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Research Article

Some Numerical Techniques for Solution of Nonlinear Regularized Long Wave Equation

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ABSTRACT

In this study, numerical solutions of the one-dimensional Regularized Long Wave (RLW) equation have been investigated. For this purpose, the RLW equation is divided into two sub equations, one linear and the other nonlinear, according to the time term. Then, algebraic equation systems have been obtained by writing the derivative approximations obtained with the help of cubic trigonometric B-spline base functions and Crank-Nicolson finite difference approximations to the derivatives in each sub-equation. To obtain numerical solutions of the RLW equation, these systems are solved the Strang splitting algorithm, Ext4, and Ext6 techniques created by Richardson extrapolation of the Strang algorithm have used to increase the accuracy of the solutions. In order to investigate the effectiveness of these methods, single solitary wave motion and the interaction of two solitary waves problems, which are most commonly used in the literature, have been taken into consideration. In addition, the stability analysis of the Strang algorithm have been investigated by the von Neumann method.

1. INTRODUCTION

In science, nonlinear partial differential equations often represent wave events that are motivated by certain physical initial/boundary conditions [1]. In recent times, solitary waves, especially soliton waves, have become both experimental and theoretically very interesting and outstanding. A soliton is a very special type of solitary wave, which has a continuous form, can be placed in a region and interaction with another soliton, and can be separated unchanged without a change of phase [11]. In this paper we examine the RLW equation in the following form as

$$U_t + U_x + \varepsilon U U_x - \mu U_{xxt} = 0, \qquad (1)$$

$U \to 0$ when $x \to \pm\infty$. Where t is the time, x is the position coordinate, $U(x,t)$ is the wave height (amplitude), and " and µ are the positive parameters.

The RLW equation was first appeared when calculating the development of the " undular bore" problem by Peregrine [2]. The RLW is a nonlinear dispersive wave equation which is a more conventional than the KdV equation in observing the wave phenomena. This equation is most commonly used in order to model physical phenomena such as pressure waves in liquid-gas bubble mixtures, longitudinal dispersive waves in elastic rods, ion-acoustic waves in plasma, thermally excited phonon packets in low temperature nonlinear crystals, shallow water waves and plasma waves [2,31].

Several researchers have solved the RLW equation using various methods and techniques. Among others, Rasoulizadeh et al. [3] developed a method for the numerical solution of the RLW equation by means of an implicit method based on the -weighted and finite difference methods. Oruç et al. [4] obtained the numerical solutions of the RLW equation using Strang splitting approach combined with Chebyshev wavelets. Irk et al. [5] used quartic trigonometric B-spline finite element method to solve the RLW equation numerically. Yağmurlu and Karakaş in the papers [6] and [7] applied the trigonometric cubic B-spline collocation method to get numerical solution of the equal width equation and modified equal width wave equation. Kutluay et al. [8] used cubic hermite B-spline collocation method to solve modified equal width wave equation. Kutluay et al. [9] utilized a robust quantic hermite collocation method for the numerical solution of the one dimensional heat equation. for Dağ et al. [10] applied a collocation method based on the trigonometric cubic B-spline function to obtain numerical solutions of the RLW equation. Kutluay [40] and Esen [25] solved the equation using both finite difference and finite elements method. Mei and Chen [30] used explicit multistep method. Gardner et al. [29] solved the regularized long wave equation numerically by Galerkin method with quadratic B-spline finite elements. Dağ et al. [31] obtained numerical solution of the RLW equation using a splitting up technique and both quadratic and cubic B-splines. Saka and Dağ [32] developed a new algorithm based on the collocation method using splitting. Oruç et al. [38] utilized Haar wavelet method and Islam and et al. [41] presented a meshfree technique using the radial basis functions (RBFs) in order to obtain the numerical solutions of the equation.

Moreover; Dağ et al. in the papers [23] and [28] developed cubic B-spline collocation and quintic B-Spline Galerkin finite element methods for obtaining numerical solutions of the present equation. Besides, while Zaki [26] applied a combination of the splitting method and cubic B-spline finite elements, and Raslan [39] used cubic B-spline collocation method for approximate solutions of the equation.

2. Formulation of Splitting Methods

One way of dealing with complex problems is" divide and conquer". In the context of evolution type equations, the operator splitting idea has been a very successful approach. The underlying idea behind such an approach is that all model evolution operators are formally written as the sum of the evolution operators for each term that is being modeled. In other words, when one splits the model into a series of sub-equations, simpler and more practical algorithms for each sub-equation occur. Then the applied numerical method is applied to each sub-problem and numerical schemes are obtained and these schemes are combined by operator splitting [13]. We are going to dwell on the situation in which we have the following the Cauchy problem

$$\frac{dU(t)}{dt} = AU(t) + BU(t), t \in [0,T], U(0) = U_0. \quad (2)$$

Where, an initial function U0 ∈ X is given, A and B are assumed to be a bounded linear operator in the Banach space X together with $A, B : X \to X$. There is also a norm associated with the space X denoted by $\|.\|_X$, and if both A and B are matrixes, then it is called Euclidean norm [14].

