Transversal Lightlike Submersions

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Abstract — In this paper, we introduce the concept of transversal lightlike submersions from semi-Riemannian manifolds onto semi-Riemannian manifolds. Specifically, we present the concepts of transversal r-lightlike and isotropic transversal lightlike submersions and examine the geometry of foliations formed by these submersions through various examples. In this way, we demonstrate certain points where transversal r-lightlike submersions differ from semi-Riemannian submersions. Furthermore, we investigate O'Neill’s tensors for transversal r-lightlike submersions and examine the integrability of certain distributions by employing these tensor fields. Thus, valuable information regarding such submersions’ geometric structures and properties can be provided, paving the way for new research avenues. We finally discuss the need for further research.

Keywords — Transversal submersion, Riemannian submersion, lightlike manifold, lightlike submersion

1. Introduction

Riemannian submersions are foundational mappings within the realm of differential geometry, serving as potent instruments for unraveling the geometric properties of Riemannian manifolds. These submersions allow for methodically examining the interactions between several manifolds. Their importance spans several fields in pure mathematics and theoretical physics, providing a detailed framework for examining the complex interactions between different geometries and providing deep insights into the structure of the physical universe.

O’Neill [1] and Gray [2] introduced the theory of Riemannian submersion, which has subsequently become the subject of numerous studies [3-12]. Consequently, it has become a useful tool for clarifying the structure of Riemannian manifolds. It is well known that when \( M_1 \) and \( M_2 \) are Riemannian manifolds, the fibers become Riemannian manifolds; however, it has been noted that the fibers of \( f \) may not be semi-Riemannian when the manifolds are semi-Riemannian [13].

Şahin has recently introduced and studied the concept of submersion from lightlike manifolds onto semi-Riemannian manifolds in [14], along with the submersion from semi-Riemannian manifolds onto lightlike manifolds in [13], providing significant insights into the geometric relationship between these disparate manifold types.

Hereinafter, we will initially provide an overview of a lightlike manifold and subsequently introduce the concept of transversal submersion from semi-Riemannian manifolds to semi-Riemannian manifolds. We will investigate specific examples to assess the possibility of constructing such a submersion and draw conclusions based on our analysis. Thus, the concept of transversal lightlike submersion will pave the way for innovative research...
2. Lightlike Manifolds

Let $V$ be a real vector space and $g_1$ be a bilinear form on $V$. If there exists a non-zero vector $\xi$ in $V$ such that $g_1(\xi, v) = 0$, for every $v \in V$, then $g_1$ is considered degenerate on $V$; otherwise, it is termed as non-degenerate. On the other hand, if $g_1(v, v) > 0$, then $g_1$ is said to be positively defined on $V$; conversely, $g_1(v, v) < 0$, then $g_1$ is said to be negatively defined on $V$. Consequently, a positive or negative defined $g_1$ is deemed to be non-degenerate [15].

Consider $V$ as a vector space and suppose that there exists a symmetric bilinear form $g_1$ on $V$. In this case, there exist bases $\{e_i\}$ on $V$ that

\[
\begin{align*}
1 \leq i \leq r & \quad ; \quad g_1(e_i, e_i) = 0 \\
1 \leq j \leq q & \quad ; \quad g_1(e_j, e_j) = -1 \\
1 \leq k \leq p & \quad ; \quad g_1(e_k, e_k) = 1 \\
i \neq j & \quad ; \quad g_1(e_i, e_j) = 0
\end{align*}
\]

These bases are referred to as orthonormal bases, and the triplet $(r, q, p)$ is the type of the bilinear form $g_1$ [16]. Let $(M_1, g_1)$ denote a real differentiable $n$ dimensional manifold, where $g_1$ is a symmetric tensor field of type $(0,2)$. Assume that $M_1$ is paracompact. The radical space of $T_x M_1$ denoted by $\text{Rad} T_x M_1$ and given as

\[
\text{Rad} T_x M_1 = \{ \xi \in T_x M_1 ; g_1(\xi, X) = 0, X \in T_x M_1 \}
\]

The nullity degree of $g_1$ corresponds to the dimension of $T_x M_1$. Suppose $\text{Rad} T M_1$ corresponds to the radical subspace of $\text{Rad} T_x M_1$ for every $x \in M_1$. In this case, $\text{Rad} T M_1$ becomes the radical distribution of $M_1$, and if $0 < r \leq n$, this manifold $M_1$ is called a lightlike manifold [13]. The non-degenerate symmetric bilinear form $g_1$ on $V$ is referred to as a semi-Euclidean metric, in this case, $V$ is termed as a semi-Euclidean space [15].

We note that $g_1(v, v) > 0$ or $v = 0$, then $v$ is defined as spacelike. Similarly, if $g_1(v, v) < 0$, then $v$ is defined as timelike. Moreover, if $g_1(v, v) = 0$ and $v \neq 0$, then $v$ is defined as lightlike (null, isotropic), where $v \in V, V$ is a semi-Euclidean space [15].

Afterward, we present the concepts of Riemannian and lightlike submersions, necessary for providing transversal lightlike submersions.

3. Riemannian Submersions

In this section, the definition of Riemann submersions is provided, along with some significant information concerning these submersions.

Let $(M_1, g_1)$ and $(M_2, g_2)$ be $m$ and $n$ dimensional Riemannian manifolds, respectively, and $f : M_1 \to M_2$ be a submersion. In this case, $\text{rank} f = \dim M_2 < \dim M_1$. For any $x \in M_2$, the fiber $F_x = f^{-1}(x)$ is a submanifold of $M_1$ with dimension $r = (m - n)$. Submanifolds $f^{-1}(x)$ are called submersion fibers.

The integrable distribution $\mathcal{V}$ of the submersion $f : M_1 \to M_2$ in $(M_1, g_1)$ is defined by $\mathcal{V}_p = \ker f_d p$, where $p \in M_1$. $\mathcal{V}_p$ is called the vertical distribution of submersion. Furthermore, $\mathcal{H}_p = (\mathcal{V}_p)^\perp$ is orthogonal to and complements the vertical distribution. We refer to the $\mathcal{H}$ distribution as the horizontal distribution of the submersion [17, 18].
Consider \((M_1, g_1)\) and \((M_2, g_2)\) as Riemannian manifolds. A differentiable mapping \(f\) is referred to as a Riemannian submersion if it satisfies the following conditions:

\(i.\) \(f\) has maximal rank.

\(ii.\) For any \(p \in M_1, f_p\) preserves the length of \(X_p \in \Gamma(H_p)\), where \(H_p\) represents the horizontal vectors [1].

