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THE SPACE OF FUZZY MINIMAL PRIME FILTERS IN AN ALMOST DISTRIBUTIVE LATTICE

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Abstract

In this paper, we establish and describe the notions of fuzzy minimal prime filters of an Almost Distributive Lattice (ADL). We prove that a fuzzy filter is the point wise infimum of all minimal prime fuzzy filters (all fuzzy minimal prime filters) of an ADL. Mainly, we introduce the topological space on the set of all fuzzy minimal prime filters of an ADL (denoted by M(R)). For an ADL R, we prove that an open set M(R) is a base for the topology and it constitutes a base for the subspace topology on a closed set N and they are the only compact open sets.

Keywords: Almost distributive lattice (ADL), fuzzy filter, prime fuzzy filter, minimal prime fuzzy filter, fuzzy prime filter and fuzzy minimal prime filter

1. Introduction

The concept of a fuzzy subset of a set X was introduced by Zadeh [8] as a function from X into [0,1] of real numbers and Goguen [1] replaced the valuation set [0,1] by of a complete lattice L in an attempt to make a generalized study of fuzzy set theory by fuzzy sets. Subsequently, (Liu [2]) worked on Fuzzy invariant subgroups and ideals. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [7]. Later, the notion of L-fuzzy filters of ADLs and, prime and maximal L-fuzzy filters of an ADL were introduced by Raj, Natnael and Swamy[5, 6]. Furthermore, Ming [3] introduced the concepts of fuzzy topological space.

In this paper, we study fuzzy prime filters contains fuzzy minimal prime filters of an ADL. Mainly, we extend M.H. Stone Theorem on prime filters of distributive lattice to fuzzy prime filters (prime fuzzy filters) as well as fuzzy minimal prime filters (minimal prime fuzzy filters) of an ADL. Also, we characterize all minimal prime fuzzy filters in terms of minimal prime filters of an ADL and minimal meet elements of a frame. Finally, we introduce the hull kernel topology on the set of all fuzzy minimal prime filters of an ADL, denoted by $M(R)$. For a fuzzy filter η of an ADL R, open set of $M(R)$ is of the form $M(R)(\eta) = \{\varphi \in M(R): \eta \nleq \phi\}$ and $N(\eta) = \{\varphi \in M(R): \eta \leq \varphi\}$ is a closed set. We prove that an open set $\{M(R)(x_s): x \in R \text{ and } 0 \neq s \in L\}$ is a base for the topology on $M(R)$ and the family $\{M(R)(x_s) \cap N^t: x \in R \text{ and } s \nleq t\}$ Constitute a base for the sub space topology on N and they are the only compact open sets (called N^t) in $M(R)$. Also, we show the space $M(R)$ is a T0-space.

Throughout this paper, R stands for an ADL $(R, \wedge, \vee, 0)$ with a maximal element and L stands for a complete lattice $(L, \wedge, \vee, 0, 1)$ satisfying the infinite meet distributive law and this type of a lattice is called a frame.

2. Preliminaries

In this section, we recall some definitions and basic results mostly taken from [7] and [5].

Definition 2.1 An algebra $R = (R, \wedge, \vee, 0)$ of type $(2,2,0)$ is called an Almost Distributive Lattice(abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in R$.

1. $0 \wedge a = 0$
2. $a \vee 0 = a$
3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
5. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
6. $(a \vee b) \wedge b = b$.

Any bounded below distributive lattice is an ADL. Any non empty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element **0** in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases}$$

and

$$a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL ([7]).

Definition 2.2 Let R be an ADL. For any a and $b \in R$, define $a \leq b$ iff $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on R with respect to which X is the smallest element in R.

Theorem 2.3 The following hold for any a, b and c in an ADL R.

(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $a \wedge b \leq b \leq b \vee a$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$
(5) $a \wedge b = b \Leftrightarrow a \vee b = a$
(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)
(7) $a \vee (b \vee a) = a \vee b$
(8) $a \leq b \Longrightarrow a \wedge b = a = b \wedge a (\Leftrightarrow a \vee b = b = b \vee a)$
(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$
(12) $a \wedge b = \inf\{a, b\}$ iff $a \wedge b = b \wedge a$ iff $a \vee b = \sup\{a, b\}$.

