Define

$$\operatorname{Ric}_{f,-}^{N,K} := \sup\{(N-1)K - \lambda(x), 0\},$$

which measures how much the inequality $\operatorname{Ric}_f^N \geq (N-1)K$ fails at $x \in M$. Next we define the normalized L^q norm of the Ricci curvature excess

$$\| \operatorname{Ric}_{f,-}^{N,K} \|_q^* := \left(\frac{1}{\operatorname{vol}_f(M)}\int_M |\operatorname{Ric}_{f,-}^{N,K}|^q e^{-f}dV\right)^{\frac{1}{q}}$$

where $\operatorname{vol}_f(M) := \int_M e^{-f}dV$. Note $\| \operatorname{Ric}_{f,-}^{N,K} \|_q^* = 0$ implies the pointwise bound $\operatorname{Ric}_f^N \geq (N-1)K$.

Comparison geometry under an assumption on the integral norm of the Ricci curvature has been investigated thoroughly, notably by Petersen and Wei [5] extending the major comparison results to the integral setting. In the weighted integral setting, the work Wu [12] extends the comparison results.

1.2. *Weighted p-Laplacian*

Consider the weighted L^p-Rayleigh quotient

$$\mathcal{R}[u] := \frac{\int_M |\nabla u|^p e^{-f}dV}{\int_M |u|^p e^{-f}dv}.$$

Compute the variation

$$\frac{d}{dt}\mathcal{R}[u + tv]|_{t=0} = \frac{p\int_M |\nabla u|^{p-2}\langle\nabla u, \nabla v\rangle e^{-f}dV}{\int_M |u|^p e^{-f}dV} - p\frac{\int_M |\nabla u|^p e^{-f}}{\int_M |u|^p e^{-f}}\frac{\int_M |u|^{p-2}\langle u, v\rangle e^{-f}dV}{\int_M |u|^p e^{-f}dV}.$$

Let $\lambda = \mathcal{R}[u]$ be a critical point. Then

$$\lambda\int_M |u|^{p-2}u, ve^{-f}dV = -\int_M \operatorname{div}(|\nabla u|^{p-2}(\nabla u)e^{-f})vdV + \int_{\partial M} |\nabla u|^{p-2}(\nabla_n u)v^{-f}dV.$$

This leads us to define the weighted p-Laplacian

$$\Delta_{p,f} u := e^{f} \operatorname{div}(e^{-f}|\nabla u|^{p-2}\nabla u)$$

defined in the distribution sense for $u \in W^{1,p}(M)$, and the corresponding eigenvalue equation

$$\Delta_{p,f} u = -\lambda |u|^{p-2} u,$$

again in the distribution sense. Its relation with the weighted 2-Laplacian is

$$\Delta_{p,f}(u) = |\nabla u|^{p-2}\Delta_f u + \langle \nabla(|\nabla u|^{p-2}), \nabla u\rangle.$$

The first nonzero eigenvalue of $\Delta_{p,f}$ on a closed manifold is given by the variational characterization

$$\lambda_{1,p,f} := \inf\left\{ \mathcal{R}[u] \mid u \in W^{1,p}(M, e^{-f}dV_g)\backslash\{0\}, \int_M |u|^{p-2} u e^{-f} dV_g = 0 \right\}.$$

The infimum is achieved by an eigenfunction $u \in C^{1,\alpha}(M)$ as show in [8].

It was proved in [7] that the following Lichnerowicz estimate for the unweighted case.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold. For $q > \frac{n}{2}$, $p \geq 2$, and $K > 0$, there exists $\varepsilon = \varepsilon(n,p,q,K)$ such that if $\| \operatorname{Ric}^K_- \|^*_q \leq \varepsilon$, then*

$$\lambda^{\frac{2}{p}} \geq \frac{\sqrt{n}(p-2)+n}{(p-1)(\sqrt{n}(p-2)+n-1)}[(n-1)K - 2\| \operatorname{Ric}^K_- \|^*_q].$$

Note that when $\operatorname{Ric} \geq (n-1)K$ and $p = 2$, this recovers the lower bound (1.1).

In the weighted setting, Wang and Li [9] proved the following

Theorem 1.3. *Assume $p \geq 2$ and $K > 0$. Let (M, g, μ) be a closed smooth metric measure space. If $\operatorname{Ric}^N_f \geq (N-1)Kg$, then,*

$$\lambda_{1,p} \geq \frac{1}{(p-1)^{p-1}} (NK)^{\frac{p}{2}}.$$

Remark 1.2. In fact, [9] gives various lower bound estimates under a point-wise lower bound of Ric^N_f. They use a linearized p-Laplace operator and work with a Bochner-type formula for the linearized operator.

We will prove the following Lichnerowicz type estimate for the p-Laplacian with control on the weighted integral curvature.

