Communications in Advanced Mathematical Sciences
Vol. 7, No. 2, 80-87, 2024
Research Article
e-ISSN: 2651-4001
https://doi.org/10.33434/cams.1442975

On Some Properties of Banach Space-Valued Fibonacci Sequence Spaces

Yılmaz Yılmaz[1]*, Seçkin Yalçın[1]

Abstract
In this work, we give some results about the basic properties of the vector-valued Fibonacci sequence spaces. In general, sequence spaces with Banach space-valued cannot have a Schauder Basis unless the terms of the sequences are complex or real terms. Instead, we defined the concept of relative basis in [1] by generalizing the definition of a basis in Banach spaces. Using this definition, we have characterized certain important properties of vector-term Fibonacci sequence spaces, such as separability, Dunford-Pettis Property, approximation property, Radon-Riesz Property and Hahn-Banach extension property.

Keywords: Approximation property, Dunford-Pettis property, Fibonacci sequence spaces, Radon-Riesz property, Vector-Valued sequence spaces
2010 AMS: 46B50, 46B20, 46B25, 46B26, 46A22, 46A16

[1] *Department of Mathematics, İnönü University, Malatya, Türkiye*
yilmaz.yilmaz@inonu.edu.tr, ORCID: 0000-0003-1484-782X
seckin.yalcin@inonu.edu.tr, ORCID: 0000-0002-1673-3319
***Corresponding author**
Received: 26 February 2024, **Accepted:** 06 May 2024, **Available online:** 05 June 2024
How to cite this article: Y. Yılmaz, S. Yalçın, *On some properties of Banach space-valued Fibonacci sequence spaces*, Commun. Adv. Math. Sci., 7(2) (2024), 80-87.

1. Introduction

Banach spaces with a Schauder basis have many important advantages. The representation of such spaces with the help of the basis and the ability to approximate the element in countable steps with the help of this representation provide the opportunity to solve many structural and numerical problems. But in general, vector-valued sequence and function spaces do not generally have a Schauder basis. The concept of basis, which we defined in [1] tells us that some of these types of spaces have this type of basis and allows us to examine the structural properties of the space.

In this work we examine certain properties of some Banach space-valued Fibonacci sequence spaces. Their scalar-valued versions are defined and investigated in [2]-[6]. Fibonacci numbers have several applications in the field of Science, Engineering and Architecture. Fibonacci sequence is a sequence in which each number is the sum of the two preceding ones. Numbers that are part of the Fibonacci sequence are known as Fibonacci numbers . Starting from 1 and 1, the sequence begins

$$1,1,2,3,5,8,13,21,34,55,89,144,....$$

The Fibonacci numbers may be defined by the recurrence relation $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. We refer to [3] for detailed studies concerning Fibonacci numbers. Fibonacci matrix is define by the Fibonacci numbers as $\mathscr{F} = (f_{nk})$ such that

$$f_{n,k}=\begin{cases}\frac{-f_{n+1}}{f_n}, & if\ \ k=n-1\\ \frac{f_n}{f_{n+1}}, & if\ \ k=n\\ 0, & otherwise\end{cases}.$$

More explicitly,

$$\mathscr{F}=\begin{bmatrix} f_1/f_2 & 0 & 0 & 0 & \cdots\\ -f_3/f_2 & f_2/f_3 & 0 & 0 & \cdots\\ 0 & -f_4/f_3 & f_3/f_4 & 0 & \cdots\\ 0 & 0 & -f_5/f_4 & f_4/f_5 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

By using this sub-triangular infinite matrix Kara [4] introduced the sequence spaces $\ell_p(F)$, $1\le p<\infty$, and $\ell_\infty(F)$ such that

$$\ell_p(\mathscr{F})=\left\{u=(u_n)\in w:\sum_{n=0}^{\infty}\left|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right|^p<\infty\right\}$$

and

$$\ell_\infty(\mathscr{F})=\left\{u=(u_n)\in w:\sup_n\left|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right|<\infty\right\}.$$