An element $m \in R$ is said to be maximal if, for any $x \in R$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$, for all $x \in R$.

Definition2.4 Let F be a non empty subset of R. Then F is called a filter of R if $a, b \in F \Rightarrow a \wedge b \in F$ and $x \vee a \in F$, for all $x \in R$.

As a consequence, for any filter F of R, $a \vee x \in F$ for all $a \in F$ and $x \in R$. For any $S \subseteq R$, the smallest filter of R containing S is called the filter generated by S in R and is denoted by $[S\rangle$. It is known that

$$[S\rangle = \left\{b \vee \left(\bigwedge_{i=1}^{n} x_i\right) : n \geq 0, x_i \in S \text{ and } b \in R\right\}$$

When $S = \{x\}$, we write $[x\rangle$ for $[\{x\}\rangle$. Note that $[x\rangle = \{a \vee x : a \in R\}$.

Theorem 2.5 Let ϕ be a fuzzy subset of R and m maximal elements in R. Then the following are equivalent to each other, for all $x, y \in R$.
(1) ϕ is a fuzzy filter of R
(2) $\phi(m) = 1$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$
(3) $\phi(m) = 1$ and $\phi(x \vee y) \geq \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) \geq \phi(x) \wedge \phi(y)$.

Theorem 2.6 Let ϕ be a fuzzy filter of R and F a non-empty subset of R. Then for any $x, y \in R$, we have the following.
(1) ϕ is an isotone mapping, in the sense that $x \leq y \Rightarrow \phi(x) \leq \phi(y)$
(2) $x \sim y \Rightarrow \phi(x) = \phi(y)$
(3) $\phi(x \vee y) = \phi(y \vee x)$
(4) $x \in [F\rangle \Rightarrow \phi(x) \geq \bigwedge_{i=1}^{n} \phi(a_i)$, for some $a_1, a_2, \ldots, a_n \in F$
(5) $x \in [y\rangle \Rightarrow \phi(x) \geq \phi(y)$.

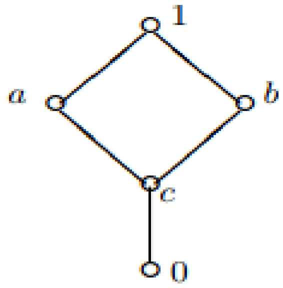
3. Fuzzy minimal prime filters

In this section, we introduce the concepts of minimal prime fuzzy filters and fuzzy minimal prime filters of an ADL. Mainly, we extend M.H. Stone Theorem on prime filters of distributive lattice of fuzzy prime filters (prime fuzzy filters) as well as fuzzy minimal prime filters (minimal prime fuzzy filters) of an ADL.

Let us recall from [6] that a proper fuzzy filter ϕ of R is called a fuzzy prime filter of R if, for any x and $y \in R$, $\phi(x \vee y) = \phi(x)$ or $\phi(y)$.

Theorem 3.1 [6] Let ϕ be any proper fuzzy filter of R. Then the following are equivalent to each other.
(1) For each $t \in L$, $\phi_t = R$ or ϕ_t is a prime filter of R
(2) ϕ is a fuzzy prime filter of R
(3) For any $x, y \in R$, $\phi(x \vee y) \leq \phi(x) \vee \phi(y)$ and hence $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and either $\phi(x) \leq \phi(y)$ or $\phi(y) \leq \phi(x)$.

Example 3.2 Let $R = \{0, a, b, c, 1\}$be a lattice represented by the Hasse diagram given below:



Define fuzzy subsets ϕ and η from R to [0,1] by $\phi(a) = \phi(1) = 1, \phi(0) = \phi(b) = \phi(c) = 0.5$ and $\eta(1) = 1, \eta(x) = 0.75$ for all $x \in R - \{1\}$. For any x and $y \in R$, we have $x \leq y$ implies $\phi(x) \leq \phi(y)$ and hence ϕ is an isotone. Also, we observe that $\phi(x \vee y) = \phi(x)$ or $\phi(y)$, for any x and $y \in R$. Thus, ϕ is a fuzzy prime filter of ϕ. On the other hand, it can be easily verified that η is a fuzzy filters of R but not fuzzy prime filter of R; since $\eta(a \vee b) = \eta(1) = 1 \neq \eta(a)$ and $\eta(b)$.