2.1. Strang splitting algorithm

In splitting methods, the given equation is generally divided into several parts and each part is solved separately, independently of the main equation, over $[t_n, t_{n+1}]$ time intervals. Such methods are generally called time splitting or fractional step methods [15]. Strang [12] has proposed a symmetrizing splitting scheme

$$\begin{aligned} \frac{dU^*}{dt} &= AU^*, U^*(0) = U_0, on \left[0, \frac{\Delta t}{2}\right] \\ \frac{dU^{**}}{dt} &= BU^{**}, U^{**}(0) = U^*\left(\frac{\Delta t}{2}\right), on\ [0, \Delta t] \quad (3) \\ \frac{dU^{***}}{dt} &= A, U^{***}(0) = U^{**}(\Delta t), on \left[0, \frac{\Delta t}{2}\right] \end{aligned}$$

where the final values are obtained by $U^{***}(\Delta t/2)$. This scheme is called $(A - B - A)$ and the scheme $(B - A - B)$ can be derived in a similar manner. This scheme has a local splitting error

$$le = \left[e^{A\frac{\Delta t}{2}} e^{B\Delta t} e^{A\frac{\Delta t}{2}} - e^{(A+B)\Delta t}\right] \quad (4)$$

$$= O(\Delta t) \quad (5)$$

is a second-order scheme and is used in practice for many applications [20].

2.2. Construction of Ext4 and Ext6 algorithms

In extrapolation techniques, a simple low-order method (basic method) is applied for different time steps t. Then, higher order methods are obtained by taking an appropriate combination of the results [21]. Now let's explain how to obtain a higher order R method. To do this, let $R(t)$ be an approximation to R with step length t when $t \to 0, R(t) \to R$ and let's assume

$$R(\Delta t) = R + a_2\Delta t^2 + a_4\Delta t^4 + a_6\Delta t^6 + \cdots \quad (6)$$

If the time step t is divided into k substeps then the basic method is applied k times [22, 16], i.e.

$$\left(R\left(\frac{\Delta t}{k}\right)\right)^k = R\left(\frac{\Delta t}{k}\right) \circ R\left(\frac{\Delta t}{k}\right) \circ R\left(\frac{\Delta t}{k}\right) \ldots \circ R\left(\frac{\Delta t}{k}\right).$$

In this study, the Strang method is used as the basic method to obtain ***Ext4*** and ***Ext6*** techniques. To obtain the ***Ext4*** method, the expressions $R(t)$ and $R\left(\frac{\Delta t}{2}\right)$ are used in (6)

$$R(\Delta t) = R + a_2\Delta t^2 \quad (7)$$

$$R\left(\frac{\Delta t}{2}\right) = R + a_2\frac{\Delta t^2}{4}. \quad (8)$$

If a_2 is eliminated in the equations (7) and (8) and necessary operations performed for R

$$R = \frac{4}{3}\left(R\left(\frac{\Delta t}{k}\right)\right)^2 - \frac{1}{3}R(\Delta t) + O(\Delta t^4) \quad (9)$$

If the approximation (9) is applied to Strang's algorithm $S_{\Delta t} = e^{A\frac{\Delta t}{2}} e^{B\Delta t} e^{A\frac{\Delta t}{2}}$, **Ext4** algorithm is obtained as follows

$$\boldsymbol{Ext4} = \frac{4}{3}\left(S_{\frac{\Delta t}{2}}\right)^2 - \frac{1}{3}S_{\Delta t}$$

$$= \frac{4}{3}\varphi^{[A]}_{\frac{\Delta t}{4}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{4}} - \frac{1}{3}\frac{4}{3}\varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\Delta t} \circ \varphi^{[A]}_{\frac{\Delta t}{2}},$$

where $\varphi_{\Delta t}$ is a numerical method. Similarly, the following equations are used to obtain the **Ext6** method

$$R(\Delta t) = R + a_2\Delta t^2 + a_4\Delta t^4 \quad (10)$$

$$R\left(\frac{\Delta t}{2}\right) = R + a_2\frac{\Delta t^2}{4} + a_4\frac{\Delta t^4}{4} \quad (11)$$

$$R\left(\frac{\Delta t}{3}\right) = R + a_2\frac{\Delta t^2}{9} + a_4\frac{\Delta t^4}{81} \quad (12)$$

If the necessary operations are performed for R after a_2 and a_4 are eliminated in the equations (10), (11), (12)

$$R = \frac{81}{40}R\left(\frac{\Delta t}{3}\right)^3 - \frac{16}{15}R\left(\frac{\Delta t}{2}\right)^2 + \frac{1}{24}R(\Delta t) + O(\Delta t^6). \quad (13)$$