The first condition in the definition ensures that the mapping is a submersion. The second condition states that the \(f\) derivative transformation at the point \(p \in M_1\) is a linear isometry from the horizontal space \(H_p\) to the tangent space \(T_{f(p)}M_2\). Therefore, \(g_{1p}(u,v) = g_{2f(p)}(f_pu, f_pv)\), holds for \(u,v \in H_p, p \in M_1\) [17]. Furthermore, given that \(X\) is horizontal and \(f\)-related to a vector field \(\tilde{X}\) on \(M_2\), that is, \(f_*(X) = \tilde{X}_f(p)\) for any \(p \in M_1\) a vector field \(X\) on \(M_1\) is considered basic [1].

**Proposition 3.1** Let \((M_1, g_1)\) and \((M_2, g_2)\) be Riemannian manifolds, where \(f: (M_1, g_1) \to (M_2, g_2)\) is a Riemannian submersion, and let \(\nabla\) and \(\nabla'\) denote the Levi-Civita connections of \(M_1\) and \(M_2\), respectively. Suppose the basic vector fields \(X\) and \(Y\) on \(M_1\) are \(f\)-related to the vector fields \(X'\) and \(Y'\). In this case, the following equations are obtained [18]:

1. \(g_1(X,Y) = g_2(X', Y') \circ f\)
2. The basic vector field \(h[X, Y]\) corresponds to \([\tilde{X}, \tilde{Y}]\)
3. The basic vector field \(h(\nabla_X Y)\) corresponds to \(\nabla'_{X'} Y'\)

### 4. Lightlike Submersions

Sahin and Gündüzalp previously introduced several concepts related to lightlike submersions in [13]. Let \((M_1, g_1)\) be a semi-Riemannian manifold, \((M_2, g_2)\) be an \(r\)-lightlike manifold. Consider a differentiable submersion \(f: M_1 \to M_2\), where \(f_*\) denotes the derivative transformation. The kernel of \(f_*\) at the point \(p \in M_2\), denotes as \(\ker f_*\), is defined as [13]:

\[
\ker f_* = \{X \in T_p(M_1): f_*(X) = 0\}
\]

**Case 4.1** \(0 < \dim \Delta < \min\{\dim (\ker f_*), \dim (\ker f_*^\perp)\}\) In this case, \(\Delta\) is the radical subspace of \(T_pM_1\).

Thus, a quasi-orthonormal basis of \(M_1\) along \(\ker f_*\) is constructed as described in [15]. Since \(\ker f_*\) is a real lightlike vector space, there exists a non-degenerate subspace that complements \(\Delta\) [15]. Then,

\[
\ker f_* = \Delta \perp S(\ker f_*)
\]

and similarly,

\[
(\ker f_*)^\perp = \Delta \perp S(\ker f_*)^\perp
\]

where \(S(\ker f_*)^\perp\) denotes the complementary subspace of \(\Delta\) in \((\ker f_*)^\perp\). Given the expression \(T_pM_1 = S(\ker f_*) \perp (S(\ker f_*)^\perp)\), since \(S(\ker f_*)\) is non-degenerate in \(T_pM_1\), it can be observed that \((S(\ker f_*))^\perp\) is the complementary subspace of \(S(\ker f_*)\) in \(T_pM_1\). Additionally, since \(S(\ker f_*)\) and \((S(\ker f_*))^\perp\) are non-degenerate, we can observe that

\[
(S(\ker f_*))^\perp = S(\ker f_*)^\perp \perp (S(\ker f_*)^\perp)^\perp
\]

Then, according to Proposition 2.4 in [15], it is known that “There exists a quasi-orthonormal basis of \(\ker f_*\).” Therefore, we have the following expressions:
\[
\begin{align*}
(g_1(\xi_i, \xi_j) &= g_1(N_i, N_j) = 0 \quad ; \quad g_1(\xi_i, N_j) = \delta_{ij} \\
g_1(W_a, \xi_j) &= g_1(W_a, N_j) = 0 \quad ; \quad g_1(W_a, W_a) = \varepsilon_2 \delta_{\alpha\beta}
\end{align*}
\]

where \(i, j \in \{1, \ldots, r\}\) and \(\alpha, \beta \in \{1, \ldots, t\}\). Here \(\{N_i\}\) represents differentiable null vector fields of \((S(\ker f_1))^{\perp}\), \(\{\xi_i\}\) is the basis of \(\Delta\), and \(\{W_a\}\) is the basis of \(S(\ker f_1)\). The set of vector fields \(\{N_i\}\) is denoted by \(\text{ltr}(\ker f_1)\), and consider the following subspace:

\[
\text{tr}(\ker f_1) = \text{ltr}(\ker f_1) \perp S(\ker f_1) \perp
\]

It should be noted that \(\text{ltr}(\ker f_1)\) and \((\ker f_1)\) are not orthogonal to each other. The space \(\ker f_1\), denoted as \(\mathcal{V}\), is referred to as the vertical space of \(T_p M_1\), while \(\text{tr}(\ker f_1)\), denoted as \(\mathcal{H}\), is called the horizontal space of \(T_p M_1\), as is usual in the theory of Riemannian submersions. Thus, we have the decomposition:

\[
T_p M_1 = \mathcal{V}_p \oplus \mathcal{H}_p
\]

We notice that \(\mathcal{V}\) and \(\mathcal{H}\) are not orthogonal [13].

**Definition 4.2** [13] Let \((M_1, g_1)\) be a semi-Riemannian manifold and \((M_2, g_2)\) be an \(r\)-lightlike manifold. Consider a submersion \(f: M_1 \to M_2\) satisfying the following conditions:

- (i) \(\dim \Delta = \dim \{(\ker f_1) \cap (\ker f_1) \perp \} = r\), \(0 < r < \min \{\dim (\ker f_1), \dim (\ker f_1) \perp \}\).
- (ii) \(f_1\) preserves the length of horizontal vectors, i.e. \(g_1(X, Y) = g_2(f_1 X, f_1 Y)\) for \(X, Y \in \Gamma(\mathcal{H})\). In this case, we can say that \(f\) is an \(r\)-lightlike submersion.

**Case 4.3.** [13] \(\dim \Delta = \dim (\ker f_1) < \dim (\ker f_1) \perp\). Then, \(\mathcal{V} = \Delta\) and \(\mathcal{H} = S(\ker f_1) \perp \perp \text{ltr}(\ker f_1)\). Thus, we name \(f\) an isotropic submersion.