Let us recall that for any fuzzy subset ϕ of R and $t \in L$, we define the fuzzy subsets $\phi \vee t$ and $\phi \wedge t$ by $(\phi \vee t)(x) = \phi(x) \vee t$ and $(\phi \wedge t)(x) = \phi(x) \wedge t$, for all $x \in R$.

Theorem 3.3 Let ϕ be a fuzzy filter of R. Then $\phi \vee t$ is a fuzzy filter of R, for all $t \in L$.

Proof. Let m be a maximal element in R and $t \in L$. Then $(\phi \vee t)(m) = \phi(m) \vee t = 1 \vee t = 1$. Also, for any x and $y \in R$,

$$\begin{aligned}(\phi \vee t)(x \wedge y) &= \phi(x \wedge y) \vee t \\ &= (\phi(x) \wedge \phi(y)) \vee t (\text{since } \phi \text{ is a fuzzy filter}) \\ &= (\phi(x) \vee t) \wedge (\phi(y) \vee t) \\ &= (\phi \vee t)(x) \wedge (\phi \vee t)(y).\end{aligned}$$

Therefore, $\phi \vee t$ is a fuzzy filter of R.

Unlike $\phi \vee t$, $\phi \wedge t$ is not a fuzzy filter of R, unless $t = 1$. For $\phi \wedge t$ to be a fuzzy filter, it is necessary that $(\phi \wedge t)(m) = 1$ and hence $t = 1 \wedge t = \phi(m) \wedge t = 1$.
Also, note that if $t = 1$, then $\phi \vee t$ is the constant map $\bar{1}$. Therefore, for $\phi \vee t$ to be a proper fuzzy filter, it is necessary that $t < 1$.

Theorem 3.4 Let ϕ be a fuzzy prime filter of R and an element t in L such that $\phi(0) \leq t < 1$. Then $\phi \vee t$ is a fuzzy prime filter of R.

Proof. By the above theorem, $\phi \vee t$ is a fuzzy filter of R. Also, $(\phi \vee t)(0) = \phi(0) \vee t = t < 1$. Therefore, $\phi \vee t$ is a proper fuzzy filter of R. For any x and $y \in R$,

$$\begin{aligned}(\phi \vee t)(x \wedge y) &= \phi(x \wedge y) \vee t \\ &= \phi(x) \vee t \text{ or } \phi(y) \vee t \text{ (since } \phi \text{ is a fuzzy prime filter)} \\ &= (\phi \vee t)(x) \text{ or } (\phi \vee t)(y).\end{aligned}$$

Therefore, $\phi \vee t$ is a fuzzy prime filter of R.

Let us recall that any proper filter of R is contained in a prime filter of R. Now, we extend this result to fuzzy prime filters of R in the following.

Theorem 3.5 If ϕ is a proper fuzzy filter of R such that $\mathrm{Sup}\{\phi(x): x \in R \text{ and } \phi(x) < 1\} < 1$. Then there exists a fuzzy prime filter η of R such that $\phi \leq \eta$.

Proof. Let $= \mathrm{Sup}\{\phi(x): x \in R \text{ and } \phi(x) < 1\} < 1$ and $F = \{x \in R: \phi(x) = 1\}$.
Then F is a filter of R (since F isthe1-cut ϕ_1). Since ϕ is a proper fuzzy filter of ϕ, it follows that F is a proper filter of R. Then, there exists a prime filter J of R such that $F \subseteq J$. Thus, χ_J is a fuzzy prime filter of R, where χ_J is the characteristic mapping corresponding to J. By theorem 3.4, $\chi_J \vee t$ is a fuzzy prime filter of R. Note that $\chi_J \vee t = t^J$. Now, for any $x \in R$,

$$x \in J \Rightarrow \phi(x) \leq 1 = t^J(x)$$

$$\text{and } x \notin J \Rightarrow x \notin F \Rightarrow \phi(x) < 1 \Rightarrow \phi(x) \leq t = t^J(x).$$

Therefore, $\phi(x) \leq t^J(x)$, for all $x \in R$. Thus, $\phi \leq t^J$ and $t^J$is a fuzzy prime filter of R.

The following theorem provides a method for constructing fuzzy prime filter of a given ADL and this straight verification which is analogous from prime filter of an ADL.