If the approximation (13) is applied to Strang's algorithm $S_{\Delta t} = e^{A\frac{\Delta t}{2}} e^{B\Delta t} e^{A\frac{\Delta t}{2}}$, **Ext6** algorithm is obtained as follows

$$Ext6 = \frac{81}{40}\left(S_{\frac{\Delta t}{3}}\right)^3 - \frac{16}{15}\left(S_{\frac{\Delta t}{2}}\right)^2 + \frac{1}{24}S_{\Delta t}$$

$$= \frac{81}{40}\varphi^{[A]}_{\frac{\Delta t}{6}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \circ \varphi^{[A]}_{\frac{\Delta t}{3}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \circ \varphi^{[A]}_{\frac{\Delta t}{3}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \circ \varphi^{[A]}_{\frac{\Delta t}{6}}$$

$$-\frac{16}{15}\varphi^{[A]}_{\frac{\Delta t}{4}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{4}} + \frac{1}{24}\varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\Delta t} \circ \varphi^{[A]}_{\frac{\Delta t}{2}}.$$

Since these methods contain negative coefficients, it is not known exactly they are stable for what kind of problems. However, it has been shown by Dia and Schatzman [16] that the **Ext4** technique is stable with dimensional splitting for linear parabolic problems at finite time intervals.

3. Method of Solution

To examine the numerical behavior of the RLW equation (1), the solution domain is constrained on a closed interval $[a, b]$. The homogenous boundary conditions

$$U(a,t)=0, U(b,t)=0, t\geq 0 \quad (14)$$
$$U_x(a,t)=0, U_x(b,t)=0$$

and the initial condition

$$U(x,0)=f(x), a\leq x\leq b$$

are taken as stated above, and $f(x)$ is a predefined function. Let us assume that the space solution domain is $[a,b]$ and a uniform discretization of this domain by the nodal points x_m, $m=0,1,\dots,N$, is given by $a=x_0<x_1<\dots<x_N=b$. Dividing the solution region into elements of equal length ensures that the calculated error norms are smaller. If we define the distance between two consecutive points as $h=x_{m+1}-x_m$ and $T_m(x)$, $m=-1(1)N+1$, then trigonometric cubic B-spline functions on the domain $[a,b]$ can be expressed in terms of nodal points xm as follows

$$T_m(x)=\frac{1}{r}\begin{cases} p^3(x_{m-2}), & x\in[x_{m-2},x_{m-1}] \\ p(x_{m-2})\big(p(x_{m-2})q(x_m)+q(x_{m+1})p(x_{m-1})\big) & \\ +q(x_{m+2})p^2(x_{m-1}), & x\in[x_{m-1},x_m] \\ q(x_{m+2})\big(p(x_{m-1})q(x_{m+1})+q(x_{m+2})p(x_m)\big) & \\ +p(x_{m-2})q^2(x_{m+1}), & x\in[x_m,x_{m+1}] \\ q^3(x_{m+2}), & x\in[x_{m+1},x_{m+2}] \\ 0, otherwise & \end{cases} \quad (15)$$

as stated by [17, 18]. Where $p(x_m)=\sin(\frac{x-x_m}{2})$, $q(x_m)=\sin(\frac{x_m-x}{2})$ and $r=sin(h/2)sin(h)\,sin(3h/2)$. It is obvious that the set $\{T_{-1}(x),T_0(x),\dots,T_{N+1}(x)\}$ constitutes a base on the domain $[a,b]$. If we assume that the function $U(x,t)$ is defined on the domain [a, b], then the function $U(x,t)$ can be approximated as follows in terms of trigonometric cubic B-spline functions and time dependent parameters $\delta_m(t)$ as follows

$$U(x,t)\cong\sum_{m=-1}^{N+1}\delta_m(t)T_m(x). \quad (16)$$

Where time-dependent parameters $\delta_m(t)$ are going to be determined using the Eq. (1) and its auxiliary conditions. Since the Eq. (1) contains the terms U, U' and U'', we need the values of U, its first and second order derivatives in terms of trigonometric cubic B-spline functions. Using the approximations (15) and (16), the nodal values of U, U' and U'' are obtained in terms of the time-dependent parameters m(t) as follows

$$\begin{aligned} U_m&=U(x_m)=\alpha_1\delta_{m-1}+\alpha_2\delta_m+\alpha_1\delta_{m+1}, \\ U'_m&=U'(x_m)=\beta_1\delta_{m+1}-\beta_1\delta_{m-1}, \quad (17)\\ U''_m&=U''(x_m)=\gamma_1\delta_{m-1}+\gamma_2\delta_m+\gamma_1\delta_{m+1}, \end{aligned}$$

with the coefficients

$$\alpha_1=\frac{sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, \alpha_2=\frac{2}{1+\cos(2h)},$$

$$\beta_1=-\frac{3}{4\sin\left(\frac{3h}{2}\right)},\ \gamma_2=\frac{3cos^2\left(\frac{h}{2}\right)}{sin^2\left(\frac{h}{2}\right)(2+4\cos(h))},$$

$$\gamma_1=\frac{3(1+3\cos(h))}{16sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}.$$

Where $'$ and $''$ denote the first and second order derivatives with respect to the space variable x, respectively.