Main Theorem. *Let (M, g, f) be an n-dimensional closed weighted manifold with smooth weight function f. Let $\lambda_{1,p}$ be the first nontrivial eigenvalue of the p-Laplacian with $p \geq 2$. For $q > \frac{n}{2}$, $N > 0$ and $K > 0$, there exists $\varepsilon = \varepsilon(n,N,p,q,K) > 0$ and $\eta = \eta(n,p) > 0$ such that for $\| \operatorname{Ric}^{N,K}_{f,-} \|_q \leq \varepsilon$, we have*

$$\lambda_{1,p} \geq \left(\frac{(N-1)K - \varepsilon}{(p-1)(1-C(p,\eta))} \right)^{\frac{p}{2}}.$$

Here $C(p,\eta) = \frac{1}{n(1+(p-2)\eta)(t+C_\eta)}$, $t = \frac{(1+\frac{N}{n}-C_\eta)+\sqrt{(C_\eta-1-\frac{N}{n})^2+4C_\eta}}{2}$. $C_\eta = \frac{p-2}{4\eta n(1+(p-2)\eta)}$ and $\eta > 0$ such that $\sqrt{\frac{1}{4n}+\frac{1}{4p^2}} - \frac{1}{2p} > \eta$.

Remark 1.3. Note that the role of the dimension n is replaced by N coming from the N-Bakry-Émery term.

Remark 1.4. When $p = 2$ and $\varepsilon = 0$, we have $\lambda_1 \geq \left(\frac{n+N}{n+N-1}\right)(N-1)K$. To recover the Lichnerowicz estimate, we need to note that if we assume that $\operatorname{Ric}^N_f \geq (n-1)K$, then the estimate will change to $\lambda_1 \geq \left(\frac{n+N}{n+N-1}\right)(n-1)K$, then setting $N = 0$, we obtain $\lambda_1 \geq nK$.

The main issue is the delicate computation required from the additional $\langle \nabla u, \nabla f\rangle$ term which comes in the weighted setting and the integral curvature term. By taking advantage of an additional nonnegative Hessian term which is thrown out in the pointwise lower bound case, we can control this additional integral curvature term.

2. Proof of Main Theorem

We will use the following Bochner-type formula for $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$.

Lemma 2.1 (Bochner-type formula)**.** *For $p > 0$,*

$$\begin{aligned} &\frac{1}{p}\Delta_f(|\nabla u|^p)e^{-f} \\ &\quad = (p-2)|\nabla u|^{p-2}|\nabla|\nabla u||^2 e^{-f} + |\nabla u|^{p-2}\{|\operatorname{Hess} u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \operatorname{Ric}_f(\nabla u, \nabla u)\}e^{-f}. \end{aligned} \tag{2.1}$$

The proof is a direct computation which we include for completeness.

Proof.

$$\begin{aligned} \frac{1}{p}\Delta_f(|\nabla u|^p)e^{-f} &= \frac{1}{p}\operatorname{div}(e^{-f}\nabla|\nabla u|^p) \\ &= \frac{1}{p}\Delta(|\nabla u|^p)e^{-f} - \frac{1}{p}\langle \nabla|\nabla u|^p, \nabla f\rangle e^{-f} \\ &= (p-2)|\nabla u|^{p-2}|\nabla|\nabla u||^2 e^{-f} + |\nabla u|^{p-2}\{|\operatorname{Hess} u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \operatorname{Ric}(\nabla u, \nabla u)\}e^{-f} \\ &\quad - \frac{1}{p}\langle \nabla|\nabla u|^p, \nabla f\rangle e^{-f} \\ &= (p-2)|\nabla u|^{p-2}|\nabla|\nabla u||^2 e^{-f} + |\nabla u|^{p-2}\{|\operatorname{Hess} u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \operatorname{Ric}_f(\nabla u, \nabla u)\}e^{-f}. \end{aligned}$$

□

Taking the normalized integral of the Bochner formula we get

$$\begin{aligned} 0 = (p-2)&\fint_M |\nabla u|^{p-2}|\nabla|\nabla u||^2 e^{-f}dV + \fint_M |\nabla u|^{p-2}|\operatorname{Hess} u|^2 e^{-f}dV \\ &+ \fint_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_f u\rangle e^{-f}dV + \fint_M |\nabla u|^{p-2}\operatorname{Ric}_f(\nabla u, \nabla u)e^{-f}dV. \end{aligned} \tag{2.2}$$

We will analyze each term. First we can integrate the Laplacian term so that

$$\begin{aligned} \fint_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_f u\rangle e^{-f}dV &= -\fint_M \operatorname{div}(e^{-f}|\nabla u|^{p-2}\nabla u)\Delta_f u dV \\ &= -\fint_M \Delta_{p,f}u\Delta_f u e^{-f}dV. \end{aligned}$$