Theorem 3.6 Let C be a chain such that $1 \in C$ and C is closed under arbitrary supremums. Let $\{F_t\}_{t\in C}$ be a class of filters of R such that $F_t = R$ or F_t is a prime filter of R, for each $t \in C$. Also, suppose that, $\bigcap_{t\in\triangle} F_t = F_{\mathrm{Sup}\, S,}$ for any $S \subseteq C$. Define a fuzzy subset ϕ of R by $\phi(x) = \vee\{t \in C: x \in F_t\}$, for any $x \in R$. Then ϕ is a fuzzy prime filter of R if ϕ is proper.

In the following, we extend an important theorem of M. H. Stone to fuzzy prime filters of ADLs.

Theorem 3.7 Let η be a fuzzy ideal and ϕ a fuzzy filter of R such that $\eta \wedge \phi \leq \bar{t}$ (the constant fuzzy subset attaining the value t) where, t is a prime element in L. Then there exists a fuzzy prime filter (prime fuzzy filter) ν of R such that $\phi \leq \nu$ and $\eta \wedge \nu \leq \bar{t}$.

Proof. We are given that $\eta(x) \wedge \phi(x) \leq t$, for all $x \in R$. Put $I = \{x \in R: \eta(x) \nleq t\}$ and $F = \{x \in R: \phi(x) \leq t\}$. Clearly, I is an ideal and F is a filter of R. Since t is prime and $\eta(x) \wedge \phi(x) \leq t$, it follows that $\eta(x) \leq t$ or $(x) \wedge \phi(x) \leq t$ and hence $x \notin I$ or $x \notin F$. Therefore, $I \cap F$ is empty set. Then, there exists a prime filter J of R such that $F \subseteq J$ and $I \cap J$ is empty set. Since t is a prime element in L and J is a prime filter of R, then t^J is a prime fuzzy filter of R and hence, t^J is a fuzzy prime filter of R. Now, for any $x \in R$,

$$x \notin J \Longrightarrow x \notin F \Longrightarrow \phi(x) \leq t = t^J(x)$$

$$\text{and } x \in J \Longrightarrow \phi(x) \leq 1 = t^J(x).$$

Therefore, $\phi \leq t^J$. Finally, we prove that $\eta \wedge t^J \leq t$. Now,

$$x \in J \Longrightarrow x \notin I \text{ (since } I \cap J = \emptyset) \Longrightarrow \eta(x) \wedge t^J(x) = \eta(x) \wedge 1 = \eta(x) \leq t = \bar{t}(x)$$

$$\text{and } x \notin J \Longrightarrow \eta(x) \wedge t^J(x) = \eta(x) \wedge t \leq t = \bar{t}(x).$$

Therefore, $\eta \wedge t^J \leq \bar{t}$.

Note that the close interval $[0,1]$ of real numbers is a frame and in which every element is prime.

Corollary 3.8 Let $L = [0,1]$ such that $0 \leq t \leq 1$. Let η be a fuzzy ideal of R and ϕ be a fuzzy filter of R such that $\eta(x) \wedge \phi(x) \leq t$, for all $x \in R$. Then there exists a fuzzy prime filter (prime fuzzy filter) φ of R such that $\phi \leq \varphi$ and $\eta(x) \wedge \varphi(x) \leq t$, for all $x \in R$.
In the following, we introduce the notion of minimal prime fuzzy filters of R and discuss certain properties of these. Also, we characterize all minimal prime fuzzy filters of R in terms of minimal prime filters of R and minimal prime elements in L.

Definition 3.9 Let η be a fuzzy filter of R. A prime fuzzy filter φ of R containing η is said to be minimal prime fuzzy filter belonging to η if there is no prime fuzzy filter of R containing η and properly contained in φ.

The following can be easily proved by using Zorn's Lemma.

Theorem 3.10 Let ϕ be a prime fuzzy filter of R containing a fuzzy filter φ. Then there exists a minimal prime fuzzy filter η of R such that $\varphi \leq \eta \leq \phi$.

Corollary 3.11 Every prime fuzzy filter of R contains a minimal prime fuzzy filter.

Theorem 3.12 Let ϕ be a fuzzy filter of R. Then ϕ is a minimal prime fuzzy filter of R if and only if $\phi = t^F$, for some minimal prime filter F of R and a minimal prime element t in L.