**Case 4.4.** [13] \(\dim \Delta = \dim (\ker f_1) \perp < \dim (\ker f_1)\). Then, \(\mathcal{V} = S(\ker f_1) \perp \perp \mathcal{H} = \text{ltr}(\ker f_1)\). Thus, we name \(f\) co-isotropic submersion.

**Case 4.5.** [13] \(\dim \Delta = \dim (\ker f_1) \perp = \dim (\ker f_1)\). Then, \(\mathcal{V} = \Delta\) and \(\mathcal{H} = \text{ltr}(\ker f_1)\). Thus, we name \(f\) totally lightlike submersion.

Therefore, in conjunction with this information, we present a new concept.

### 5. Transversal Lightlike Submersions

In this section, we will introduce the concept of transversal submersion and provide four related examples. Through these examples, we will explore the existence of various types of submersions. Additionally, we will introduce O’Neill tensors for transversal submersions, which will lead to different results regarding their overall properties.

Firstly we note that a basic vector field on \(M_1\) is a horizontal vector field \(X\) that is \(f\)-related to vector field \(\tilde{X}\) on \(M_2\), meaning that \(f_1(X_p) = \tilde{X}_{f(p)}\) for all \(p \in M_1\) (Where \(f_1\) is a derivative map). Every vector field \(\tilde{X}\) on \(M_2\) has a unique horizontal lift \(X\) to \(M_1\), and \(X\) is basic. Therefore, the correspondence \(X \leftrightarrow \tilde{X}\) establishes a one-to-one relationship between fundamental vector fields on \(M_1\) and arbitrary vector fields on \(M_2\) [13]. Thus, we can give the following definition.

**Definition 5.1** Consider \((M_1, g_1)\) and \((M_2, g_2)\) be a semi-Riemannian manifold and let \(f: M_1 \to M_2\) be a submersion. If the condition
\[ g_1(X,Y) = g_2(f_*(X), f_*(Y)) \] 

holds for all \( X, Y \in \Gamma(S(\ker f_*)^\perp) \), we call the mapping \( f \) as a transversal submersion.

Therefore,
- \( f \) has maximal rank,
- At each point \( p \) in \( M_1 \), the \( f_*p \) mapping preserves the lengths of horizontal vectors; that is, \( g_{2f}(X,Y) = g_{2f}(f_*X, f_*Y) \). This implies that at a point \( p \) in \( M_1 \), the \( f_* \) derivative transformation states a linear isometry from \( \Gamma(S(\ker f_*^*)^\perp) \) space onto \( T_{f*p}M_2 \).

Note that for \( p \in M_2 \), \( f^{-1}(p) \) is a submanifold with \( \dim M_1 - \dim M_2 \).

**Definition 5.2** Consider \((M_1, g_1)\) and \((M_2, g_2)\) as semi-Riemannian manifolds and let \( f: M_1 \to M_2 \) be a transversal submersion. According to Definition 4.2 in Case 4.1, \( f \) is characterized as a transversal \( r \)-lightlike submersion. Furthermore, as per Definition 4.2 in Case 4.3, \( f \) is denoted as an isotropic transversal lightlike submersion.

We will give examples of transversal \( r \)-lightlike and isotropic transversal lightlike submersions.

**Example 5.3** Consider \( \mathbb{R}^6 \) and \( \mathbb{R}^3 \) to be \( \mathbb{R}^6 \) and \( \mathbb{R}^3 \) endowed with semi-Riemannian metrics. Define these metrics as follows:

\[ g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 \]

and

\[ g_2 = -(dy_1)^2 + \frac{1}{2}(dy_2)^2 + \frac{1}{2}(dy_3)^2 \]

where \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) and \( \{y_1, y_2, y_3\} \) are the canonical coordinates on \( \mathbb{R}^6 \) and \( \mathbb{R}^3 \), respectively.

Moreover, we define the following map:

\[ f: \mathbb{R}_1^6 \to \mathbb{R}_1^3 \]

\[ (x_1, x_2, x_3, x_4, x_5, x_6) \to (x_1 - x_5, x_2 + x_6, x_3 + x_4) \]

The kernel of \( f_* \) is then given by

\[ \ker f_* = \text{Span}\left\{ W_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, W_3 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\} \]

Thus,

\[ (\ker f_*)^\perp = \text{Span}\left\{ T_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, T_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, T_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\} \]

Therefore,

\[ W_1 = T_1, \quad \Delta = \ker f_* \cap (\ker f_*)^\perp = \text{Span}\{W_1\} \]

Then,

\[ \text{Itr}(\ker f_*) = \text{Span}\left\{ N = \frac{1}{2}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}\right)\right\} \]
Using $N = \frac{1}{g_1(V,V)} \left\{ V - \frac{g_1(V,V)}{2g_1(V,V)} \xi \right\}$ from Equation (1.5) in [15], it is easy to check that $g_1(N,W_1) = 1$, $g_1(N,W_2) = 0$, and $g_1(N,W_3) = 0$. Thus, we give the vertical and horizontal spaces as:

$$V = \text{Span}\{W_1, W_2, W_3\}, \mathcal{H} = \text{Span}\{T_2, T_3, N\}$$

Furthermore, since $f_*(T_2) = 2 \frac{\partial}{\partial y_2}, f_*(T_3) = 2 \frac{\partial}{\partial y_3}$, and $f_*(N) = -\frac{\partial}{\partial y_1}$, we obtain that

$$g_1(T_2, T_2) = g_2(f_*(T_2), f_*(T_2)) = 2, g_1(T_3, T_3) = g_2(f_*(T_3), f_*(T_3)) = 2$$

$$g_1(N, N) = 0, g_2(f_*(N), f_*(N)) = -1$$

Here, we state that the lengths of the vectors in $S(\ker f)_1$ are conserved, but we cannot say the same for $\text{ltr}(\ker f)$. In this case, the mapping $f$ is a transversal $1$–lightlike submersion.

