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Araştırma Makalesi

Investigating Solutions of the Noyes-Whitney Dynamic Equation via Proportional Fractional Derivative

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Abstract: The dynamics of solid material dissolving in a solvent are fundamentally described by the Noyes-Whitney equation. For the purpose of simulating intricate processes with memory effects and non-local behaviors, fractional calculus offers a strong foundation. We explore the effects of memory and non-locality on dissolution kinetics by solving the Noyes-Whitney equation using fractional derivatives. By means of mathematical analysis, we provide insights into the dissolving processes in chemical engineering and pharmaceutical applications by clarifying the behavior of the Noyes-Whitney equation with proportional fractional derivative. In this study, after discussing the characteristics and theories of the proportional fractional derivative on a time scale, we solve the proportional fractional Noyes-Whitney dynamic equation in the presence of the initial condition and give several examples on various time scales via the proportional fractional derivative.

Key words: Noyes-Whitney dynamic equation, fractional calculus, proportional fractional derivative.

Noyes-Whitney Dinamik Denkleminin Çözümlerinin Oransal Kesirli Türeve Göre İncelenmesi

Öz: Katı maddenin bir çözücü içinde çözünmesinin dinamiği temel olarak Noyes-Whitney denklemi ile tanımlanır. Karmaşık süreçleri hafıza etkileri ve yerel olmayan davranışlarla simüle etmek amacıyla kesirli analiz güçlü bir temel sunar. Noyes-Whitney denklemini kesirli türevler kullanarak çözerek hafızanın ve yerel olmamanın çözünme kinetiği üzerindeki etkilerini araştırıyoruz. Matematiksel analiz yoluyla, Noyes-Whitney denkleminin orantılı kesirli türevle davranışını açıklığa kavuşturarak kimya mühendisliği ve farmasötik uygulamalardaki çözünme süreçlerine ilişkin bilgiler sağlıyoruz. Bu çalışmada oransal kesirli türevin özelliklerini ve teorilerini zaman ölçeğinde verdikten sonra oransal kesirli Noyes-Whitney dinamik denklemini başlangıç koşulunun varlığında ve oransal kesirli türev üzerinden çeşitli zaman ölçeklerinde birkaç örnek vererek çözüyoruz.

Anahtar kelimeler: Noyes-Whitney dinamik denklemi, kesirli hesap, oransal kesirli türev.

1. Introduction

Numerous industrial and scientific operations depend heavily on the kinetics of a solid substance's dissolution in a solvent. Based on variables including surface area, diffusion coefficient, and concentration gradient, the Noyes-Whitney equation [15,23,26,30,32,33] offers a standard model for explaining dissolution rates. We introduce a proportional fractional derivative to the Noyes-Whitney equation in this study, inspired by the non-local behaviors and memory effects found in dissolution events.

The classical fractional derivative operators [1,10,11,17,19,24,25,27,34] are extended by the new conformable fractional derivative, named proportional fractional derivative [4,5,18], with parameters κ_0 and κ_1. If P^0 is the unit operator and P^1 is the classical differential operator, then the differential operator P^β is called a proportional derivative where $\beta \in [0,1]$. In order to overcome some constraints of the current fractional calculus operators and offer a more adaptable framework for characterizing the behavior of complex systems with fractional-order dynamics, the conformable fractional derivative was developed.

With respect to parameters κ_0 and κ_1, the proportional fractional derivative is defined as follows:

Definition 1.1 [5] Assume that $\beta \in [0,1]$, $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \to \mathbb{R}_0^+$ are continuous functions and that

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$$\begin{cases} \lim_{\beta\to 0^+} \kappa_0(\beta,t) = 0, \quad \lim_{\beta\to 0^+} \kappa_1(\beta,t) = 1, \\ \lim_{\beta\to 1^-} \kappa_0(\beta,t) = 1, \quad \lim_{\beta\to 1^-} \kappa_1(\beta,t) = 0, \\ \kappa_0(\beta,t) \neq 0,\ \beta \in (0,1],\ \ \kappa_1(\beta,\ t) \neq 0,\ \beta \in [0,1), \end{cases} \tag{1.1}$$

are true. In this situation, if the function φ is differentiable at t and $\varphi'=\frac{d}{dt}\varphi$, then the differential operator P^β defined by

$$P^\beta\varphi(t)=\kappa_1(\beta,t)\varphi(t) + \kappa_0(\beta,t)\varphi'(t), \tag{1.2}$$

is said to be proportional.