For any Banach space V, let us define following V-valued Fibonacci sequence spaces

$$\ell_p(\mathscr{F},V)=\left\{u=(u_n)\in w(V):\sum_{n=0}^{\infty}\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V^p<\infty\right\}$$

and

$$\ell_\infty(\mathscr{F},V)=\left\{u=(u_n)\in w(V):\sup_n\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V<\infty\right\}$$

For $V=\mathbb{K}$, the real or complex number, then $\ell_p(\mathscr{F},V)=\ell_p(\mathscr{F})$ and $\ell_\infty(\mathscr{F},V)=\ell_\infty(\mathscr{F})$. It is easy to prove that $\ell_p(\mathscr{F},V)$ and $\ell_\infty(\mathscr{F},V)$ are Banach spaces with norms

$$\|u\|_{\ell_p(\mathscr{F},V)}=\left(\sum_{n=0}^{\infty}\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V^p\right)^{1/p}$$

and

$$\|u\|_{\ell_\infty(\mathscr{F},V)}=\sup_n\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V,$$

respectively.

$\mathscr{F}$ is an invertible triangle matrix, that is $\mathscr{F}^{-1}$ exists and it defines an isomorphism from $\ell_\infty(V)$ onto $\ell_\infty(\mathscr{F},V)$ and from $\ell_p(V)$ onto $\ell_p(\mathscr{F},V)$.

We will see in the sequel that $\ell_1(\mathscr{F},V)$ has Dunford-Pettis property and moreover will prove that $\ell_p(\mathscr{F},V)$ have the approximation property for $1\le p<\infty$ in some conditions.

Let us give some known required results from Banach space theory.

Suppose that U and V are Banach spaces. A linear operator S from U into V is compact if $S(B)$ is a relatively compact (means $\overline{S(B)}$ is compact) subset of V whenever B is a bounded subset of U. The collection of all compact linear operators from U into V is denoted by $K(U,V)$, or by just $K(U)$ if $U=V$. The range of a compact linear operator from a Banach space into a Banach space is closed if and only if the operator has finite rank, that is, the range of the operator is finite-dimensional [7]. A Banach space U has the *approximation property* if, for every Banach space V, the set of all finite-rank operators in $B(V,U)$ is dense in $K(V,U)$ [8]. The spaces c_0 and ℓ_p, $1\le p<\infty$, have the approximation property [7].

Let us remember that for any sequence (x_n) in a Banach space U converges weakly to U, or briefly $x_n\overset{w}{\to}x$, whenever $f(x_n)\to f(x_n)$ for each $f\in U^*$, the dual of U. We refer the reader to [7] for the definition of weak topology and weak convergence in detail. Suppose that U and V are Banach spaces. A linear operator S from U into V is weakly compact if $S(B)$ is a relatively weakly compact subset of V whenever B is a bounded subset of U.

Proposition 1.1. *[7] Suppose that S is a linear operator from a Banach space U into a Banach space V. Then S is weakly compact if and only if for any bounded sequence (x_n) in U has a subsequence $\left(x_{n_j}\right)_{j=0}^{\infty}$ such that $\left(Sx_{n_j}\right)$ converges weakly.*

Let us give important definitions of D.Hilbert. Suppose that U and V are Banach spaces. A linear operator S from U into V is *completely continuous* or a Dunford-Pettis operator if $S(K)$ is a compact subset of V whenever K is a weakly compact subset of U [9] .

Definition 1.2. *Suppose that U and V are Banach spaces. A Banach space U has the* Dunford-Pettis property *if, for every Banach space V, each weakly compact linear operator from U into V is completely continuous [7].*

Proposition 1.3. *[7] ℓ_1 has the Dunford-Pettis Property.*

Theorem 1.4. *(R.S.Phillips, [10]) Let V be a linear subspace of the Banach space U and suppose $T : V \to \ell_\infty$ is a bounded linear operator. Then T may be extended to a bounded linear operator $S : U \to \ell_\infty$ having the same norm as T.*