**Example 5.4** Consider $\mathbb{R}_2^6$ and $\mathbb{R}_2^3$ be $\mathbb{R}^6$ and $\mathbb{R}^3$ endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$

and

$$g_2 = -(dy_1)^2 - (dy_2)^2 + \frac{1}{2}(dy_3)^2$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3\}$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively. We define the following map:

$$f : \mathbb{R}_2^6 \rightarrow \mathbb{R}_2^3$$

$$(x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (x_1 + x_4, x_2 + x_5, x_3 + x_6)$$

The kernel of $f_*$ is then

$$\ker f_* = \text{Span}\left\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, W_3 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}\right\}$$

Thus,

$$(\ker f_*)_\perp = \text{Span}\left\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, T_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, T_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}\right\}$$

Therefore, we have $W_1 = T_1$ and $W_2 = T_2$,

$$\Delta = (\ker f_*) \cap (\ker f_*)_\perp = \text{Span}\{W_1 = T_1, W_2 = T_2\}$$

Then,

$$\text{ltr}(\ker f_*) = \text{Span}\left\{N_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}\right), N_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}\right)\right\}$$

Moreover, we have

$$f_*(T_3) = 2 \frac{\partial}{\partial y_3}, f_*(N_1) = \frac{\partial}{\partial y_1}, f_*(N_2) = \frac{\partial}{\partial y_2}$$

Then, we obtain that

$$g_1(T_3, T_3) = g_2(f_*(T_3), f_*(T_3)) = 2$$

$$g_1(N_1, N_1) = 0, g_2(f_*(N_1), f_*(N_1)) = -1, g_1(N_2, N_2) = 0, g_2(f_*(N_2), f_*(N_2)) = -1$$

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Thus, $f$ is a transversal 2-lightlike submersion.

**Example 5.5** Let $\mathbb{R}^4_1$ and $\mathbb{R}^3_1$ be $\mathbb{R}^4$ and $\mathbb{R}^3$ endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$$

and

$$g_2 = -(dy_1)^2 + (dy_2)^2 + (dy_3)^2$$

where $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3\}$ are the canonical coordinates on $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. We define the following map:

$$f : \mathbb{R}^4_1 \rightarrow \mathbb{R}^3_1$$

$$(x_1, x_2, x_3, x_4) \rightarrow (x_1 + x_2, x_3, x_4)$$

The kernel of $f$, is then

$$\ker f = \text{Span}\left\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right\}$$

Thus,

$$(\ker f)^\perp = \text{Span}\left\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, T_2 = \frac{\partial}{\partial x_3}, T_3 = \frac{\partial}{\partial x_4}\right\}$$

Hence, we have

$$\Delta = \ker f \cap (\ker f)^\perp = \text{Span}\{W_1\}$$

Then,

$$\text{ltr}(\ker f) = \text{Span}\left\{N = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\right\}$$

Moreover,

$$f_*(N) = \frac{\partial}{\partial y_1}, f_*(T_2) = \frac{\partial}{\partial y_2}, f_*(T_3) = \frac{\partial}{\partial y_3}$$

Thus, we obtain that

$$g_1(T_2, T_2) = 1, g_2(f_*(T_2), f_*(T_2)) = 1, g_1(T_3, T_3) = 1, g_2(f_*(T_3), f_*(T_3)) = 1$$

$$g_1(N, N) = 0, g_2(f_*(N), f_*(N)) = -1$$

Hence, $f$ is isotropic transversal 1-lightlike submersion.

**Example 5.6** Let $\mathbb{R}^6_2$ and $\mathbb{R}^4_2$ be $\mathbb{R}^6$ and $\mathbb{R}^4$ endowed with semi-Riemannian metrics. Define these metrics as follows:

$$g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$

and

$$g_2 = -(dy_1)^2 - (dy_2)^2 + (dy_3)^2 + (dy_4)^2$$

where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3, y_4\}$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^4$, respectively. We define the following map:
Proof. We introduce the concepts of transversal lightlike submersion as defined in Definition 5.1. Let \( f : (M_1, g_1) \rightarrow (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). Then, for all \( X, Y \in \Gamma(S(\ker f))^\perp \)

\[
g_1(X, Y) = g_2(\tilde{\nabla}X, Y) \circ f
\]

Proof. The proof can be made easily from the isometry condition in Definition 5.1 using (5.1).

Theorem 5.9 Let \( f : (M_1, g_1) \rightarrow (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). Then \( h\nabla_X Y \) is the fundamental vector field corresponding to \( \nabla_X \tilde{Y} \), for \( X, Y \in \Gamma(S(\ker f))^\perp \).

Proof. Since \( M_1 \) is a semi-Riemannian manifold with the Levi-Civita connection, the Koszul equality holds, leading to

\[
2g_1(\nabla_X Y, Z) = X(g_1(Y, Z)) + Y(g_1(X, Z)) - Z(g_1(X, Y)) + g_1([X, Y], Z) + g_1([Z, X], Y) - g_1([Y, Z], X)
\]
where $X, Y, Z \in \Gamma(S(\ker f))$. By utilizing Lemma 5.8, we obtain $(g_1(Y, Z)) = \bar{X}g_2(\bar{Y}, \bar{Z}) \circ f$. Similarly, if we generalize this equality, we have
\[
2g_1(\nabla_X Y, Z) = \bar{X}g_2(\bar{Y}, \bar{Z}) \circ f + \bar{Y}g_2(\bar{Z}, \bar{X}) \circ f - \bar{Z}g_2(\bar{X}, \bar{Y}) \circ f + g_2([\bar{X}, \bar{Y}], \bar{Z}) \circ f + g_2([\bar{Z}, \bar{Y}], \bar{X}) \circ f - g_2([\bar{Y}, \bar{Z}], \bar{X}) \circ f
\]
Considering that $M_2$ is a semi-Riemannian manifold, it has a Levi-Civita connection. From here, we can state that $\nabla$ satisfies Koszul’s equality. Then,
\[
g_1(\nabla_X Y, Z) = g_2(\nabla_X \bar{Y}, \bar{Z}) \circ f
\]
Therefore, we deduce that $h\nabla_X Y$ represents the fundamental vector field associated with $\nabla_X \bar{Y}$.

**Remark 5.10** Let $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal $r$-lightlike submersion between semi-Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$, in this case, the expression in Theorem 5.9 does not apply to $N_1, N_2 \in \Gamma(\text{ltr}(\ker f))$.

**Remark 5.11** Let $f: (M_1, g_1) \to (M_2, g_2)$ be a transversal $r$-lightlike submersion between semi-Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$. In this case, for any $U \in \Gamma(S(\ker f))$ and $X \in \Gamma(S(\ker f))$, $[X, U]$ is a vertical vector field.

**Theorem 5.12** Consider the semi-Riemannian manifolds, $M_1$ and $M_2$, equipped with the metrics $g_1$ and $g_2$, respectively. Let $f: M_1 \to M_2$ be a transversal $r$-lightlike submersion. In this case, we have $g_1(\nabla_{N_1} N_2, N_3) = -g_1(N_2, \nabla_{N_1} N_3)$ for all $N_1, N_2, N_3$ in $\Gamma(\text{ltr}(\ker f))$, where $\nabla$ represents the Levi-Civita connection.