Proof. Suppose $\phi = t^F$, for some minimal prime filter F of R and a minimal prime element t in L. Then ϕ is a prime fuzzy filter of R. Let η be a prime fuzzy filter of R and $\eta \leq \phi$. Again $\eta = s^G$, for some prime filter G of R and a prime element s in L. Therefore, $s^G \leq t^F$ and it follows that, $s \leq t$ and $G \subseteq F$. By the minimality of F and t, we get that $s = t$ and $G = F$. Therefore, $\eta = \phi$ and hence ϕ is a minimal prime fuzzy filter of R. Conversely suppose that ϕ is a minimal prime fuzzy filter of R. Since ϕ is prime, there exists a prime filter F of R and a prime element t in L such that $\phi = t^F$. Let G be a prime filter of R such that $G \subseteq F$. Then t^G is a prime fuzzy filter of R and $t^G \subseteq t^F = \phi$. By the minimality of ϕ, we get $t^G = t^F$. Therefore, $G = F$ and hence F is a minimal prime filter of R. Let s be a prime element in L and $s \leq t$. Then $s^F \leq t^F$. Again by minimality of $\phi, s^F = t^F$ and hence $s = t$. Thus t is a minimal prime element in L.

Corollary 3.13 The mapping $(F, t) \longmapsto t^F$ establishes a one-to-one correspondence between the pairs (F, t), where F is a minimal prime filter of R and t is a minimal prime element in L, and the minimal prime fuzzy filters of R.

If the smallest element 0 in L is prime, then 0 will be the only minimal prime element in L. Also, note that $\chi_F = 0_F$, for any filter F of R. The following is a consequence of 3.12.

Theorem 3.14 Let 0 be a prime element in L. Then a fuzzy filter ϕ of R is a minimal prime fuzzy filter of R if and only if $\phi = \chi_F$, for some minimal prime filter F of R. Moreover, $F \mapsto \chi_F$ is a one-to-one correspondence between the set of minimal prime filters of R onto the set of minimal prime fuzzy filters of R.

Next, we characterize all fuzzy minimal prime filters of R. By an fuzzy minimal prime filter of R we mean, as usual, a minimal element in the poset of all fuzzy prime filters of R under the point-wise ordering.

Theorem 3.15 [6] Let R be an ADL in which every maximal is ∨-irreducible element and let ϕ be an L-fuzzy prime filter of R. Then ϕ is an L-fuzzy minimal prime filter of R if and only if the t-cut ϕ_t is a minimal prime filter of R, for all $t \in L$.
By using Zorn's Lemma we can prove that there exists a fuzzy minimal prime filter belonging to a fuzzy filter, whenever ϕ is contained in a fuzzy prime filter η of R. Thus we have the following and this is a consequence of 3.7.

Theorem3.16 Let ψ be a fuzzy ideal of R and η be a fuzzy filter of R such that $\psi \wedge \eta \leq \bar{t}$, where t is a prime element in L. Then there exists a fuzzy minimal prime filter (minimal prime fuzzy filter) φ of R belonging to η such that $\psi \wedge \varphi \leq t$.

Let us recall that the intersection of all minimal prime filters belonging to a filter F of R is a filter itself. Now, we extend this result to the case of fuzzy filters in the following.

Theorem 3.17 Suppose that every element of L is meet of prime elements of L and η is a fuzzy filter of R. Then $\eta = \wedge\{\psi$: ψ is a fuzzy minimal prime filter (minimal prime fuzzy filter) of R belonging to $\eta\}$.

Proof. Put $\omega = \wedge\{\psi$: ψ is a fuzzy minimal prime filter (minimal prime fuzzy filter) of R belonging to $\eta\}$. It can be easily verified that $\eta \leq \omega$. On the other hand, suppose $\omega(x) \nleq \eta(x)$, for some $x \in R$. By assumption $\eta(x) = \bigwedge_{i\in\triangle} t_i$, where each t_i is a prime element in L. It follows that, $\omega(x) \nleq \bigwedge_{i\in\triangle} t_i$. Then there exists $j \in \triangle$ such that $\omega(x) \nleq t_j$ and $\eta(x) \leq t_j$. Consider a fuzzy subset φ of R defined by

$\varphi(y) = \chi_{(x]}(y)$. Thus, φ is a fuzzy ideal of R. Now we prove $\eta \wedge \varphi \leq t_i$, for all $i \in \triangle$.