It is possible to modify the behavior and characteristics of the proportional fractional derivative operator by varying the parameters κ_0 and κ_1. With respect to linearity, commutativity, and the chain rule, the proportional fractional derivative with parameters κ_0 and κ_1 inherits some of the beneficial characteristics of conventional fractional derivatives. Furthermore, when modeling complicated systems [8,31] with fractional-order dynamics, it provides increased flexibility and adaptability. There are several applications of this fractional derivative in the domains of signal processing, physics, engineering, biology, and finance [3,7,9,16,21,22,28]. Among the phenomena displaying fractional-order behavior include viscoelasticity, anomalous diffusion, fractional-order control systems, and so on. The features and behavior of this derivative are investigated through mathematical modeling and analysis, utilizing experimental validation, numerical simulations, and analytical techniques. The fractional derivative operator can be tailored to the unique properties of the system under study because to the flexibility offered by the parameters κ_0 and κ_1. In comparison to conventional fractional calculus operators, the proportional fractional derivative with parameters κ_0 and κ_1 offers greater flexibility and versatility, making it a useful tool for characterizing and comprehending the dynamics of complex systems with fractional-order behavior.

A time scale $\mathbb{T}$ is a closed, nonempty subset of the real numbers that denotes the domain of evolution of a dynamic process. It offers a comprehensive framework for researching dynamic processes and systems that change throughout different kinds and durations of time. Time scales having irregular or non-uniform time intervals can be modeled using continuous, discrete, or hybrid time scales. Since its introduction by Stefan Hilger in 1988 [20], time scale calculus has grown to be an essential tool for the study of dynamic systems with a wide range of temporal features. Numerous disciplines, including engineering, physics, biology, economics, and finance, can benefit from the use of time scale calculus. It is applied to population dynamics, mathematical biology, control theory, signal processing, and other fields to model and understand dynamic systems.

A basic equation used to characterize the rate at which a solid material dissolves into a solvent is the Noyes-Whitney equation, sometimes referred to as the Noyes-Whitney equation of dissolution kinetics. It offers a numerical connection between the dissolution rate and other process-influencing variables.

The aim of the study is to obtain analytical solutions of the fractional derivative Noyes-Whitney Dynamic Equation with proportional delay. The formula for the Noyes-Whitney problem is

$$\frac{dR}{dt} = \delta(R_s - R), \quad R(0) = 0. \tag{1.3}$$

The solubility of the substance, or the concentration of its saturated solution, is represented by R; the concentration at the expiration of the time t is represented by R_s; and δ is a constant. This indicates that the solution profile, as derived from the integration of Eq. (1.3), is exponential and reaches the plateau value R_s in an indefinite amount of time:

$$R = R_s\left(1 - e^{-\delta t}\right). \tag{1.4}$$

In this article, we consider the proportional fractional Noyes-Whitney dynamic equation

$$P^\beta R(t) = \delta(R_s - R)(t). \tag{1.5}$$

Firstly, we will give some properties and theories about the proportional fractional derivative on a time scale, and then we will find the solution of Eq. (1.5) with the initial condition and give some examples on different time scales.

2. Preliminaries

Some basic definitions and features of proportional fractional calculus theories will be covered in this part. Firstly, let us denote on a time scale $\mathbb{T}$ by μ, ρ and σ the graininess function, the backward and forward jump operators, respectively, and additionally note that $\mathbb{T}^k=\mathbb{T}-\{m\}$ if there is a maximum m point of $\mathbb{T}$; else, $\mathbb{T}^k=\mathbb{T}$. Detailed information about time scale calculation can be found in [2,6,12-15,23,26,30,33].

The proportional delta derivative of the function $\varphi:\mathbb{T}\to\mathbb{R}$ of order $\beta\in[0,1]$ at point $t\in\mathbb{T}^k$ will now be defined.