In some cites the operator T in the above theorem is known as a Hahn-Banach operator and then it is said that ℓ_∞ has the *Hahn-Banach extension property.*

2. Some Properties of Banach Space-Value Fibonacci Sequences

Definition 2.1. *[1] Let U and V be Banach spaces and $\mathbb{A}$ be a set. A family $\{\eta_a : a \in \mathbb{A}\}$ of continuous linear functions $\eta_a : V \to U$ is called* Y-basis for *U if the following condition is satisfied. There exist a unique family $\{R_a : a \in \mathbb{A}\}$ of linear functions R_a from U onto V and a subset $\mathscr{D}$ of $\mathscr{F}$, directed by some relation $\ll$, such that, for each $x \in U$, the net $(\pi_F(x) : \mathscr{D})$ converges to x in U. Where, for each $F \in \mathscr{D}$,*

$$\pi_F(x) = \sum_{a \in F} (\eta_a \circ R_a)(x),$$

and $\mathscr{F}$ is the family of all finite subsets of the index set $\mathbb{A}$. Furthermore, $\{\eta_a\}$ is called a Y-Schauder basis for *U whenever each R_a is continuous.*

Definition 2.2. *[1] The family $\{R_a : a \in \mathbb{A}\}$ is called* associate family of functions *(A.F.F.) to the V-basis $\{\eta_a : a \in \mathbb{A}\}$.*

Let $\{\eta_a : a \in \mathbb{A}\}$ be a V-basis for U. Clearly, the finite summation $\pi_F(x)$ defines an operator π_F on U for each $F \in \mathscr{D}$. This operator is called F*-projection* on U corresponding V-basis and it is continuous whenever $\{\eta_a\}$ is a V-Schauder basis.

Remark 2.3. *Let V be a Banach space on the field $\mathbb{C}$ possessing a basis $\{x_n\}$ (in the classical manner). Then the sequence $\{\eta_n\}$ of the functions*

$$\eta_n : \mathbb{C} \to V : \eta_n(t) = tx_n$$

is a $\mathbb{C}$-basis for V in the sense of above Definition. Indeed; take $\mathbb{A} = \mathbb{N}$ and

$$\mathscr{D} = \{\{1\}, \{1,2\}, \{1,2,3\}, \ldots\}$$

with the relation inclusion again, and $\{R_n\}$ as the sequence of associate coordinate functionals (g_n) to the basis $\{x_n\}$. Then $(\pi_F(x) : \mathscr{D})$ converges to x in U iff

$$\sum_{k=1}^{n} (\eta_k \circ R_k)(x) = \sum_{k=1}^{n} g_k(x) x_k,$$

converges to $x = \sum_{n=1}^{\infty} g_n(x) x_n$.

Theorem 2.4. *Let V be a Banach space for which a family $\{\eta_a : a \in \mathbb{A}\}$ be a V-basis for some Banach space V. Then, V is separable if $\mathbb{A}$ is countable [1].*

Main Results

Let us give some main results on V-valued Fibonacci sequence spaces in this section

Theorem 2.5. *Let V be a Banach space. Then $\ell_p(\mathscr{F},V)$ has an unconditional V-Schauder basis.*

Proof. Take $\mathbb{A}=\mathbb{N}$ and consider

$$\begin{array}{rcl} I_n & : & V \to \ell_p(V) \\ I_n(z) & = & (0,...,0,z,0,...) \end{array}$$

for and remember the Fibonacci matrix $\mathscr{F}$. Then obviously each $\mathscr{F}I_n$ defines a bounded linear operator from V into $\ell_p(\mathscr{F},V)$. Now the linear operator sequence

$$\{\mathscr{F}I_n : n\in\mathbb{N}\}$$

is a V-Schauder basis for $\ell_p(\mathscr{F},V)$. Let us prove this. First of all consider the sequence of coordinate projections

$$P_n : \ell_p(\mathscr{F},V)\to V;\; P_n(x)=x_n,$$

as $\{R_n : n\in\mathbb{N}\}$ in the basis definition, and take $\mathscr{D}$ as the family of all F finite subsets of $\mathbb{N}$ which is directed by the inclusion relation $\subseteq$. Then we must show that the net $(\pi_F(x):\mathscr{D})$ converges to x in $\ell_p(\mathscr{F},V)$ where

$$\pi_F(x)=\sum_{n\in F}(\mathscr{F}I_nP_n)(x)=\sum_{n\in F}\mathscr{F}I_n(x_n).$$