The time split RLW equation is taken as follows

$$U_t-\mu U_{xxt}+U_x=0 \quad (18)$$
$$U_t-\mu U_{xxt}+\varepsilon UU_x=0. \quad (19)$$

If the values of U_m, U'_m and U''_m at nodal points x_m are used in (18) and (19) and basic necessary operations are performed, we obtain the following first order ordinary differential equation systems

$$\begin{aligned} &\alpha_1\dot{\delta}_{m-1}+\alpha_2\dot{\delta}_m+\alpha_1\dot{\delta}_{m+1}-\mu(\gamma_1\dot{\delta}_{m-1}+\gamma_2\dot{\delta}_m+\gamma_1\dot{\delta}_{m+1}) \\ &\quad+\beta_1(\delta_{m+1}-\delta_{m-1})=0, \quad (20)\\ &\alpha_1\dot{\delta}_{m-1}+\alpha_2\dot{\delta}_m+\alpha_1\dot{\delta}_{m+1}-\mu(\gamma_1\dot{\delta}_{m-1}+\gamma_2\dot{\delta}_m+\gamma_1\dot{\delta}_{m+1}) \\ &\quad+\varepsilon z_m\beta_1(\delta_{m+1}-\delta_{m-1})=0, \quad (21) \end{aligned}$$

where $\cdot$ denotes derivation with respect to t and the value of z_m is taken as follows for linearization process

$$z_m=\alpha_1\delta_{m-1}+\alpha_2\delta_m+\alpha_1\delta_{m+1}.$$

Instead of the parameter δ_m, $\frac{\delta_m^{n+1}+\delta_m^n}{2}$ is written and instead of time-varying parameters $\dot{\delta}_m$, $\frac{\delta_m^{n+1}-\delta_m^n}{\Delta t}$ is written in Eqs. (20) and (21), the following equations

$$\begin{aligned} &a_1\delta_{m-1}+b_1\delta_m+c_1\delta_{m+1} \\ &\qquad=c_1\delta_{m-1}+b_1\delta_m+a_1\delta_{m+1}, \quad (22)\\ &a_2\delta_{m-1}+b_2\delta_m+c_2\delta_{m+1} \\ &\qquad=c_2\delta_{m-1}+b_2\delta_m+a_2\delta_{m+1} \quad (23) \end{aligned}$$

$$a_1=\alpha_1-\mu\gamma_1-\frac{\beta_1\Delta t}{2}, b_1=\alpha_2-\mu\gamma_2,$$

$$c_1=\alpha_1-\mu\gamma_1+\frac{\beta_1\Delta t}{2}, a_2=\alpha_1-\mu\gamma_1-\frac{\varepsilon z_m\beta_1\Delta t}{2},$$

$$b_2=\alpha_2-\mu\gamma_2, c_2=\alpha_1-\mu\gamma_1+\frac{\varepsilon z_m\beta_1\Delta t}{2}$$

are obtained. The equations given in (22) and (23) consist of $(N+1)$ equations and $(N+3)$ unknown j parameters. Using the boundary conditions $U(a,t)=0$ and $U(b,t)=0$, we obtain the following equalities for parameters δ_{-1} and δ_{N+1}

$$\delta_{-1}=\frac{\alpha_2}{\alpha_1}\delta_0-\delta_1, \delta_{N+1}=-\delta_{N-1}-\frac{\alpha_2}{\alpha_1}\delta_N. \quad (24)$$

If the parameters δ_{-1} and δ_{N+1} are eliminated from systems (22) and (23) using identities (24), $(N+1)\times(N+1)$ dimensional tridiagonal band matrix systems are obtained. A unique solution of these systems can be obtained using the Thomas algorithm. In order to solve these systems, it is necessary to use $\delta^0{}_m$ initial parameters in (22) and (23) after the initial parameters $U(x,0)=f(x)$ are obtained. If we call (22) and (23) systems A and B respectively, then the results will be obtained using the splitting scheme $(A-B-A)$ as stated in (3).