Let us define

$$\mathfrak{J}(\mathbb{T})=\{\varphi:\mathbb{T}\to\mathbb{R} : P^{\beta}\varphi(t) \text{ exists and is finite for all } t\in\mathbb{T}^k\},$$

as the collection of all proportional delta differentiable functions [29].

Theorem 2.1 [29] Let $\varphi:\mathbb{T}\to\mathbb{R}$ be a function, $t\in\mathbb{T}^k$; κ_0 and κ_1 be continuous functions that fulfill the conditions (1.1). In this case

$$P^{\beta}\varphi(t)=\kappa_0(\beta,t)\,\varphi^{\Delta}(t)+\kappa_1(\beta,t)\,\varphi(t), \tag{2.1}$$

defines the β-th order proportional derivative of φ at point t where $\beta\in[0,1]$.

Lemma 2.2 [29] If $\varphi_1,\varphi_2:\mathbb{T}\to\mathbb{R}$ are proportional delta differentiable at the point $t\in\mathbb{T}^k$ and κ_0 and κ_1 satisfy the conditions (1.1) and are continuous functions, then the following properties are provided:

(i) $P^{\beta}[\rho\varphi_1+\varsigma\varphi_2]=\rho P^{\beta}[\varphi_1]+\varsigma P^{\beta}[\varphi_2]$, all $\rho,\varsigma\in\mathbb{R}$;

(ii) $P^{\beta}[\varphi_1\varphi_2]=\varphi_1{}^{\sigma}P^{\beta}[\varphi_2]+P^{\beta}[\varphi_1]\varphi_2-\varphi_1{}^{\sigma}\varphi_2\kappa_1(\beta,.)$;

(iii) $D^{\alpha}\left[\frac{\varphi_1}{\varphi_2}\right]=\frac{P^{\beta}[\varphi_1]\,\varphi_2{}^{\sigma}-\varphi_1P^{\beta}[\varphi_2]}{\varphi_2\varphi_2{}^{\sigma}}+\frac{\varphi_1{}^{\sigma}}{\varphi_2{}^{\sigma}}\kappa_1(\beta,.)$, $\quad\varphi_2\varphi_2{}^{\sigma}\neq 0$.

Definition 2.3 [29] Let $\beta\in[0,1]$ and $\kappa_0,\kappa_1:[0,1]\,x\,\mathbb{T}\to\mathbb{R}_0^+$ be continuous functions that fulfill (1.1). $p:\mathbb{T}\to\mathbb{R}$ is regarded as being β-regressive if the requirement

$$1+\frac{p(\tau)-\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\mu(\tau)\neq 0, \quad \text{all } t\in\mathbb{T}^k,$$

is hold. The collection of all β-regressive and rd-continuous functions on $\mathbb{T}$ is represented by $\mathfrak{R}_{\beta}=\mathfrak{R}_{\beta}(\mathbb{T})$.

Definition 2.4 [29] Let $\beta\in(0,1]$, $p\in\mathfrak{R}_{\beta}$. Assume that κ_0,κ_1 are continuous functions and p/κ_0, κ_1/κ_0 delta integrable functions on $\mathbb{T}$, and that (1.1) is satisfied.

$$\tilde{E}_p(t,s)=\exp\left[\int_s^t\frac{1}{\mu(\tau)}Log\left(1+\frac{p(\tau)-\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\mu(\tau)\right)\Delta\tau\right], \tag{2.2}$$

$$\tilde{E}_0(t,s)=\exp\left[\int_s^t\frac{1}{\mu(\tau)}Log\left(1-\frac{\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\mu(\tau)\right)\Delta\tau\right],\ s,t\in\mathbb{T},$$

defines the proportional exponential function on $\mathbb{T}$ for operator P^{β}, where Log is the fundamental logarithm function.

$$\tilde{E}_p(t,s)=\exp\left[\int_s^t\left(\frac{p(\tau)-\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\right)\Delta\tau\right], \quad \mu(t)=0, \tag{2.3}$$