Obviously convergence of the above net corresponds to the convergence of the partial sums sequence of the series $\sum\limits_{n=0}^{\infty}\mathscr{F}I_n(x_n)$. Now, consider an arbitrary $\varepsilon>0$. We must find a finite subset $F_0=F_0(\varepsilon)\in\mathscr{D}$ such that, for each finite set $F\supseteq F_0$,

$$\|x-\pi_F(x)\|_{\ell_p(\mathscr{F},V)}\leq\varepsilon.$$

Since $x\in\ell_p(\mathscr{F},V)$ there exists an $n_0(\varepsilon)$ such that $\sum\limits_{n>n_0}^{\infty}\|(\mathscr{F}x)_n\|_V^p<\varepsilon$. Now take F_0 as

$$F_0=\left\{n\in\mathbb{N}:\sum_{n>n_0}^{\infty}\|(\mathscr{F}x)_n\|_V^p>\varepsilon\right\},$$

Then we get

$$\|x-\pi_F(x)\|_{\ell_p(\mathscr{F},V)}=\|\{x_n:n\in\mathbb{N}\setminus F\}\|_{\ell_p(\mathscr{F},V)}\leq\varepsilon,$$

for each finite $F\supseteq F_0$.This implies $(\pi_F(x):\mathscr{D})\to x$ in $\ell_p(\mathscr{F},V)$.

Let us show the uniqueness of the sequence $\{P_n\}$. Suppose

$$\sum_{n\in\mathbb{N}}(\mathscr{F}I_nP_n)(x)=\sum_{n\in\mathbb{N}}\left(\mathscr{F}I_nP_n'\right)(x)$$

and write

$$\pi_F^{\circ}(x)=\sum_{n\in\mathbb{N}}\left(\mathscr{F}I_n\left(P_n-P_n'\right)\right)(x),\quad F\in\mathscr{D}.$$

Remember that

$$\|\pi_F^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}=\left(\sum_{n\in F}\left\|\left(\mathscr{F}I_n\left(P_n-P_n'\right)\right)(x)\right\|^p\right)^{1/p}$$

and

$$\|\pi_F^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}\leq\|\pi_G^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}$$

for $F \subseteq G$. Since $(\pi_F(x) : \mathscr{D}) \to x$ in $\ell_p(\mathscr{F}, V)$ we get

$$\lim_{F \in \mathscr{D}} \|\pi_F^\circ(x)\|_{\ell_p(\mathscr{F},V)} = 0.$$

By this observation we have $(P_n - P_n')(x) = 0$ for each n and for every $x \in \ell_p(\mathscr{F}, V)$. This implies, $P_n = P_n'$ for each n. This gives the uniqueness of the basis.

Further, each P_n is continuous because $\|x_n\|_V \leq \|x\|_{\ell_p(\mathscr{F},V)}$. This proves that sequence $\{\mathscr{F} I_n : n \in \mathbb{N}\}$ is a V-Schauder basis for $\ell_p(\mathscr{F}, V)$. □

Theorem 2.6. *For $1 \leq p < \infty$, the Banach space $\ell_p(\mathscr{F}, V)$ has the approximation property if and only if V has.*