**Proof.** Since $\nabla$ is the Levi-Civita connection for $N_1, N_2, N_3$ in $\Gamma(\text{ltr}(\ker f))$, we have
\[
g_1(\nabla_{N_1} g_1(N_2, N_3) = N_1(g_1(N_2, N_3)) - g_1(\nabla_{N_1} N_2, N_3) - g_1(N_2, \nabla_{N_1} N_3)
\]
\[
g_1(\nabla_{N_1} N_2, N_3) = -g_1(N_2, \nabla_{N_1} N_3)
\]

Let $M_1$ and $M_2$ be semi-Riemannian manifolds, $f: M_1 \to M_2$ be transversal submersion and $E, F$ arbitrary vector fields on $M_1$. Also, let the projections $h: TM_1 \to H$ and $v: TM_1 \to V$ denote the natural projections associated with the decomposition of $TM_1 = H \oplus V$. Moreover, $\nabla$ represents the Levi-Civita connection of $(M_1, g_1)$. We define the fundamental tensor field $T$ of type $(1,2)$;
\[
T^E_F = h\nabla^E_F v + v\nabla^E_F h F
\]
has the following properties:

i. $T$ exchanges the role of horizontal and vertical subspaces

ii. $T$ is vertical: $T^E_E = T^v_v$

The tensor field $A$;
\[
A^E_F = v\nabla^E_F h F + h\nabla^E_F v F
\]
has the following properties:

i. $A$ exchanges the role of horizontal and vertical subspaces

ii. $A$ is horizontal: $A^X_X = A^h_h$
Lemma 5.13 Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, we obtain the followings:

i. \( T_0 V = h \nabla_0 V \)  
ii. \( T_0 \xi = h \nabla_0 \xi \)  
iii. \( T_\xi V = h \nabla_\xi V \)  
iv. \( T_\xi_1 \xi_2 = h \nabla_\xi_1 \xi_2 \)

where \( U, V \in \Gamma(S(\ker f)) \), \( \xi, \xi_1, \xi_2 \in \Gamma(\Delta) \).

Proof. Here, we will consider two situations:

i. If we use (5.2) for transversal \( r \)-lightlike submersion, we can express it as
\[
T_0 V = h \nabla_0 V
\]  
(5.8)

ii. Considering elements for which the multiplication equations by \( T_0 V \) is non-zero, examine the following equations:
\[
U g_1(V, \xi) = g_1(\nabla_0 V, \xi) + g_1(V, \nabla_0 \xi) \\
g_1(\nabla_0 V, \xi) = -g_1(V, \nabla_0 \xi) \\
g_1(h \nabla_0 V, \xi) = -g_1(V, v \nabla_0 \xi) - g_1(V, h \nabla_0 \xi) \\
g_1(T_0 V, \xi) = -g_1(V, v \nabla_0 \xi)_{\neq 0}
\]  
(5.9)

where \( U, V \in \Gamma(S(\ker f)) \), \( \xi \in \Gamma(\Delta) \). In this case, \( T_0 V \neq 0 \).

\[
U g_1(V, X) = g_1(\nabla_0 V, X) + g_1(V, \nabla_0 X) \\
g_1(\nabla_0 V, X) = -g_1(V, \nabla_0 X) \\
g_1(h \nabla_0 V, X) = -g_1(V, v \nabla_0 X) - g_1(V, h \nabla_0 X) \\
g_1(T_0 V, X) = -g_1(V, v \nabla_0 X)_{\neq 0}
\]  
(5.10)

In this case, \( T_0 V \neq 0 \) where \( U, V \in \Gamma(S(\ker f)) \), \( X \in \Gamma(S(\ker f)^\perp) \). Consequently, from (5.9) and (5.10), we derive the non-zero equality expressed as \( T_0 V = h \nabla_0 V \). Similarly, we can establish the proofs for (5.5)-(5.7).

Corollary 5.14 Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, utilizing Lemma 5.13, we can deduce the expression:
\[
T_{W_1} W_2 = h \nabla_{W_1} W_2
\]  
(5.11)

where \( W_1, W_2 \in \Gamma(\ker f) \).

Lemma 5.15 Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, we have the following equations:

i. \( T_0 X = v \nabla_0 X \)
ii. \( T_0 N = v \nabla_0 N \)
iii. \( T_\xi X = v\nabla_\xi X \)

iv. \( T_\xi N = v\nabla_\xi N \)

where \( U \in \Gamma(S(\ker f_\ast)), \xi \in \Gamma(\Delta), X \in \Gamma(S(\ker f_\ast)^\perp), N \in \Gamma(\tr(\ker f_\ast)) \).

**Proof.** The proof of the first equation is done in a similar way to the proof of Lemma 5.13, using (5.2) and the Levi-Civita connection. Other equations can be obtained similarly easily.

**Corollary 5.16** Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)–lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, by using Lemma 5.15, we obtain

\[
T_W F = v\nabla_W F
\]

where \( W \in \Gamma(\ker f_\ast), F \in \Gamma(\tr(\ker f_\ast)) \).

**Lemma 5.17** Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)–lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, we have

i. \( A_X U = h\nabla_X U \)

ii. \( A_X \xi = h\nabla_X \xi \)

iii. \( A_N U = h\nabla_N U \)

iv. \( A_N \xi = h\nabla_N \xi \)

where \( U \in \Gamma(S(\ker f_\ast)), \xi \in \Gamma(\Delta), X \in \Gamma(S(\ker f_\ast)^\perp), N \in \Gamma(\tr(\ker f_\ast)) \).

**Proof.** The proof of the first equation is done in a similar way to the proof of Lemma 5.13, using (5.3) and the Levi-Civita connection. Other equations can be obtained similarly easily.

**Corollary 5.18** Let \((M_1, g_1), (M_2, g_2)\) be semi-Riemannian manifold and \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)–lightlike submersion. In this case, based on Lemma 5.17, we obtain \( A_F W = h\nabla_F W \), where \( F \in \Gamma(\tr(\ker f_\ast)), W \in \Gamma(\ker f_\ast) \).

**Lemma 5.19** Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, we have

i. \( A_X Y = v\nabla_X Y \quad (5.12) \)

ii. \( A_X N = v\nabla_X N \quad (5.13) \)

iii. \( A_N X = v\nabla_N X \quad (5.14) \)

iv. \( A_{N_1} N_2 = v\nabla_{N_1} N_2 \quad (5.15) \)

where \( X, Y \in \Gamma(S(\ker f_\ast)^\perp), N, N_1, N_2 \in \Gamma(\tr(\ker f_\ast)) \).