Definition 2.5 [29] Let $p:\mathbb{T}\to\mathbb{R}$ and $\beta\in(0,1]$. Let us use $\mathfrak{R}_{\beta}^{+}$ to define all positive β-regressive components of $\mathfrak{R}_{\beta}$, that is,

$$\mathfrak{R}_{\beta}^{+}=\left\{p\in\mathfrak{R}_{\beta}: 1+\frac{p(\tau)-\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\mu(\tau)>0, \text{ all } t\in\mathbb{T}\right\}.$$

Theorem 2.6 [29] If $p\in\mathfrak{R}_{\beta}^{+}$ and $\beta\in(0,1]$, the following properties are true:

(i) $\tilde{E}_p(\sigma(t),s)=\left(1+\frac{p(t)-\kappa_1(\beta,t)}{\kappa_0(\beta,t)}\mu(t)\right)\tilde{E}_p(t,s)$;

(ii) $\tilde{E}_p(t,s)=\frac{1}{\tilde{E}_p(s,t)}$;

(iii) $\tilde{E}_p(t,s)\tilde{E}_p(s,r)=\tilde{E}_p(t,r)$;

(iv) $\tilde{E}_p^{\Delta}(t,s)=\left(\frac{p(t)-\kappa_1(\beta,t)}{\kappa_0(\beta,t)}\right)\tilde{E}_p(t,s)$.

Lemma 2.7 [29] Let $\beta\in(0,1]$ and $p\in\Re_\beta$. For fixed $s\in\mathbb{T}$,

$$P^\beta\left[\tilde{E}_p(.\,,s)\right]=p(t)\tilde{E}_p(.\,,s),$$

and for the proportional exponential function $\tilde{E}_0$,

$$P^\beta\left[\int_a^t\frac{\varphi(\tau)\tilde{E}_0(t,\sigma(\tau))}{\kappa_0(\beta,\tau)}\,\Delta\tau\right]=\varphi(t). \tag{2.4}$$

Definition 2.8 [29] Assume that $\varphi\in C_{rd}(\mathbb{R})$, $\beta\in(0,1]$, and $t_0\in\mathbb{T}$. The indefinite proportional integral (anti derivative) is defined as

$$\int P^\beta\varphi(t)\Delta_\alpha\tau=\varphi(t)+c\tilde{E}_0(t,t_0),\ \forall t\in\mathbb{T},c\in\mathbb{R},$$

with respect to Lemma 2.7

$$\int_a^t\varphi(\tau)\tilde{E}_0(t,\sigma(\tau))\,\Delta_\beta\tau=\int_a^t\frac{\varphi(\tau)\tilde{E}_0(t,\sigma(\tau))}{\kappa_0(\beta,\tau)}\,\Delta\tau,\quad \Delta_\beta\tau=\frac{1}{\kappa_0(\beta,\tau)}\,\Delta\tau, \tag{2.5}$$

describes the indefinite proportional integral (anti derivative) of φ on $[a,b]_{\mathbb{T}}$.

Lemma 2.9 [29] Let $\beta\in(0,1]$, $\varphi\in C_{rd}(\mathbb{R})$, and κ_0, κ_1 be continuous functions and satisfy (1.1). Then,

$$P^\beta\left[\int_a^t\varphi(\tau)\tilde{E}_0(t,\sigma(\tau))\,\Delta_\beta\tau\right]=\varphi(t). \tag{2.6}$$

Lemma 2.10 [29] If $\varphi\in\Im(\mathbb{T})$,

$$\int_a^t P^\beta\left[\varphi(\tau)\right]\,\tilde{E}_0(t,\sigma(\tau))\,\Delta_\beta\tau=\left[\varphi(\tau)\,\tilde{E}_0(t,\sigma(\tau))\right]_{\tau=a}^t.$$

Definition 2.11 [4] It is assumed that $\varphi:\mathbb{T}\to\mathbb{C}$ is regulated. Afterward, for each $g\in\mathrm{H}_\beta(f)$, where $\mathrm{H}_\beta(f)$ is the set of all complex numbers that fulfill

$$\kappa_0+f^\sigma g(\mu-\kappa_1)\neq 0,\qquad g-\kappa_1\in\Re_\beta,$$

$$\mathrm{h}=-gf^\sigma\left(1+\mu\frac{\mathrm{h}-\kappa_1}{\kappa_0}\right),$$

the proportional fractional Laplace transform of φ is given as

$$\mathrm{L}_\beta(\varphi)(u)=\int_0^\infty\varphi(u)f^\sigma(u)\tilde{E}_{\mathrm{h}}(t,0)\,\Delta_{\beta,\infty}\tau.$$