3.1. Initial Condition

The initial vector $\delta^0{}_m$ will be formed using the initial condition $U(x,0)=f(x)$ as follows

$$
\begin{gathered}
U(x_i,0) = U_N(x_i,0), i = 0(1)N \\
U_m = \alpha_1 \delta^0{}_{m-1} + \alpha_2 \delta^0{}_m + \alpha_1 \delta^0{}_{m+1}, m = 0(1N) \\
U_0 = \alpha_1 \delta^0{}_{-1} + \alpha_2 \delta^0{}_0 + \alpha_1 \delta^0{}_1 \\
U_1 = \alpha_1 \delta^0{}_0 + \alpha_2 \delta^0{}_1 + \alpha_1 \delta^0{}_2 \\
\vdots \\
U_N = \alpha_1 \delta^0{}_{N-1} + \alpha_2 \delta^0{}_N + \alpha_1 \delta^0{}_{N+1}.
\end{gathered} \tag{25}
$$

This system consists of $(N+1)$ equations and $(N+3)$ unknown $\delta^0{}_m$ parameters. The parameters $\delta^0{}_{-1}$ and $\delta^0{}_{N+1}$ are calculated from (25) using the boundary conditions $U'_N(a,0) = 0$ and $U'_N(b,0) = 0$

$$\beta_1 \delta^0{}_1 - \beta_1 \delta^0{}_{-1} = 0, \beta_1 \delta^0{}_{N+1} - \beta_1 \delta^0{}_{N-1} = 0.$$

Now, newly obtained $(N+1) \times (N+1)$ dimensional solvable matrix is obtained for $\delta^0{}_m$ parameters.

$$
\begin{bmatrix}
\alpha_2 & 2\alpha_1 & 0 & & \\
\alpha_1 & \alpha_2 & \alpha_1 & & \\
 & & \ddots & & \\
 & & \alpha_1 & \alpha_2 & \alpha_1 \\
 & & & 2\alpha_1 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
\delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0
\end{bmatrix}
=
\begin{bmatrix}
U_0 \\ U_1 \\ \vdots \\ U_{N-1} \\ U_N
\end{bmatrix}.
$$

3.1. Von Neumann Stability Analysis

The (22) and (23) numerical schemes have been considered by the Fourier von Neumann [19] method. In this method $\delta_m^n = \xi^n e^{i\beta mh}$ is taken, where $i = \sqrt{-1}$, β is mode number, ξ is amplification factor and h is the space step in the method. In Eq. (1) in the term uu_x, since we take $u = z_m$ for linearization purpose, it will behave as a local constant. Let us assume that the amplification factors related to the schemes in (22) and (23) be ρ_A and ρ_B, respectively. If we write $\delta_m^n = \xi_{A/2}^n e^{i\beta mh}$ in Eq.(22), we obtain

$$\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right) = \frac{X - iY}{X + iY},$$

$$X = b_1 + (c_1 + a_1)cos\beta h, Y = (c_1 - a_1)isin\beta h.$$

Thus, since $\left|\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right)\right| \leq 1$ is valid, the linearized scheme is unconditionally stable. In a similar way, if we take $\delta_m^n = \xi_B^n e^{i\beta mh}$ in Eq. (23), we obtain $\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right| \leq 1$. Since the scheme (3) is as follows

$$\rho_S(\xi) = \rho_A^{n+1/2} \rho_B^{n+1} \rho_A^{n+1/2},$$

$$|\rho_S(\xi)| \leq \left|\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right)\right| \left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right| \left|\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right)\right| \leq 1$$

the solution for Eq. (1) obtained using Strang splitting scheme is unconditionally stable.

4. Numerical Examples and Their Results

We have considered two test problems to observe the effectiveness of the method. The solution of each problem with cubic B-spline collocation method gives $(N+1) \times (N+1)$ tridiagonal band matrix systems which can be easily and effectively solved by Thomas algorithm. In order to see the difference between numerical solution and analytic solution, we have used the error norms defined as to measure the difference and thus to see how well the wave position and amplitude estimate of the method are

$$L_2 = \sqrt{h \sum_{j=1}^{N} \left[U_j^{exact} - U_j\right]^2}, L_\infty = \max_j \left|U_j^{exact} - U_j\right|.$$

The RLW equation given in (1) satisfies three invariants known as mass, momentum and energy given as follows

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} U dx \approx h \sum_{j=1}^{N} U_j^n, \\
I_2 &= \int_{-\infty}^{+\infty} [U^2 + \mu (U_x)^2] dx \approx h \sum_{j=1}^{N} \left[\left(U_j^n\right)^2 + \mu\left((U_x)_j^n\right)^2\right], \\
I_3 &= \int_{-\infty}^{+\infty} [U^3 + 3U^2] dx \approx h \sum_{j=1}^{N} \left[\left(U_j^n\right)^3 + 3\left(U_j^n\right)^2\right].
\end{aligned} \tag{26}
$$