**Proof:** The proof of (5.12) is done in a similar way to the proof of Lemma 5.13, using (5.3) and the Levi-Civita connection. Other equations can be obtained similarly easily.

**Corollary 5.20** Let \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, utilizing Lemma 5.19, we derive \( A_{F_1} F_2 = v\nabla_{F_1} F_2 \), where \( F_1, F_2 \in \Gamma(\tr(\ker f_\ast)) \).

**Lemma 5.21** Let \((M_1, g_1), (M_2, g_2)\) be semi-Riemannian manifolds, \( \nabla \) be Levi-Civita connection in \( M_1, T \) and \( A \) be tensor fields, \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion. In this case,
The following equations:

\[ i. \nabla_{W_1} W_2 = T_{W_1} W_2 + v \nabla_{W_1} W_2 \]  \hspace{1cm} (5.16)

\[ ii. \nabla_W \xi = T_W \xi + v \nabla_W \xi \]  \hspace{1cm} (5.17)

\[ iii. \nabla_U W = T_U W + v \nabla_U W \]  \hspace{1cm} (5.18)

\[ iv. \nabla_U V = T_U V + v \nabla_U V \]  \hspace{1cm} (5.19)

\[ v. \nabla_U \xi = T_U \xi + v \nabla_U \xi \]  \hspace{1cm} (5.20)

\[ vi. \nabla_\xi W = T_\xi W + v \nabla_\xi W \]  \hspace{1cm} (5.21)

\[ vii. \nabla_{\xi_1} \xi_2 = T_{\xi_1} \xi_2 + v \nabla_{\xi_1} \xi_2 \]  \hspace{1cm} (5.22)

\[ viii. \nabla_\xi V = T_\xi V + v \nabla_\xi V \]  \hspace{1cm} (5.23)

where \( W, W_1, W_2, \in \Gamma(\ker f_i) \), \( \xi, \xi_1, \xi_2 \in \Gamma(\mathcal{V}) \), \( U, V \in \Gamma(S(f)) \).

**Proof.** Here, we will prove only (5.16). For any vector fields \( W_1, W_2 \in \Gamma(\ker f_i) \), we can establish the equation

\[ \nabla_{W_1} W_2 = v \nabla_{W_1} W_2 + h \nabla_{W_1} W_2. \]

Using Lemma 5.13 and Corollary 5.14, we obtain the equation \( \nabla_{W_1} W_2 = v \nabla_{W_1} W_2 + T_{W_1} W_2 \). The proof of the remaining equations can also be carried out similarly. \( \Box \)

**Lemma 5.22** Let \((M_1, g_1), (M_2, g_2)\) be semi-Riemannian manifolds, \( \mathcal{V} \) be Levi-Civita connection in \( M_1, T \) and \( A \) be tensor fields, \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion. In this case, we obtain the following equations:

\[ i. \nabla_W N = T_W N + h \nabla_W N \]

\[ ii. \nabla_U N = T_U N + h \nabla_U N \]

\[ iii. \nabla_\xi N = T_\xi N + h \nabla_\xi N \]

\[ iv. \nabla_U F = T_U F + h \nabla_U F \]

\[ v. \nabla_\xi F = T_\xi F + h \nabla_\xi F \]

\[ vi. \nabla_W F = T_W F + h \nabla_W F \]

where \( W \in \Gamma(\ker f_i) \), \( U, V \in \Gamma(S(f)) \), \( N \in \Gamma(\ker f_i) \), \( F \in \Gamma(\ker f_i) \).

**Proof.** Here, we will prove only second equation. For any vector fields \( U \in \Gamma(S(f)) \), and \( N \in \Gamma(\ker f_i) \), we can establish the equation \( \nabla_U N = v \nabla_U N + h \nabla_U N. \) Using Lemma 5.15, we can establish \( \nabla_U N = T_U N + h \nabla_U N. \) Other equations can be obtained in a similar way.

**Lemma 5.23** Let \((M_1, g_1), (M_2, g_2)\) be semi-Riemannian manifolds, \( \mathcal{V} \) be Levi-Civita connection in \( M_1, T \) and \( A \) be tensor fields, \( f: (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion. In this case, we obtain the following equations:

\[ i. \nabla_X W = A_X W + v \nabla_X W \]

\[ ii. \nabla_X \xi = A_X \xi + v \nabla_X \xi \]

\[ iii. \nabla_N W = A_N W + v \nabla_N W \]

\[ iv. \nabla_N U = A_N U + v \nabla_N U \]

\[ v. \nabla_N \xi = A_N \xi + v \nabla_N \xi \]

\[ vi. \nabla_F W = A_F W + v \nabla_F W \]

\[ vii. \nabla_F U = A_F U + v \nabla_F U \]
Similarly, $
abla_{v} \xi = A_{v} \xi + v \nabla_{v} \xi$

where $U \in \Gamma(S(\ker f))$, $\xi \in \Gamma(\Delta)$, $W \in \Gamma(\ker f)$, $X \in \Gamma(S(\ker f)^{-1})$, $N \in \Gamma(\text{tr}(\ker f))$, $F \in \Gamma(\text{tr}(\ker f))$.

**Proof.** Here we will prove only fourth equation. For any vector fields $N \in \Gamma(\text{tr}(\ker f))$ and $U \in \Gamma(S(\ker f))$, we can establish the equation $\nabla_{v} U = v \nabla_{v} U + h \nabla_{v} U$. Using Lemma 5.17, we can establish the equation $\nabla_{v} U = A_{v} U + v \nabla_{v} U$. Other equations can be obtained in a similar way.

**Lemma 5.24** Let $(M_{1}, g_{1}), (M_{2}, g_{2})$ be semi-Riemannian manifolds, $\nabla$ be Levi-Civita connection in $M_{1}$, $T$ and $A$ be basic tensor fields, $f: (M_{1}, g_{1}) \to (M_{2}, g_{2})$ be a transversal $r$ - lightlike submersion. In this instance, we obtain the following equations:

- $\nabla_{X} N = A_{X} N + h \nabla_{X} N$ \hspace{1cm} (5.24)
- $\nabla_{N_{1}} N_{2} = A_{N_{1}} N_{2} + h \nabla_{N_{1}} N_{2}$ \hspace{1cm} (5.25)
- $\nabla_{F_{1}} F_{2} = A_{F_{1}} F_{2} + h \nabla_{F_{1}} F_{2}$ \hspace{1cm} (5.26)
- $\nabla_{X} F = A_{X} F + h \nabla_{X} F$ \hspace{1cm} (5.27)
- $\nabla_{N} F = A_{N} F + h \nabla_{N} F$ \hspace{1cm} (5.28)
- $\nabla_{F} N = A_{F} N + h \nabla_{F} N$ \hspace{1cm} (5.29)
- $\nabla_{X} Y = A_{X} Y + h \nabla_{X} Y$ \hspace{1cm} (5.30)

where $X, Y \in \Gamma(S(\ker f)^{-1})$, $N_{1}, N_{2} \in \Gamma(\text{tr}(\ker f))$, $F_{1}, F_{2} \in \Gamma(\text{tr}(\ker f))$.