Theorem 2.12 [4] Assume that $\varphi_1,\varphi_2:\mathbb{T}\to\mathbb{C}$ are regulated; $\gamma_1,\gamma_2\in\mathbb{C}$. Thus, for $u\in\mathrm{H}_\beta(\varphi_1)\cap\mathrm{H}_\beta(\varphi_2)$,

$$\mathrm{L}_\beta(\gamma_1\varphi_1+\gamma_2\varphi_2)(u)=\gamma_1\mathrm{L}_\beta(\varphi_1)(u)+\gamma_2\mathrm{L}_\beta(\varphi_2)(u).$$

Lemma 2.13 [4] **(i)** $\mathrm{L}_\beta(1)(u) = \frac{1}{u}\tilde{E}_0(\infty,0), \quad u \in \mathrm{H}_\beta(1),$
(ii) $\mathrm{L}_\beta\left(\tilde{E}_\mathrm{q}(\infty,0)\right) = \frac{\tilde{E}_0(\infty,0)}{u-q}.$

3. Main Results

Using the Laplace transform, solutions of the proportional fractional Noyes-Whitney problem

$$\begin{cases} P^\beta R(t) = \delta(R_s - R)(t), & (3.1)\\ R(0) = R_0, & (3.2)\end{cases}$$

for various time scales will be found in this section.

Theorem 3.1 The solutions of the proportional fractional Noyes-Whitney problem (3.1)-(3.2) is

$$R(t) = R_s + (R_0 - R_s)\tilde{E}_{-\delta}(t,0), \tag{3.3}$$

where $R_s(t) = R_s$.

Proof: Using the initial condition (3.2) and the proportional fractional Laplace transform of both sides of Eq. (3.1) can result in the discovery that

$$\begin{aligned}\mathrm{L}_\beta(P^\beta R)(u) &= \mathrm{L}_\beta(\delta(R_s - R))(u)\\ &= \delta(R_s - R)\mathrm{L}_\beta(1)(u)\\ &= \delta R_s \mathrm{L}_\beta(1)(u) - \delta \mathrm{L}_\beta(R)(u),\end{aligned}$$

and then using lemma 2.13

$$u\mathrm{L}_\beta(R)(u) - R_0\tilde{E}_0(\infty,0) = \delta R_s \frac{\tilde{E}_0(\infty,0)}{u} - \delta \mathrm{L}_\beta(R)(u),$$

$$\mathrm{L}_\beta(R)(u) = \frac{\delta R_s + R_0 u}{u(u+\delta)}\tilde{E}_0(\infty,0),$$

$$= \left(\frac{R_s}{u} + \frac{R_0 - R_s}{u+\delta}\right)\tilde{E}_0(\infty,0).$$

By applying the inverse proportional fractional Laplace transform to both sides,

$$R(t) = R_s + (R_0 - R_s)\tilde{E}_{-\delta}(t,0),$$

is found.

Example 3.2 Let $\mathbb{T} = \mathbb{Z}, \kappa_1(\beta,t) = (1-\beta)3^{\beta/2}, \ \kappa_0(\beta,t) = \beta 3^{(1-\beta)/2}$. In this case, the solution of the problem

$$\begin{cases} P^{1/2}R(t) = -\frac{3}{2}(R_s - R)(t), & (3.4)\\ R(0) = R_0, & (3.5)\end{cases}$$

is

$$R(t) = R_s + (R_0 - R_s)\exp[tLog2].$$

Solution Since $\mu(t) = 1$, from the definition of the proportional fractional exponential function (4), we have

$$\tilde{E}_{-\delta}(t,0) = \exp\left[\int_0^t Log\left(1 - \frac{-\frac{3}{2}+\frac{3}{2}}{\frac{3}{2}}\right)\Delta u\right]$$