Proof. Suppose T be a compact linear operator from a Banach space V into $\ell_p(\mathscr{F}, V)$. We will find a sequence (T_n) of bounded linear operators of finite-rank from V into $\ell_p(\mathscr{F}, V)$. For any $x \in V$, $Tx \in \ell_p(\mathscr{F}, V)$ and for any bounded sequence (x_n) in V, the sequence (Tx_n) has a convergent subsequence $\left(Tx_{n_j}\right)_{j=0}^\infty$ in $\ell_p(\mathscr{F}, V)$ since T is compact. Hence

$$\begin{aligned} \left\|Tx_{n_i} - Tx_{n_j}\right\|^p_{\ell_p(\mathscr{F},V)} &= \left\|T\left(x_{n_i} - x_{n_j}\right)\right\|^p_{\ell_p(\mathscr{F},V)} \\ &= \sum_{m=0}^\infty \left\| \frac{f_m}{f_{m+1}} T\left(x_{n_i} - x_{n_j}\right)_m - \frac{f_{m+1}}{f_m} T\left(x_{n_i} - x_{n_j}\right)_{m-1} \right\|^p_V \end{aligned}$$

If we remember the definition of the space $\ell_p(\mathscr{F}, V)$,

$$\left\|T\left(x_{n_i} - x_{n_j}\right)\right\|^p_{\ell_p(\mathscr{F},V)} = \left\|(\mathscr{F}T)\left(x_{n_i} - x_{n_j}\right)\right\|^p_{\ell_p(V)}.$$

Now V has the approximation property if and only if $\ell_p(V)$ has. Hence

$$\left\|(\mathscr{F}T)\left(x_{n_i} - x_{n_j}\right)\right\|^p_{\ell_p(V)} \to 0 \text{ as } i, j \to \infty.$$

This means the operator $\mathscr{F}T : V \to \ell_p(V)$ is well-defined and compact. The matrix transformation $\mathscr{F}$ is clearly bounded linear and so is $\mathscr{F}T$. Since $\ell_p(V)$ have the approximation property, there exits a sequence $(A_m)_{m=0}^\infty$ of bounded linear operators of finite-rank from V to $\ell_p(V)$ such that $\|\mathscr{F}T - A_m\| \to 0$ as $m \to \infty$. Now the sequence $\left(\mathscr{F}^{-1}A_m\right)_{m=0}^\infty$ is the desired sequence of finite-rank operators from V to $\ell_p(\mathscr{F}, V)$. Easily we can see that each $\mathscr{F}^{-1}A_m$ is bounded linear and has the finite-rank. Further

$$\begin{aligned} \left\|T - \mathscr{F}^{-1}A_m\right\| &= \sup_{\|x\|=1} \left\|\left(T - \mathscr{F}^{-1}A_m\right)x\right\|_{\ell_p(\mathscr{F},V)} \\ &= \sup_{\|x\|=1} \left\|Tx - \left(\mathscr{F}^{-1}A_m\right)x\right\|^p_{\ell_p(\mathscr{F},V)} \\ &= \sup_{\|x\|=1} \left\|\mathscr{F}Tx - \mathscr{F}\left(\mathscr{F}^{-1}A_m\right)x\right\|^p_{\ell_p(V)} \\ &= \sup_{\|x\|=1} \|(\mathscr{F}T - A_m)x\|^p_{\ell_p(V)} \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

This completes the proof. □

Theorem 2.7. *$\ell_1(\mathscr{F}, V)$ has the Dunford-Pettis Property if and only if V has.*

Proof. Let T be any weakly compact linear operator from $\ell_1(\mathscr{F}, V)$ into V and compose T with $\mathscr{F}^{-1}$. Then $T\mathscr{F}^{-1}$ is obviously a bounded linear operator from $\ell_1(V)$ into V. Further it is weakly compact if and only if V is. Let us prove this: Suppose U is a bounded in $\ell_1(V)$. By the boundedness of the matrix operator $\mathscr{F}^{-1}$ we have $\mathscr{F}^{-1}(U)$ is a bounded subset of $\ell_1(\mathscr{F}, V)$. Therefore

$$T\left(\mathscr{F}^{-1}(U)\right) = \left(T\mathscr{F}^{-1}\right)(U)$$

is a relatively weakly compact set in V. As a result $T\mathscr{F}^{-1} : \ell_1(V) \to V$ is a weakly compact operator if and only if V is. Now, since $\ell_1(V)$ has the Dunford-Pettis Property if and only if V has, we get $T\mathscr{F}^{-1}$ is completely continuous. Let W be a weakly compact subset of $\ell_1(\mathscr{F}, V)$. Then $\mathscr{F}(W)$ is a weakly compact subset of $\ell_1(V)$, and so