4.1. Single Solitary Movement

Analytical solution for single soliton wave solution of RLW equation is

$$U(x,t) = 3c \sec h^2 [k(x - x_0 - vt)]$$

Where $k = \frac{1}{2}(\frac{\varepsilon c}{\mu(1+\varepsilon c)})$, $v = 1 + \varepsilon c$ is wave velocity and $3c$ is wave amplitude. The following initial

$$U(x,0) = 3c \sec h^2 [k(x - x_0)]$$

and the boundary conditions

$$U(a,t) = U(b,t) = 0$$

are used at the boundaries. Analytic values of the invariants for this problem are

$$I_1 = \frac{6c}{k}, I_1 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, I_3 = \frac{36c^2}{k}\left(1 + \frac{4c}{5}\right) \tag{27}$$

given by Zaki [26]. In order to be able to make a comparison with previous studies, all calculations were taken as $\varepsilon = 1, \mu = 1, x_0 = 0$ and $\Delta t = 0.1$. In Table 1, the values of the error norms L_2, L_∞ and the invariant values calculated with the $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques are given at different times in the region $-40 \leq x \leq 60$. As it is clearly seen from Table 1, the invariants remain almost the same as time progresses. Moreover, the error norms L_2, L_∞ calculated by $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ are smaller than those with $\boldsymbol{S_{\Delta t}}$. In Table 2, several comparisons have been made with some of those in the literature at time $t = 20$ for the values of $c = 0.03, h = 0.1, 0.125$. As can be clearly seen from Table 2, the results obtained with $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques are in good harmony with other results found in the literature. Figure 1 and 2 show 2 and 3 dimensional behavior of the numerical solution and the exact solution. As can be clearly seen from Figure 1 and 2, $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques preserve the physical structure of the problem quite well.

Table 1: A comparison of the invariants and error norms calculated at various times for values of $h = 0.1, 0.125, \Delta t = 0.1, c = 0.1, \varepsilon = \mu = 1$ in the region $-40 \leq x \leq 60$ for single solitary wave.

h	Time	Method	I_1	I_2	I_3	$L_2 \times 10$	$L_\infty \times 10$
	$t = 0$	$S_{\Delta t}$	3.979927	0.810462	2.579007	0.00000	0.00000
		Ext4	3.979927	0.810462	2.579007	0.00000	0.00000
		Ext6	3.979927	0.810462	2.579007	0.00000	0.00000
	$t = 4$	$S_{\Delta t}$	3.979953	0.810462	2.579007	0.29086	0.79795
		Ext4	3.979953	0.810463	2.579008	0.28673	0.78652
		Ext6	3.979999	0.810480	2.579066	0.28705	0.76386
	$t = 8$	$S_{\Delta t}$	3.979971	0.810462	2.579007	0.58210	1.62771
		Ext4	3.979971	0.810463	2.579008	0.57383	1.60407
0.125		***Ext6***	3.980062	0.810498	2.579124	0.57354	1.55669
	$t = 12$	$S_{\Delta t}$	3.979984	0.810462	2.579007	0.87407	2.47927
		Ext4	3.979984	0.810463	2.579008	0.86164	2.44339
		Ext6	3.980121	0.810516	2.579183	0.85984	2.36902
	$t = 16$	$S_{\Delta t}$	3.979986	0.810462	2.579007	1.16709	3.34710
		Ext4	3.979986	0.810463	2.579008	1.15046	3.29862
		Ext6	3.980169	0.810533	2.579241	1.14629	3.19580
	$t = 20$	$S_{\Delta t}$	3.979962	0.810463	2.579007	1.46140	4.22716
		Ext4	3.979962	0.810463	2.579008	1.44054	4.16731
		Ext6	3.980190	0.810551	2.579299	1.43318	4.03364
	$t = 4$	$S_{\Delta t}$	3.979954	0.810462	2.579007	0.18742	0.51431
		Ext4	3.979954	0.810463	2.579008	0.18665	0.51186
		Ext6	3.979954	0.810463	2.579008	0.18397	0.50423
	$t = 8$	$S_{\Delta t}$	3.979973	0.810462	2.579007	0.37508	1.04901
0.1		***Ext4***	3.979973	0.810463	2.579008	0.37353	1.04404
		Ext6	3.979973	0.810463	2.579008	0.36816	1.02848
	$t = 12$	$S_{\Delta t}$	3.979988	0.810462	2.579007	0.56319	1.59752
		Ext4	3.979988	0.810463	2.579008	0.56085	1.59032
		Ext6	3.979988	0.810463	2.579008	0.55278	1.56656
	$t = 16$	$S_{\Delta t}$	3.979993	0.810462	2.579007	0.75197	2.15660
		Ext4	3.979993	0.810463	2.579008	0.74883	2.14697
		Ext6	3.979993	0.810463	2.579008	0.73803	2.11515
	$t = 20$	$S_{\Delta t}$	3.979970	0.810462	2.579007	0.94157	2.72364
		Ext4	3.979970	0.810463	2.579008	0.93764	2.71180
		Ext6	3.979970	0.810463	2.579008	0.92410	2.67198