$$= \exp\left[\int_0^t Log2\,\Delta u\right]$$

$$= \exp\left[Log2\sum_0^{t-1} 1\right]$$

$$= \exp[tLog2].$$

Hence, the solution of (3.4)-(3.5) is given that

$$R(t) = R_s + (R_0 - R_s)\exp[tLog2].$$

Example 3.3 Let $\mathbb{T} = 3^{\mathbb{N}^0}, \kappa_1(\beta, t) = (1-\beta),\ \kappa_0(\beta, t) = \beta t^{\frac{3\beta}{2}}$. In this case, the solution of the problem is:

$$\begin{cases} P^{1/3}R(t) = -(R_s - R)(t), & (3.6) \\ R(1) = R_1, & (3.7) \end{cases}$$

$$R(t) = R_s + (R_1 - R_s)3^k.$$

Solution: Since $\mu(t) = 2t$, we get

$$\tilde{E}_{-\delta}(t,1) = \exp\left[\int_s^t \frac{1}{2u} Log\left(1 - \frac{-1+\frac{1}{3}}{\frac{2}{3}t}2t\right)\Delta u\right]$$

$$= \exp\left[\int_0^t \frac{1}{2u} Log3\,\Delta u\right]$$

$$= \exp\left[Log3\sum_1^{t/3} 1\right]$$

$$= \exp\left[Log3\sum_{3^0}^{3^{k-1}} 1\right]$$

$$= \exp[kLog3],$$

$$= 3^k, \qquad t = 3^k,\ k \in \mathbb{N},$$

and the solution can be obtained as

$$R(t) = R_s + (R_1 - R_s)3^k.$$

4. Conclusion

Following a discussion of the properties and theories of the proportional fractional derivative on a time scale, we solve the proportional fractional Noyes-Whitney dynamic equation in this study while the initial condition is present. We then provide multiple examples using the proportional fractional derivative on different time scales. We provide light on the behavior of the Noyes-Whitney equation with proportional fractional derivative, so offering insights into dissolving processes in chemical engineering and medicinal applications.