**Proof.** Here we will prove only first equation. For any vector fields $X \in \Gamma(S(\ker f)^{-1})$ and $N \in \Gamma(\text{tr}(\ker f))$, we can establish the equation $\nabla_{X} N = v \nabla_{X} N + h \nabla_{X} N$. Using Lemma 5.19, we can establish $\nabla_{X} N = A_{X} N + h \nabla_{X} N$. Other equations can be obtained in a similar way.

**Corollary 5.25.** Let $f: (M_{1}, g_{1}) \to (M_{2}, g_{2})$ be a transversal $r$ - lightlike submersion between semi-Riemannian manifolds $(M_{1}, g_{1})$ and $(M_{2}, g_{2})$. Then.

- $g_{1}(T_{U} N_{1}, N_{2}) = -g_{1}(N_{1}, T_{U} N_{2})$
- $g_{1}(T_{\xi} N_{1}, N_{2}) = -g_{1}(N_{1}, T_{\xi} N_{2})$
- $g_{1}(A_{N} \xi_{1}, \xi_{2}) = -g_{1}(\xi_{1}, A_{N} \xi_{2})$
- $g_{1}(A_{X} N_{1}, N_{2}) = -g_{1}(N_{1}, A_{X} N_{2})$

where $U \in \Gamma(S(\ker f))$, $\xi_{1}, \xi_{2} \in \Gamma(\Delta)$, $X \in \Gamma(S(\ker f)^{-1})$, $N_{1}, N_{2} \in \Gamma(\text{tr}(\ker f))$.

**Proof.** Provide the proof solely for the first equality.

Since $(M_{1}, g_{1})$ is a semi-Riemannian manifold the torsion-free metric connection used here is the Levi-Civita connection. For $U \in \Gamma(S(\ker f))$, $N_{1}, N_{2} \in \Gamma(\text{tr}(\ker f))$, we have

\[ U g_{1}(N_{1}, N_{2}) = g_{1}(\nabla_{U} N_{1}, N_{2}) + g_{1}(N_{1}, \nabla_{U} N_{2}) \]
\[ g_{1}(U \nabla_{U} N_{1}, N_{2}) = -g_{1}(N_{1}, U \nabla_{U} N_{2}) \]

By using Lemma 5.15 and Lemma 5.22, we can derive the expression $g_{1}(T_{U} N_{1}, N_{2}) = -g_{1}(N_{1}, T_{U} N_{2})$. Similarly, by using Lemma 5.23 and Lemma 5.24, we can obtain other equations.
Theorem 5.26. Let \( f : (M_1, g_1) \rightarrow (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, \( \ker f \) is integrable for \( W_1, W_2 \in \Gamma(\ker f) \).

Proof. Since \( W_{1p}, W_{2p} \) are elements of \( \Gamma(\ker f) \), we have \( f_*(W_1) = \overline{W}_1 = 0 \) and \( f_*(W_2) = \overline{W}_2 = 0 \). From equation in Definition 20 [1], we obtain \([\overline{W}_1, \overline{W}_2]g_2 = f_*(\{W_1, W_2\}) \circ g_2\). Therefore, \([W_1, W_2] \) belongs to \( \Gamma(\ker f) \), then \( \ker f \) is integrable. □

Remark 5.27 Let \( f : (\overline{M}_1, \overline{g}_1) \rightarrow (\overline{M}_2, \overline{g}_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((\overline{M}_1, \overline{g}_1)\) and \((\overline{M}_2, \overline{g}_2)\), where \( \overline{V} \) is the Levi-Civita connection corresponding to \( \overline{g}_1 \) on the manifold \( M_1 \). \( S(\ker f) \) and \( \text{tr}(\ker f) \) denote the corresponding screen distribution and transversal lightlike vector bundle of \( M_1 \), respectively. By utilizing the expression \( T\overline{M}_1 = \ker f \oplus \text{tr}(\ker f) \), we can derive \( \overline{V}_0V = v\overline{V}_0V + h\overline{V}_0V \), where \( U, V \in \Gamma(\ker f) \). Furthermore, using (5.4), we obtain \( \overline{V}_0V = \overline{V}_0V + \overline{T}_0V \), where \( \overline{T}_0V \) is associated with \( \Gamma(\text{tr}(\ker f)) \) and \( \overline{V}_0V \) is associated with \( \Gamma(\ker f) \).

Remark 5.28 Let \( P \) denote the projection morphism of \( (\ker f) \) onto \( S(\ker f) \) based on the decomposition of \( V \). By utilizing the equation \( \overline{V}_0P\overline{V} = v\overline{V}_0P\overline{V} + h\overline{V}_0P\overline{V} \), we get the following equation:

\[
\overline{V}_0P\overline{V} = \overline{V}_0P \overline{V} + \overline{T}_0P \overline{V}
\]

(5.31)

where \( \overline{T}_0P \overline{V} \) is associated with \( \Gamma(D) \), while \( \overline{V}_0P \overline{V} \) is associated with \( \Gamma(\ker f) \).

Theorem 5.29 Let \( f : (\overline{M}_1, \overline{g}_1) \rightarrow (\overline{M}_2, \overline{g}_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((\overline{M}_1, \overline{g}_1)\) and \((\overline{M}_2, \overline{g}_2)\). In this case, the necessary and sufficient condition for the integrability \( S(\ker f) \) is that \( \overline{T}_0P \overline{V} = \overline{T}_VP \overline{V} \).