$$\left(T\mathscr{F}^{-1}\right)\mathscr{F}(W) = T(W)$$

is a compact subset in V. □

Theorem 2.8. *Let V be a linear subspace of the Banach space U and suppose $T : V \to \ell_\infty(\mathscr{F}, V)$ is a bounded linear operator. Then T may be extended to a bounded linear operator $H : U \to \ell_\infty(\mathscr{F}, V)$ having the same norm as T if V has the Hahn-Banach extension property.*

Proof. Consider any bounded linear operator $T : V \to \ell_\infty(\mathscr{F}, V)$. Now $\mathscr{F}T : V \to \ell_\infty(V)$ is a bounded linear operator since the Fibonacci matrix is. Now $\ell_\infty(V)$ has the Hahn-Banach extension property since V has.

For any $x \in V$, $\mathscr{F}Tx \in \ell_\infty(V)$ and

$$\begin{aligned} \mathscr{F}Tx &= ((\mathscr{F}Tx)_1, (\mathscr{F}Tx)_2, \ldots) \\ &= ((P_1\mathscr{F}T)(x), (P_2\mathscr{F}T)(x), \ldots). \end{aligned}$$

Note that each P_n is coordinate projection from $\ell_\infty(V)$ into V such that $P_n(x) = x_n$. By the Hahn-Banach extension property of $\ell_\infty(V)$, the operator $\mathscr{F}T : V \to \ell_\infty(V)$ can be extended the bounded linear operator $S : U \to \ell_\infty(V)$ with the same norm as $\mathscr{F}T$, that is $\|S\| = \|\mathscr{F}T\|$. Let us define the operator H from U into $\ell_\infty(\mathscr{F}, V)$ such that for $x \in U$,

$$Hx = \left(\mathscr{F}^{-1}S\right)(x).$$

H is well-defined and linear since S and $\mathscr{F}^{-1}$ are. Further

$$\begin{aligned} \|Hx\|_{\ell_\infty(\mathscr{F},V)} &= \left\|\mathscr{F}^{-1}(S(x))\right\|_{\ell_\infty(\mathscr{F},V)} \\ &= \left\|\mathscr{F}\left(\mathscr{F}^{-1}(S(x))\right)\right\|_{\ell_\infty(V)} \\ &= \|S(x)\|_{\ell_\infty(V)} \\ &\leq \|S\| . \|x\| \end{aligned}$$

so that H is bounded. Now for $x \in V$,

$$\begin{aligned} \|Hx\|_{\ell_\infty(\mathscr{F},V)} &= \|S(x)\|_{\ell_\infty(V)} \\ &= \|(\mathscr{F}T)(x)\|_{\ell_\infty(V)} \\ &= \|Tx\|_{\ell_\infty(\mathscr{F},V)} \end{aligned}$$

so that H is an extension of T. Finally

$$\begin{aligned} \|H\| &= \sup_{\|x\|_U=1} \|Hx\|_{\ell_\infty(\mathscr{F},V)} \\ &= \sup_{\|x\|_U=1} \left\|\mathscr{F}^{-1}(S(x))\right\|_{\ell_\infty(\mathscr{F},V)} \\ &= \sup_{\|x\|_U=1} \left\|\mathscr{F}\left(\mathscr{F}^{-1}(S(x))\right)\right\|_{\ell_\infty(V)} \\ &= \sup_{\|x\|_U=1} \|S(x)\|_{\ell_\infty(V)} \\ &= \sup_{\|x\|_U=1} \|Tx\|_{\ell_\infty(\mathscr{F},V)} \\ &= \|T\|. \end{aligned}$$

This completes the proof. □

The following property is another desired property of Banach spaces. Now we see that $\ell_2(\mathscr{F}, V)$ has this property whenever V has, which we call it as the Radon-Riesz Property. The Radon-Riesz property is named after J. Radon and F. Riesz proved that the spaces $Lp(\Omega, \Sigma, \mu)$ for $1 < p < \infty$ have this property [11]-[13]. Radon-Riesz Property also known as the Kadets-Klee property since their further investigation and application of this concept [14]-[16].