Table 2: A comparison of the invariants and error norms calculated at $t = 20$ for values of $h = 0.1, 0.125, \Delta t = 0.1, c = 0.03, \varepsilon = \mu = 1$ in the region $-40 \leq x \leq 60$ for single solitary wave.

h	Method	I_1	I_2	I_3	$L_2 \times 10$	$L_\infty \times 10$
	$S_{\Delta t}$	2.109485	0.12730	0.388807	0.67947641	0.248333660
	Ext4	2.109485	0.127303	0.388807	0.67959141	0.248317182
	Ext6	2.109480	0.127302	0.388805	0.67885613	0.248300644
	[24](SBCM1)	2.10904	0.12730	0.38881	0.556	0.419
0.1	[24](SBCM2)	2.10904	0.12730	0.38881	0.556	0.419
	[29]	2.1050	0.12730	0.38880	0.563	0.432
	[32]	2.10948	0.12730	0.38880	0.651	0.432
	[26]	2.10760	0.127302	0.38879	0.41652	0.23197
	$S_{\Delta t}$	2.109003	0.127302	0.388806	0.64278799	2.19097274
	Ext4	2.109003	0.127302	0.388806	0.64057300	2.19069041
	Ext6	2.108997	0.127302	0.388804	0.64176027	2.19050366
	[24](SBCM1)	2.10849	0.12730	0.38881	0.444	0.419
0.125	[24](SBCM2)	2.10849	0.12730	0.38881	0.444	0.419
	[31]	2.10471	0.12730	0.38880	0.538	0.198
	[32]	2.10902	0.12731	0.38881	0.547	0.432
	[26]	2.10741	0.12723	0.38856	0.242	0.125

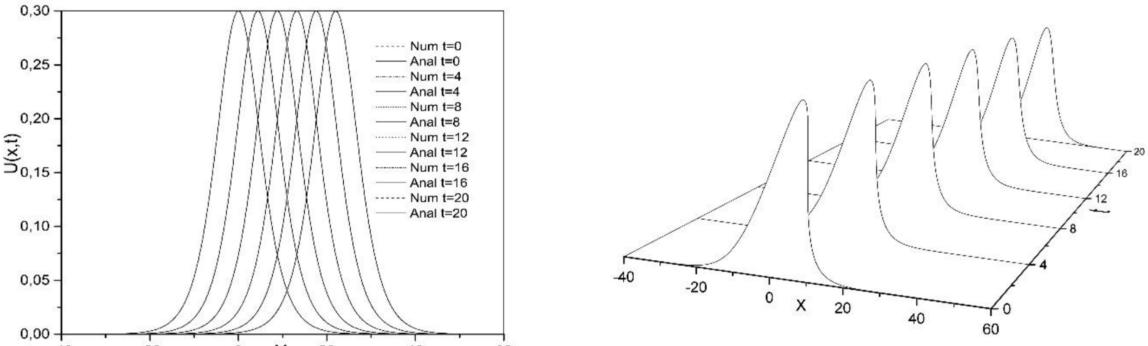

Figure 1. Physical behavior of the single solitary wave for c = 0.1.

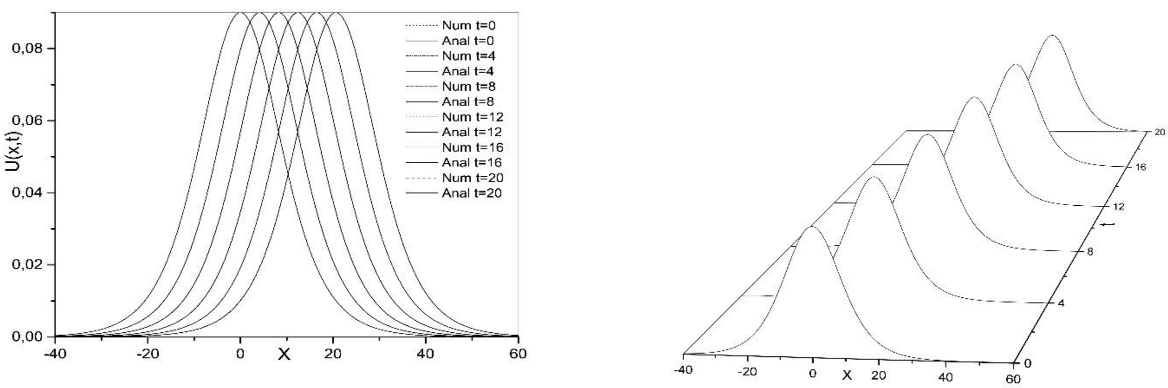

Figure 2. Physical behavior of the single solitary wave for c = 0.03.