Proof. Let \( U, V \in \Gamma(\ker f) \), \( N \in \Gamma(\text{tr}(\ker f)) \). Since \( \overline{V} \) is torsion-free, we have \( \overline{g}_1([U, V], N) = \overline{g}_1(\overline{V}_0U, N) - \overline{g}_1(\overline{V}_VU, N) \). From (5.31), it is easy to see that \( \overline{g}_1([U, V], N) = \overline{g}_1(\overline{V}_0P \overline{V} + \overline{T}_0P \overline{V}, N) - \overline{g}_1(\overline{V}_VP \overline{V} + \overline{T}_VP \overline{V}, N) \). Then, since \( \overline{g}_1(\overline{V}_0P \overline{V}, N) = 0 \) and \( \overline{g}_1(\overline{V}_VP \overline{V}, N) = 0 \), we have

\[
\overline{g}_1([U, V], N) = \overline{g}_1(\overline{T}_0P \overline{V}, N) - \overline{g}_1(\overline{T}_VP \overline{V}, N)
\]

□

Theorem 5.30 Let \( f : (M_1, g_1) \rightarrow (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, for \( W_1, W_2 \in \Gamma(\ker f) \), we have,

\[
T_{W_1}W_2 = T_{W_2}W_1
\]

Proof. Utilizing equation (5.11) for \( W_1, W_2 \in \Gamma(\ker f) \), we observe \( T_{W_1}W_2 - T_{W_2}W_1 = h[W_1, W_2] \). By employing Theorem 5.26, as \([W_1, W_2] \in \Gamma(\ker f) \), we deduce \( h[W_1, W_2] = 0 \). Consequently, this completes proof.

Theorem 5.31 Let \( f : (M_1, g_1) \rightarrow (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, if \( S(\ker f)^{-1} \) is integrable, we obtain \( A_x^Y = A_y^X \) for \( X, Y \in \Gamma(S(\ker f)^{-1}) \). Conversely, if \( A_x^Y = A_y^X \), we have \( [X, Y] \in \mathcal{H} \).

Proof. We will prove this theorem by considering two situations together.

i. Since \( \overline{V} \) is the Levi-Civita connection, the following equation holds:
\[ g_1([X,Y], U) = g_1(\nabla_X Y, U) - g_1(\nabla_Y X, U) \]

where \( X, Y \in \Gamma(S(\ker f_r))^\perp \), \( U \in \Gamma(S(\ker f_r)) \). By using (5.30), we can derive the following expression.

\[ g_1([X,Y], U) = g_1(A_X Y, U) - g_1(A_Y X, U) \]

If \( S(\ker f_r)^\perp \) is integrable, we can further simplify the equation and obtain:

\[ g_1(A_X Y, U) = g_1(A_Y X, U) \] (5.32)

\( ii. \) From the equation \( g_1([X,Y], N) = g_1(\nabla_X Y, N) - g_1(\nabla_Y X, N) \) and the integrability of \( S(\ker f_r)^\perp \), we obtain the following equation:

\[ g_1(A_X Y, N) = g_1(A_Y X, N) \] (5.33)

If we consider (5.32) and (5.33) together, it follows that \( A_X Y = A_Y X \). We can easily show that if \( A_X Y = A_Y X \), then \([X,Y] \in \mathcal{H}\).

We also note that \( A \) has the alternation property \( A_X Y = -A_Y X \) for a Riemannian submersion. However, this situation differs for transversal \( r \)-lightlike submersions.

**Theorem 5.32** Let \( f : (M_1, g_1) \to (M_2, g_2) \) be a transversal \( r \)-lightlike submersion between semi-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). In this case, if the \( \text{ltr}(\ker f_r) \) distribution is parallel in the direction of the \( S(\ker f_r)^\perp \) distribution, we obtain the equality \( A_X Y = -A_Y X \), where \( X, Y \in \Gamma(S(\ker f_r)^\perp) \), \( N \in \Gamma(\text{ltr}(\ker f_r)) \).

**Proof.** We first establish \( A_X Y = 0 \) for any \( X \in \Gamma(S(\ker f_r)^\perp) \). Let \( X, Y \in \Gamma(S(\ker f_r)^\perp) \), then we derive \( V g_1(X,X) = 2g_1(\nabla_X X, X) \), where \( V \in \Gamma(S(\ker f_r)) \). By utilizing Remark 5.11, we then have \( 2g_1(\nabla_X X, X) = 2g_1(\nabla_X V, X) \). Subsequently, it becomes apparent that \( 2g_1(\nabla_X X, N) = -2g_1(\nabla_X V, V) \). Furthermore, in accordance with (5.30), we conclude that

\[ 2g_1(\nabla_X V, X) = -2g_1(A_X X, V) \] (5.34)

On the other hand, since \( M_1 \) is a semi-Riemannian manifold, \( g_1(X,X) \) is constant on each fiber, and thus \( V g_1(X,X) = 0 \). From this, we conclude that \( g_1(A_X X, V) = 0 \). However, the condition for the result to be zero relies on two possibilities: either \( A_X X \in \Gamma(\mathcal{D}) \) or \( A_X X = 0 \). It can be observed that if \( A_X X = 0 \), then \( A_Y Y = -A_Y X \). If we consider the expression \( g_1(A_X X, N) \) for \( N \in \Gamma(\text{ltr}(\ker f_r)) \), using (5.12), we obtain

\[ g_1(A_X X, N) = g_1(\nabla_X X, N) = X g_1(X, N) - g_1(\nabla_X N, X) \]

Then, if the \( \text{ltr}(\ker f_r) \) distribution for \( N \in \Gamma(\text{ltr}(\ker f_r)) \) is parallel in the direction of the \( S(\ker f_r)^\perp \) distribution, then \( \nabla_X N \in \Gamma(\text{ltr}(\ker f_r)) \), we have

\[ g_1(A_X X, N) = 0 \] (5.35)

Thus, from (5.34) and (5.35), we obtain the expression

\[ A_X Y = -A_Y X \]

**6. Conclusion**

In this study, we introduced the concept of transversal lightlike submersions, and to illustrate the existence of such a structure, we offer illustrative examples. Our research delves into significant geometric analyses by...
examining O’Neill tensors for submersion, which we have defined as transversal lightlike submersions. In this way, various connections were obtained according to vector fields selected from certain fibers by utilizing these tensor fields, and meaningful results were obtained by investigating the integrability of certain distributions.

Thus, meaningful outcomes can be derived by computing various curvatures on the structure established for transversal submersions. Moreover, examining these submersions from two perspectives, transversal r-lightlike and isotropic transversal lightlike submersions, facilitates a geometric comparison. These investigations offer valuable insights into the intrinsic geometric properties of such mappings, potentially paving the way for new avenues of research.

**Author Contributions**

All the authors equally contributed to this work. They all read and approved the final version of the paper.

**Conflicts of Interest**

All the authors declare no conflict of interest.

**References**