Let $y \in R$. Then

$$
\begin{aligned}
y \in (x] &\Rightarrow y = x \wedge y \\
&\Rightarrow \eta(y) = \eta(x \wedge y) = \eta(x) \wedge \eta(y) \\
&\Rightarrow \eta(y) \leq \eta(x) \\
y \in (x] &\Rightarrow (\eta \wedge \varphi)(y) = \eta(y) \wedge \varphi(y) = \eta(y) \leq \eta(x) \leq t_j \\
\text{and } y \notin (x] &\Rightarrow \varphi(y) = 0 \\
&\Rightarrow \eta(y) \wedge \varphi(y) = 0 \leq t_j.
\end{aligned}
$$

Therefore, in both cases, $\eta \wedge \varphi \leq t_i$ and hence, there exists a fuzzy minimal prime filter (minimal prime fuzzy filter) ψ of R such that $\eta \leq \psi$ and $\psi \wedge \varphi \leq t_j$. But $\omega \leq \psi$, in particular, $\omega(x) \leq \psi(x) = (\omega \leq \psi)(x) \leq t_j$. It follows that, $\omega(x) \leq t_j$, which is a contradiction.Therefore, $\omega \leq \eta$. Thus, $\omega = \eta$. Note that the set F(R) of all maximal elements of R forms a filter of R and which is the smallest filter of R. If $\chi_{F(R)}$ is the characteristic map of F(R), then $\chi_{F(R)}$ is the smallest fuzzy filter of R and fuzzy minimal prime filter (minimal prime fuzzy filter) belonging to $\chi_{F(R)}$ is simply called a fuzzy minimal prime filter (minimal prime fuzzy filter) of R. Now we have the following.

Corollary 3.18 Suppose that every element of L is meet of prime elements in L. Then the point-wise infimum of all fuzzy minimal prime filters (and hence all fuzzy prime filters) of R is $\chi_{F(R)}$.

Corollary 3.19 Suppose that every element of L is meet of prime elements in L. Then the point-wise infimum of all minimal prime fuzzy filters (and hence all prime fuzzy filters) of R is $\chi_{F(R)}$.

4. Hull space

Let $M(R)$ denote the set of all fuzzy minimal prime filters of R. For any fuzzy subset η of R, let $N(\eta) = \{\phi \in M(R): \eta \leq \phi\}$ and $M(R)(\eta) = \{\phi \in M(R): \eta \nleq \phi\}$.

It can be easily seen that, for any subset η of R, $N(\eta) = N(\bar{\eta})$ and $M(R)(\eta) = M(R)(\bar{\eta})$, where $\bar{\eta}$ is the fuzzy filter of R generated by η.

Theorem 4.1 Let $\tau = \{M(R)(\eta): \eta \text{ is a fuzzy filter of } R\}$. Then the pair $(M(R), \tau)$ is a topological space.

Proof. Consider fuzzy filters η and ψ of R defined by, $\eta(x) = \chi_{\{0\}}$ and $\psi(x) = 1$, for all $x \in R$. Then $N(\eta) = M(R)$ and $N(\psi) = \emptyset$, and $M(R)(\eta) = \emptyset$ and $M(R)(\psi) = M(R)$. Thus, $\eta, \psi \in \tau$. Also, $\phi \in N(\eta) \cap N(\psi) \Leftrightarrow \eta \leq \phi$ and $\psi \leq \phi$

$$\Leftrightarrow \eta \wedge \psi \leq \phi$$
$$\Leftrightarrow \phi \in N(\eta \wedge \psi).$$

Thus τ is closed under finite intersections. Also, let $\{\eta_i: i \in I\}$ be non-empty collection of fuzzy filters of R. Then it can be easily verified that $\cap \{N(\eta_i): i \in I\} = N(\overline{\vee_{i\in I} \eta_i})$ and $\cup \{M(R)(\eta_i): i \in I\} = M(R)(\overline{\vee_{i\in I} \eta_i})$. Thus τ is closed under arbitrary union. Therefore, τ is a topology on $M(R)$.

Definition 4.2 Let $x \in R$ and $t \in L$, define $x_t: R \to L$ by

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in R$, is called a fuzzy point corresponding to x and t.