Table 3: The invariants of the interaction problem and comparison with those in Ref. [40, 39]

method	t	0	4	8	16	20	25
$S_{\Delta t}$	I_1	37.91652	37.91702	37.91732	37.91764	37.91761	37.91769
	I_2	120.5228	120.5244	120.5081	120.4223	120.5126	120.5273
	I_3	744.0812	744.0866	744.0760	743.9454	744.0848	744.0999
***Ext*4**	I_1	37.91648	37.91697	37.91721	37.91740	37.91744	37.91747
	I_2	120.5234	120.5258	120.5117	120.4292	120.5298	120.5551
	I_3	744.0813	744.0921	744.1171	744.1573	744.2281	744.3226
***Ext*6**	I_1	37.91648	37.91719	37.91763	37.91800	37.91821	37.91851
	I_2	120.5234	120.5258	120.5101	120.4258	120.5159	120.5301
	I_3	744.0813	744.0920	744.1042	744.1249	744.1126	744.1141
[40]	I_1	37.91648	37.91697	37.91719	37.91740	37.91744	37.91745
	I_2	120.3515	120.3584	120.3570	120.3886	120.3599	120.3595
	I_3	744.0814	744.0110	743.8679	742.4889	743.8638	744.0085
[39]	I_1	37.91652	37.91170	37.85975	37.52916	37.64730	38.05010
	I_2	120.5228	121.1602	119.7317	119.4185	119.8041	119.8355
	I_3	744.0815	736.9443	728.5173	725.8399	727.1948	727.4392

4.2. Interaction of Two Solitary Waves

An interaction problem is that the boundary condition is taken as $U \to 0$ when $x \to \pm\infty$ and the initial condition is taken as follows

$$U(x,0) = \sum_{j=1}^{2} 3A_j \sec h^2\left[k_j(x - x_j)\right]$$

where $A_j = \frac{4k_j^2}{1-4k_j^2}, j = 1,2.$ In order to observe this interaction problem, we worked within the region $0 \leq x \leq 120$ and with the parameters $x_1 = 15, x_2 = 35, k = 0.4, k_2 = 0.3, \varepsilon = 1, \mu = 1, h = 0.3$ and $\Delta t = 0.1$. To observe the problem of the interaction, the small waves amplitude is calculated as 1.6869 at $t = 25$, $x = 70.2$ and the large waves amplitude is calculated as $5, 3456$ at $x = 87$. In Figure 3, the interaction and separation of two solitary waves is demonstrated using the numerical results calculated with the $\boldsymbol{Ext6}$ technique. As can

interaction of two solitary waves, the wave with the larger amplitude is placed at $x_1 = 15$ and the smaller one at $x_2 = 35$. Since the speed of the wave with the larger amplitude is also higher, it is observed that it catches the smaller wave and leaves it behind as time progresses. After the

be seen from Table 3, the invariants are calculated with $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques remain almost constant throughout the calculations and the results are in good agreement with those in Refs. [40, 39].

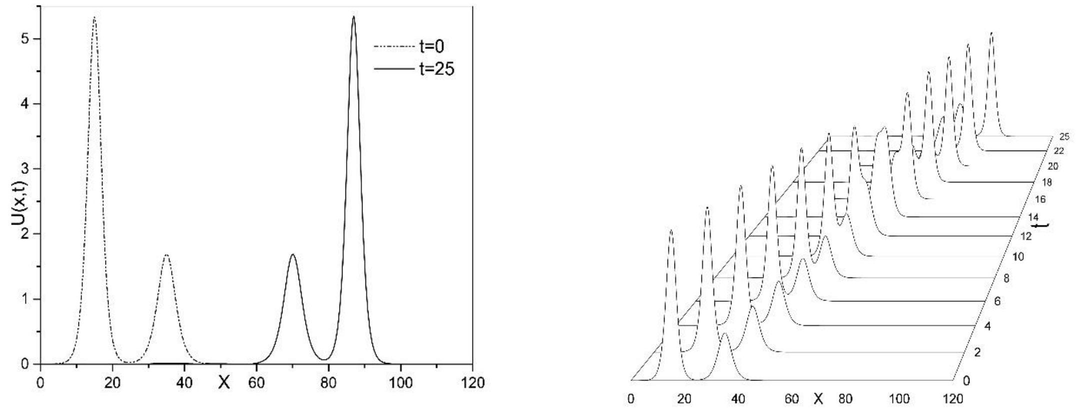

Figure 3. Physical behavior of the interaction of two solitary waves.

5. Conclusions

In this study, the trigonometric cubic B-spline collocation method is used with some splitting techniques for the numerical solutions of the regularized long wave equation. $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques have been constructed using the Strang splitting algorithm. The results obtained with all three techniques have compared with each other and with some studies in the literature. it has been seen that the numerical results calculated with $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques preserve the physical structure of the problems and the calculated error norms L_2, L_∞ are small enough. As a result, it can be said that $\boldsymbol{S_{\Delta t}}$, $\boldsymbol{Ext4}$ and $\boldsymbol{Ext6}$ techniques are effective techniques to improve the numerical results of partial differential equations

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