Definition 2.9. *[7] A normed space has the* Radon-Riesz property *or the* Kadets-Klee property*, and is called a Radon-Riesz space, if it satisfies the following condition: Whenever (x_n) is a sequence in the space and x an element of the space such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$.*

The proof of the following lemma is routine.

Lemma 2.10. *Let V is a Hilbert space. Then $\ell_2(\mathscr{F},V)$ is a Hilbert space with the inner-product*

$$\langle u,v\rangle_{\ell_2(\mathscr{F},V)} = \langle \mathscr{F}u,\mathscr{F}v\rangle_{\ell_2(V)} = \sum_{k=1}^{\infty} \langle (\mathscr{F}u)_k,(\mathscr{F}v)_k\rangle_V .$$

Theorem 2.11. *$\ell_2(\mathscr{F},V)$ has the Radon-Riesz Property whenever V is a Hilbert space possessing the Radon-Riesz Property.*

Proof. Let (u_n) be a sequence in $\ell_2(\mathscr{F},V)$ and u be an element of $\ell_2(\mathscr{F},V)$. Assume that $u_n \overset{w}{\to} u$ in $\ell_2(\mathscr{F},V)$ and assume that $\|u_n\|_{\ell_2(\mathscr{F},V)} \to \|u\|_{\ell_2(\mathscr{F},V)}$. We will prove that (u_n) norm convergent to u that is $u_n \to u$ in $\ell_2(\mathscr{F},V)$. Now the assumption $u_n \overset{w}{\to} u$ means $f(u_n) \to f(u_n)$ for each $f \in \ell_2(\mathscr{F},V)^*$. Let us show that $\|u_n - u\|_{\ell_2(\mathscr{F},V)} \to 0$ to complete the proof:

$$\begin{aligned}
\|u_n - u\|^2_{\ell_2(\mathscr{F},V)} &= \|\mathscr{F}u_n - \mathscr{F}u\|^2_{\ell_2(V)} \\
&= \langle \mathscr{F}u_n - \mathscr{F}u, \mathscr{F}u_n - \mathscr{F}u\rangle_{\ell_2(V)} \\
&= \langle \mathscr{F}u_n,\mathscr{F}u_n\rangle_{\ell_2(V)} - \langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)} \\
&\quad - \langle \mathscr{F}u,\mathscr{F}u_n\rangle_{\ell_2(V)} + \langle \mathscr{F}u,\mathscr{F}u\rangle_{\ell_2(V)} \\
&= \|\mathscr{F}u_n\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)} - \langle \mathscr{F}u,\mathscr{F}u_n\rangle_{\ell_2(V)}
\end{aligned}$$

Let $z = \mathscr{F}u \in \ell_2(V) = \ell_2(V)^*$ and let us consider $z \circ \mathscr{F}$ such that $(z\circ\mathscr{F})u = \langle \mathscr{F}u,\mathscr{F}u\rangle_{\ell_2(V)}$. Then from the properties of the matrix $\mathscr{F}$ and by the Riesz's Theorem (on $\ell_2(V)$) we have $z\circ\mathscr{F}$ is a continuous linear functional on $\ell_2(\mathscr{F},V)$ and

$$(z\circ\mathscr{F})u_n = z(\mathscr{F}u_n) = \langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)} .$$

By the assumption $u_n \overset{w}{\to} u$ we have

$$\begin{aligned}
(z\circ\mathscr{F})(u_n) &= \langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)} \\
&\to \langle \mathscr{F}u,\mathscr{F}u\rangle_{\ell_2(V)}, \text{ as } n\to\infty, \\
&= (z\circ\mathscr{F})(u) \\
&= \|\mathscr{F}u\|^2_{\ell_2(V)}
\end{aligned}$$