Theorem 4.3 For any $x \in R$ and $t, s \in L$, $M(R)(x_s) = \{t^F \in M(R): x \notin F \text{ and } s \nleq t\}$.

Proof. $\phi \in M(R)(x_s) \Rightarrow x_s \nleq t^F = \emptyset$

$$\Rightarrow x_s(y) \nleq t^F(y), \text{ for some } y \in R$$
$$\Rightarrow s \nleq t^F(y) \text{ and } y = x$$
$$\Rightarrow x \notin F, s \nleq t \text{ and } \phi = t^F.$$

On the other hand, let $t^F \in M(R)$ such that $x \notin F$ and $s \nleq t$. Then $t^F(x) = t$, so that $x_s \nleq t^F$.

From [5] recall that, for any filter F of R and $s \leq t \in L$, the fuzzy filter $\langle t, s\rangle^F$ defined by

$$\langle t, s\rangle^F(x) = \begin{cases} 1 & \text{if } x \text{ is maximal} \\ t & \text{if } x \text{ is not maximal and } x \in F \\ s & \text{if } x \text{ is not maximal and } x \notin F. \end{cases}$$

Theorem 4.4 For any $x \in R$ and $t \in L$, $M(R)(x_t) = M(R)(\phi)$, for some fuzzy filter ϕ of R.

Proof. It is clear; for, $\overline{x_t} = \langle t, 0\rangle^{\langle x]}$ and $M(R)(x_t) = M(R)(\overline{x_t})$.

Theorem 4.5 The sub family $\{M(R)(x_s): x \in R \text{ and } s \in L\}$ form a base for a topology on $M(R)$.

Proof. By the previous theorem, $\{M(R)(x_s): x \in R \text{ and } s \in L\}$ is a sub family of τ. Let $M(R)(\eta) \in \tau$ and $\phi \in M(R)(\eta)$. Then $\eta \nleq \phi$ and hence there exists $x \in R$ such that $\eta(x) \nleq \phi(x)$. Let $\eta(x) = s$. Then $x_s(x) \nleq \phi(x)$. So that $x_s \nleq \phi$ and hence $\overline{x_s} \nleq \phi$.

Therefore, $\phi \in M(R)(\overline{x_s}) = M(R)(x_s)$. Now,

$$\begin{aligned}\psi \in N(\eta) &\Rightarrow \eta \leq \psi\\ &\Rightarrow \eta(x) \leq \psi(x)\\ &\Rightarrow s \leq \psi(x)\\ &\Rightarrow x_s(x) \leq \psi(x)\\ &\Rightarrow x_s \leq \psi\\ &\Rightarrow \psi \in N(x_s).\end{aligned}$$

Therefore, $N(\eta) \subseteq N(x_s)$. Thus, $M(R)(x_s) \subseteq M(R)(\eta)$.

Definition 4.6 By the above theorem, the class $\{M(R)(x_s): x \in R \text{ and } s \in L\}$ form a base for the topology on $M(R)$ is called the fuzzy stone topology and $M(R)$ together with the fuzzy stone topology is called the fuzzy stone space of R.

Theorem 4.7 Let t be a prime element in L. Then $N^t = \{\eta \in M(R): \text{Image of } \eta = \{1, t\}\}$ is compact.

Proof. As $\{M(R)(x_s): x \in R \text{ and } 0 \neq s \in L\}$ is a base for the topology τ on $M(R)$, it can be easily seen that the family $\{M(R)(x_s) \cap N^t: x \in R \text{ and } s \nleq t\}$ constitute a base for the subspace topology on N. Now, let $\{M(R)\big((x_i)_{s_i}\big) \cap N^t: i \in \triangle, x_i \in R \text{ and } s \nleq t\}$ be an open cover of N^t. Let $s = \vee\{s_i: i \in \triangle\}$. Then the family $\{M(R)((x_i)_s) \cap N^t: i \in \triangle, x_i \in R \text{ and } s \nleq t\}$ also covers N^t. Now,

$$\begin{aligned}N^t &= \cup\{M(R)((x_i)_s) \cap N^t: i \in \triangle, x_i \in R \text{ and } s \nleq t\}\\ &= (\cup\{M(R)((x_i)_s): i \in \triangle, x_i \in R \text{ and } s \nleq t\}) \cap N^t\\ &= (M(R) - N(\cup\{(x_i)_s: i \in \triangle, x_i \in R \text{ and } s \nleq t\})) \cap N^t\\ &= N^t - N(\cup\{(x_i)_s: i \in \triangle, x_i \in R \text{ and } s \nleq t\}) \cap N^t).\end{aligned}$$