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References

[1] Abdeljewad T. On conformable fractional calculus. J Comput Appl Math 2015; 279: 57–66.
[2] Agarwal R, Bohner M, O'Regan D, Peterson A. Dynamic equations on time scales: a survey. J Comput Appl Math 2002; 141(1-2): 1-26.
[3] Alkan A. Improving homotopy analysis method with an optimal parameter for time-fractional Burgers equation. Karamanoğlu Mehmetbey Univ J Engin Natural Sci 2022; 4(2), 117-134.
[4] Anderson DR, Georgiev SG. Conformable Dynamic Equations on Time Scales. Chapman and Hall/CRC, 2020.
[5] Anderson DR, Ulness DJ. Newly defined conformable derivatives. Adv Dyn Syst Appl 2015; 10(2): 109-137.
[6] Aulbach B, Hilger S. A unified approach to continuous and discrete Dynamics. Qual Theory Differ Equ (Szeged, 1988), 37–56, Colloq Math Soc János Bolyai, 53 North-Holland, Amsterdam, 1990.
[7] Avcı HH, Anaç H. The New Conformable methods to solve conformable time-fractional generalized Burgers equation with proportional delay. Erciyes Univ Inst Sci Tech J Sci 2023; 39(2), 315-329.
[8] Bekiryazıcı Z, Merdan M, Kesemen T, Khaniyev T. Mathematical modeling of biochemical reactions under random effects. TJMCS 2016; 5, 8-18.
[9] Bektaş U, Anaç H. Q-homotopy Shehu analysis transform method of time-fractional coupled Burgers equations. Eskişehir Technical Univ J Sci Tech A-Appl Sci Eng 2023; 24(3), 177-191.
[10] Benkhettou N, Brito da Cruz AMC, Torres DFM. A fractional calculus on arbitrary time scales: Fractional differentiation and fractional integration. Signal Process 2015; 107: 230– 237.
[11] Benkhettou N, Hassani S, Torres, DFM. A conformable fractional calculus on arbitrary time scales. J King Saud Univ Sci 2016; 28(1): 93-98.
[12] Bohner M, Peterson A. Dynamic equations on time scales, An introduction with applications. Boston, MA: Birkhauser, 2001.
[13] Bohner M, Peterson A. Advances in Dynamic Equations on Time Scales. Boston: Birkhauser, 2004.
[14] Bohner M, Svetlin G. Multivariable dynamic calculus on time scales. Cham: Springer, 2016.
[15] Dokoumetzidis A, Papadopoulou V, Macheras P. Analysis of dissolution data using modified versions of Noyes–Whitney equation and the Weibull function. Pharm Res 2006; 23: 256-261.
[16] Erol AS, Anaç H, Olgun A. Numerical solutions of conformable time-fractional Swift-Hohenberg equation with proportional delay by the novel methods. Karamanoglu Mehmetbey Univ J Engin Natural Sci 2023; 5(1), 1-24.
[17] Gulsen T, Yilmaz E, Goktas S. Conformable fractional Dirac system on time scales. J Inequal Appl 2017; 2017(1): 161.
[18] Gülşen T, Acar M. Self-Adjoint Sturm-Liouville Dynamic Problem via Proportional Derivative. JIST 2023; 13(4): 2945-2957.
[19] Gülşen T, Yilmaz E, Kemaloğlu H. Conformable fractional Sturm-Liouville equation and some existence results on time scales. Turk J Math 2018; 42(3): 1348-1360.
[20] Hilger S. Analysis on measure chains a unified approach to continuous and discrete calculus. Results Math 1990; 18(1).
[21] Kartal A, Anaç H, Olgun A. Numerical solution of conformable time fractional generalized Burgers equation with proportional delay by new methods. Black Sea Sci J 2023; 13(2), 310-335.
[22] Kartal A, Anaç H, Olgun A. The new numerical solutions of conformable time fractional generalized Burgers equation with proportional delay. Gümüşhane Univ J Sci 2023; 13(4), 927-938.
[23] Katsanevakis S, Maravelias CD. Modelling fish growth: multi-model inference as a better alternative to a priori using von Bertalanffy equation. Fish fish 2008; 9(2): 178-187.
[24] Katugampola U. A new fractional derivative with classical properties, arXiv:1410.6535v2, 2014.
[25] Khalil R, Horani MAl, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math 2014; 264: 57–66.
[26] Lindberg NO, Lundstedt, T. The relationship between the dissolution rate and the particle size of prednimustine: a disagreement with the noyes-whitney equation. Drug Dev Ind Pharm 1994; 20(16): 2547-2550.
[27] Ortigueira MD, Machado JT. What is a fractional derivative? J Comput Phys 2015; 293: 4-13.
[28] Öner BK, Anaç H. The Numerical Solutions of the Conformable Time-Fractional Noyes Field Model via a New Hybrid Method. IKJM 2023; 5(2), 76-91.
[29] Segi Rahmat MR. A new definition of conformable fractional derivative on arbitrary time scales. Adv Differ Equ 2019; 2019(1): 1-16.
[30] Shi Y, Wan A, Shi Y, Zhang Y, Chen Y. Experimental and mathematical studies on the drug release properties of aspirin loaded chitosan nanoparticles. BioMed Res Int 2014; 2014: 8 pages.
[31] Tomar BS, Tirumkudulu MS, Yu W, Berchielli A, Manthena S, Doshi P. A model to predict drug release from a single-layer osmotic controlled-release tablet. J Drug Deliv Sci Technol 2023; 80, 104138.
[32] Van der Zwaan I, Frenning G. A new modelling approach for dissolution of polydisperse powders. Int J Pharma 2023; 633, 122626.
[33] Wimalasiri VW, Dunuweera SP, Dunuweera AN, Rajapakse RMG. Research article Noyes-Whitney dissolution model-based pH-sensitive slow release of paclitaxel (taxol) from human hair-derived keratin microparticle carriers. BioMed Res Int 2021; 2021, Article ID 6657482: 8 pages.
[34] Yilmaz E, Gulsen T, Panakhov ES. Existence Results for a Conformable Type Dirac System on Time Scales in Quantum Physics. Appl Comput Math Int J 2022; 21(3): 279-291.