For the Hessian term, we apply the Cauchy-Schwarz inequalities

$$\begin{aligned} (\Delta u)^2 &\le n|\operatorname{Hess} u|^2 \\ (\operatorname{Hess} u(\nabla u, \nabla u))^2 &\le |\nabla u|^4|\operatorname{Hess} u|^2 \end{aligned}$$

and re-write in terms of the weighted Laplacian so that

$$\begin{aligned} \fint_M |\nabla u|^{p-2}|\operatorname{Hess} u|^2 e^{-f}dV &\ge \fint_M |\nabla u|^{p-2}\frac{(\Delta u)^2}{n}e^{-f}dV \\ &\ge \frac{1}{n}\fint_M \frac{1}{t}|\nabla u|^{p-2}(\Delta_f u)^2 e^{-f} - \frac{1}{n}\fint_M \frac{1}{t-1}|\nabla u|^{p-2}\langle \nabla f, \nabla u\rangle^2 e^{-f} \end{aligned} \tag{2.3}$$

where in the second line we used the inequality $(a-b)^2 \ge \frac{1}{t}a^2 - \frac{1}{t-1}b^2$, $t > 1$. We then change one of the weighted Laplacian term to the weighted p-Laplacian so that

$$\fint_M |\nabla u|^{p-2}(\Delta_f u)^2 e^{-f} = \fint_M (\Delta_f u)(\Delta_{p,f}(u))e^{-f} - (p-2)\fint_M (\Delta_f u)|\nabla u|^{p-4}\operatorname{Hess} u(\nabla u, \nabla u)e^{-f},$$

and we bound the Hessian term using Young's inequality

$$|\nabla u|^{\frac{p-2}{2}}(\Delta_f u)|\nabla u|^{\frac{p-6}{2}}\operatorname{Hess}u(\nabla u,\nabla u)\le \eta|\nabla u|^{p-2}(\Delta_f u)^2+\frac{1}{4\eta}|\nabla u|^{p-6}|\operatorname{Hess}u(\nabla u,\nabla u)|^2$$

for $\eta>0$ to be chosen later. Then we have

$$(1+(p-2)\eta)\fint_M|\nabla u|^{p-2}(\Delta_f u)^2e^{-f}\ge\fint_M(\Delta_f u)(\Delta_{p,f}u)e^{-f}-\frac{(p-2)}{4\eta}\fint_M|\nabla u|^{p-2}|\operatorname{Hess}u|^2e^{-f}.$$

Substituting back to (2.3), we get

$$\begin{aligned}\fint_M|\nabla u|^{p-2}|\operatorname{Hess}u|^2e^{-f}&\ge\frac{1}{tn}\fint_M|\nabla u|^{p-2}(\Delta_f u)^2e^{-f}-\frac{1}{n(t-1)}\fint_M|\nabla u|^{p-2}\langle\nabla f,\nabla u\rangle^2e^{-f}\\&\ge\frac{1}{tn(1+(p-2)\eta)}\fint_M(\Delta_f u)(\Delta_{p,f}u)e^{-f}\\&\quad-\frac{1}{tn(1+(p-2)\eta)}\frac{(p-2)}{4\eta}\fint_M|\nabla u|^{p-2}|\operatorname{Hess}u|^2e^{-f}-\frac{1}{n(t-1)}\fint_M|\nabla u|^{p-2}\langle\nabla f,\nabla u\rangle^2e^{-f}\end{aligned}$$

or

$$\begin{aligned}&\fint_M|\nabla u|^{p-2}|\operatorname{Hess}u|^2e^{-f}\\&\quad\ge\frac{1}{n(1+(p-2)\eta)(t+C_\eta)}\fint_M(\Delta_f u)(\Delta_{p,f}u)e^{-f}-\frac{1}{n(t-1)(1+\frac{C_\eta}{t})}\fint_M|\nabla u|^{p-2}\langle\nabla f,\nabla u\rangle^2e^{-f}\end{aligned}$$

where $C_\eta=\frac{p-2}{4\eta n(1+(p-2)\eta)}$.