So that $N(\cup\{(x_i)_s: i \in \triangle, x_i \in R \text{ and } s \nleq t\}) \cap N^t$ is empty set. Let F be a prime filter of R and define a fuzzy minimal prime filter η of R by $\eta = t^F$. So that, $\text{Im}\, \eta = \{1, t\}$ and hence $\eta \in N^t$. Clearly, $\eta \notin N(\cup\{(x_i)_s: i \in \triangle, x_i \in R \text{ and } s \nleq t\})$. Hence there exists $i \in \triangle$ such that $(x_i)_s \nsubseteq \eta$ and hence $s \nleq \eta(x_i)$. Consequently, $x_i \notin F$. Hence there is no prime ideal of R containing the set $\{x_i\}_{i \in \triangle}$ in R. Put $Q = \{x_i \in R: i \in \triangle\}$. Then $\langle Q] = R$. In particular, $u \in \langle Q]$, where n is a maximal element in R and $u = (\vee_{i=1}^{n} x_i) \wedge y$, for some $x_1, x_2, \dots, x_n \in Q$ and $y \in R$. Now, $N(\cup\{(x_i)_s: i = 1, \dots, n \text{ and } s \nleq t\}) \cap N^t = \emptyset$. Thus, $\eta \notin N(\cup\{(x_i)_s: i = 1, \dots, n \text{ and } s \nleq t\})$. This show that $\eta \in M(R)(\cup\{(x_i)_s: i = 1, \dots, n \text{ and } s \nleq t\})$ and hence $j \in \{1,2, \dots, n\}$ such that $(x_j)_s \nleq \eta$ and hence $s \nleq \eta(x_j)$. So that $x_j \notin F$. Thsi shows that $\eta \in M(R)((x_j)_s)$ and hence $x \in \bigcup_{j=1}^{n} M(R)((x_j)_s)$. Therefore, $N^t \subseteq \bigcup_{j=1}^{n}\{M(R)((x_j)_s) \cap N^t: s \nleq t\}$. Thus, $\bigcup_{j=1}^{n}\{M(R)((x_j)_s) \cap N^t: s \nleq t\}$ covers N^t. Hence N^t is compact.

Let us recall that a topological space X is said to be T0- space if, for any $x \neq y \in X$ there exists an open set containing x and not containing y or vice-versa. By using this, we prove the following result.

Theorem 4.8 For any topological space $M(R)$, we have the space $M(R)$ is a T0-space
$N(\phi) = \overline{\{\phi\}}$, where $\overline{\{\phi\}}$ is the closure of ϕ in $M(R)$.

Proof. (1). Let x_t and $y_s \in M(R)$ such that $x_t \neq y_s$, for all $t, s \in L - \{0\}$. Then either $x_t \nleq y_s$ or $y_s \nleq x_t$. Thus, $y_s \in M(R)(x_t)$. Also, $x_t \notin M(R)(x_t)$ and hence $M(R)(x_t)$ is open set in $M(R)$. Therefore, $M(R)$ is T_0.

(2). Let $\phi \in M(R)$. Then $N(\phi)$ is a closed set in $M(R)$ containing ϕ and hence $\overline{\{\phi\}} \subseteq N(\phi)$. On the other inequality, if $\eta \notin \overline{\{\phi\}}$, then there exists an open set $M(R) - N(x_t) = M(R)(x_t)$ such that $\eta \in M(R) - N(x_t)$ but $\phi \notin M(R) - N(x_t)$. Therefore, $x_t \nleq \eta$ but $x_t \leq \phi$ and hence $\eta \notin N(\phi)$. Thus, $N(\phi) \subseteq \overline{\{\phi\}}$. Hecne the result.

5. Conclusion

In this work, fuzzy prime filters, fuzzy minimal prime filters, and the topological space of these ideas are discussed and different findings are obtained. In the future, we plan to investigate O-fuzzy prime filters and related ideas.

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