Dually, let us now take $z_n = \mathscr{F}u_n \in \ell_2(V)^* = \ell_2(V)$, for each n, then

$$(z_n\circ\mathscr{F})u = z_n(\mathscr{F}u) = \langle \mathscr{F}u,\mathscr{F}u_n\rangle_{\ell_2(V)} .$$

Now again each $z_n\circ\mathscr{F}$ is a continuous linear functional on $\ell_2(\mathscr{F},V)$ and again by the assumption $u_n \overset{w}{\to} u$ we have

$$\begin{aligned}
(z_n\circ\mathscr{F})(u) &= z_n(\mathscr{F}u) \\
&= \langle \mathscr{F}u,\mathscr{F}u_n\rangle_{\ell_2(V)} \\
&= \overline{\langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)}} \\
&\to \overline{\langle \mathscr{F}u,\mathscr{F}u\rangle_{\ell_2(V)}}, \text{ as } n\to\infty, \\
&= \|\mathscr{F}u\|^2_{\ell_2(V)} .
\end{aligned}$$

Eventually, by the assumption $\|u_n\|_{\ell_2(\mathscr{F},V)} \to \|u\|_{\ell_2(\mathscr{F},V)}$, we have

$$\begin{aligned}
\|u_n - u\|^2_{\ell_2(\mathscr{F},V)} &= \|\mathscr{F}u_n\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \langle \mathscr{F}u_n,\mathscr{F}u\rangle_{\ell_2(V)} - \langle \mathscr{F}u,\mathscr{F}u_n\rangle_{\ell_2(V)} \\
&\to \|\mathscr{F}u\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \|\mathscr{F}u\|^2_{\ell_2(V)} - \|\mathscr{F}u\|^2_{\ell_2(V)} \\
&= 0, \text{ as } n\to\infty.
\end{aligned}$$

$\square$

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] Y. Yilmaz, *Relative bases in Banach spaces*, Nonlinear Anal., **71** (2009), 2012–2021.

[2] E. E. Kara, M. İlkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear Multilinear Algebra, **64**(11) (2016), 2208–2223.

[3] T. Koshy, *Fibonacci and Lucus Numbers with Applications*, Wiley, New York, 2001.

[4] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **38** (2013), 1–15.

[5] S. Ercan, Ç. A. Bektaş, *Some topological and geometric properties of a new BK-space derived by using regular matrix of Fibonacci numbers*, Linear Multilinear Algebra, **65**(5) (2017), 909–921.

[6] M. Candan, *Some characteristics of matrix operators on generalized Fibonacci weighted difference sequence space*, Symmetry, **14**(7) (2022), 1283.

[7] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.

[8] A. Grothendieck, *Sur les applications lineaires faiblement compactness d'espaces du type $C(K)$*, Canad. J. Math., **5** (1953), 129-173.

[9] D. Hilbert, *Grundzii.ge einer allgemeinen Theorie der linearen lntegralgleichungen, IV*, Nachr. Kgl. Gesells. Wiss. Gottingen Math.Phys. Kl., (1906), 157-227.

[10] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc., **48** (1940), 516-541.

[11] J. Radon, *Theorie und Anwendungen der absalut additiven Mengenfunktionen*, S.-B. Akad. Wiss. Wien, **122** (1913), 1295-1438.

[12] F. Riesz, *Sur la convergence en moyenne, I*, Acta Sci. Math., **4** (1928-1929), 58-64.

[13] F. Riesz, *Sur la convergence en moyenne, II*, Acta Sci. Math., **4** (1928-1929), 182-185.

[14] M. I. Kadets, *On strong and weak convergence*, Dokl. Akad. Nauk SSSR, **122** (1958), 13-16. (Russian)

[15] V. Klee, *Mappings into normed linear spaces*, Fund. Math., **49** (1960/61), 25-34.

[16] V. Klee, *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Functional Anal. Appl., **1** (1967), 53-62.