The mixed term $\langle\nabla f,\nabla u\rangle^2$ will be combined with Ric_f to become Ric_f^N and in order to do so we want to solve for t such that

$$\frac{1}{n(t-1)(1+\frac{C_\eta}{t})}=\frac{1}{N}.$$

This is accomplished for

$$t=\frac{-(C_\eta-1-\frac{N}{n})+\sqrt{(C_\eta-1-\frac{N}{n})^2+4C_\eta}}{2}$$

with the additional requirement that $\sqrt{\frac{1}{4n}+\frac{1}{4p^2}}-\frac{1}{2p}>\eta$ so that $t>1$. With this choice (2.2) becomes

$$\begin{aligned}0&\ge(p-2)\fint_M|\nabla u|^{p-2}|\nabla|\nabla u||^2e^{-f}dV\\&\quad-(1-C(p,\varepsilon))\fint_M(\Delta_f u)(\Delta_{p,f}u)e^{-f}+\fint_M|\nabla u|^{p-2}\operatorname{Ric}_f^N(\nabla u,\nabla u)e^{-f},\end{aligned}\tag{2.4}$$

with $C(p,\varepsilon)=\frac{1}{n(1+(p-2)\eta)(t+C_\eta)}$. Note $C(p,\varepsilon)<1$.

For the curvature term,

$$\begin{aligned}\fint_M|\nabla u|^{p-2}\operatorname{Ric}_f^N(\nabla u,\nabla u)e^{-f}&\ge(N-1)K\fint_M|\nabla u|^pe^{-f}-|\operatorname{Ric}_{f,-}^{N,K}||\nabla u|^pe^{-f}\\&\ge(N-1)K\fint_M|\nabla u|^pe^{-f}-\|\operatorname{Ric}_{f,-}^{N,K}\|_q^*\left(\fint_M|\nabla u|^{\frac{pq}{q-1}}e^{-f}\right)^{\frac{q-1}{q}}.\end{aligned}$$

To estimate the last term, we will use the following Sobolev inequality

Proposition 2.1. *Given $q>\frac{n}{2}$, and $K>0$, there exists $\varepsilon=\varepsilon(n,q,K)$ such that if M^n is a complete manifold with $\|\operatorname{Ric}_{f,-}^{N,K}\|_q^*\le\varepsilon$, then there is a constant $C_s(n,q,K)$ such that*

$$\left(\fint_M u^{\frac{2q}{q-1}}e^{-f}dV\right)^{\frac{q-1}{q}}\le C_s(n,q,K)\fint_M|\nabla u|^2+2\fint_M u^2$$

for all $u\in W^{1,2}(M,e^{-f}dV_g)$.

Remark 2.1. The case for $K \in \mathbb{R}$ is established in [6], however for the $K > 0$, this can be shown using results of Wang-Wei [10] and compactness.

Applying the Sobolev inequality we get

$$\left(\fint_M (|\nabla u|^{\frac{p}{2}})^{\frac{2q}{q-1}} e^{-f}\right)^{\frac{q-1}{q}} \leq C_s \frac{p^2}{4} \fint_M |\nabla u|^{p-2} |\nabla|\nabla u||^2 e^{-f} + 2 \fint_M |\nabla u|^p e^{-f}.$$

Plugging into (2.4) we get

$$\begin{aligned}(1 - C(p,\varepsilon)) \fint_M (\Delta_f u)(\Delta_{p,f} u) e^{-f} \geq &((N-1)K - 2\| \operatorname{Ric}_{f,-}^{N,K} \|_q^*) \fint_M |\nabla u|^p e^{-f} \\ &+ ((p-2) - \frac{C_s p^2}{4} \| \operatorname{Ric}_{f,-}^{N,K})\|_q^* \fint_M |\nabla u|^{p-2} |\nabla|\nabla u||^2 e^{-f}.\end{aligned} \tag{2.5}$$

Now applying (2.5) to the first non-trivial eigenfunction of $\Delta_{p,f}$, we get on the left hand side by Hölder's inequality

$$\begin{aligned}\fint_M (\Delta_f u)(\Delta_{p,f} u) e^{-f} &\leq (p-1)\lambda_{1,p,f} \fint_M |u|^{p-2} |\nabla u|^2 e^{-f} \\ &\leq (p-1)\lambda_{1,p,f} \left(\fint_M |u|^p e^{-f}\right)^{1-\frac{2}{p}} \left(\fint_M |\nabla u|^p e^{-f}\right)^{\frac{2}{p}} \\ &= (p-1)\lambda_{1,p,f}^{\frac{2}{p}} \fint_M |\nabla u|^p e^{-f}\end{aligned}$$

where in the last line we used

$$\fint_M |u|^p e^{-f} = \frac{1}{\lambda_{1,p,f}} \fint_M |\nabla u|^p e^{-f}.$$

This implies

$$\lambda_{1,p,f} \geq \left(\frac{(N-1)K - 2\| \operatorname{Ric}_{f,-}^{N,K} \|_q}{(p-1)(1 - C(p,\varepsilon))}\right)^{\frac{p}{2}}.$$

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Author's contributions

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Affiliations

SHOO SETO
ADDRESS: Department of Mathematics, California State University, Fullerton, 800 N. State Blvd, Fullerton, CA 92831
E-MAIL: shoseto@fullerton.edu
ORCID ID: 0009-0005-6276-7949