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Some Properties of Generalized Jacobsthal-Like Sequences

Genelleştirilmiş Jacobsthal-Benzeri Dizilerin Bazı Özellikleri

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Abstract

In this article, using Jacobsthal and Jacobsthal-Lucas sequences, we define generalized Jacobsthal-Like sequences and investigate their algebraic properties like Binet's formula, generating functions, Simson formula and summation formula. We also prove some other summation formulas like sum of even and odd indices and alternating sum of generalized Jacobsthal-Like sequences.

Keywords: Generalized Jacobsthal-Like sequences, Jacobsthal sequence, Jacobsthal-Lucas sequences.

Öz

Bu makalede Jacobsthal ve Jacobsthal-Lucas dizilerini kullanarak genelleştirilmiş Jacobsthal-Benzeri dizilerini tanımlayıp Binet formülü, üreten fonksiyonlar, Simson formülü ve toplam formülü gibi cebirsel özelliklerini araştırıyoruz. Ayrıca çift ve tek indekslerin toplamı ve genelleştirilmiş Jacobsthal-Benzeri dizilerinin alterne toplamı gibi diğer toplama formüllerini de kanıtlıyoruz.

Anahtar Kelimeler: Genelleştirilmiş Jacobsthal-benzeri diziler, Jacobsthal dizisi, Jacobsthal-Lucas dizileri.

1. Introduction

For many years, extensive studies have been conducted on generalized Fibonacci-Like sequences, exploring both their characteristics and preliminary results. (Harne et al. 2014, Gupta et al. 2014, Singh et al. 2014). Using these studies on generalized Fibonacci-Like sequences, we can extend these studies to the generalized Jacobsthal-Like and other interesting sequences in a similar fashion.

Every term in the Jacobsthal sequence can be determined recursively with the initial values $J_0=0, J_1=1$. Similar is the case with Jacobsthal-Lucas sequence. See (Horadam 1996). The definitions of Jacobsthal-Like sequences associated with Jacobsthal and Jacobsthal-Lucas sequences can be found in the papers (Natividad 2016, Pakapongpun 2020).

The sequence of Jacobsthal numbers $\{J_n\}$ is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2, \quad J_0 = 0, J_1 = 1. \tag{1}$$

The sequence of Jacobsthal-Lucas numbers $\{J_n\}$ is defined by

$$j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2, \;\; j_0 = 2, j_1 = 1. \tag{2}$$

The Binet's formula for Jacobsthal sequence is given by

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{3}\left[(2^n - (-1)^n)\right] \tag{3}$$

where $\alpha = 2$ and $\beta = -1$.

Similarly, the Binet's formula for Jacobsthal-Lucas sequence is given by

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n. \tag{4}$$

In this paper, we present various properties of the generalized Jacobsthal-Like sequence defined by

$$V_n = V_{n-1} + 2V_{n-2}, \quad n \geq 2 \tag{5}$$

with $V_0 = 2$ and $V_1 = 1 + m$, m being a fixed positive integer.

Here the initial conditions V_0 and V_1 are the sum of m times the initial conditions of Jacobsthal sequence and the initial conditions of Jacobsthal-Lucas sequence respectively.

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The relation between Jacobsthal sequence and generalized Jacobsthal-Like sequence can be written as

$$V_n = mJ_n + j_n, \quad n \geq 0. \tag{6}$$

Then, the terms of the sequence $\{V_n\}$ are given by

$\{V_n\} = \{2, 1+m, 5+m, 7+3m, 17+5m, 31+11m, \ldots\}$.

2. Preliminary results of Generalized Jacobsthal-Like sequence

The first, we introduce some basic results of generalized Jacobsthal-Like sequence and Jacobsthal sequence.

The corresponding characteristic equation of relation (1) is

$$x^2 - x - 2 = 0 \text{ and its roots are } \alpha = 2 \text{ and } \beta = -1. \tag{7}$$

Using these two roots, we obtain Binet's formula of recurrence relation (5)

$$V_n = \frac{m}{3}(\alpha^n - \beta^n) + (\alpha^n + \beta^n)$$
$$= \frac{m}{3}(2^n - (-1)^n) + (2^n + (-1)^n).$$

Generating function of $\{V_n\}$ is defined as

$$\sum_{k=0}^{\infty} V_k x^k = \frac{2 + (m-1)x}{1 - x - 2x^2}. \tag{8}$$

3. Properties of Generalized Jacobsthal-Like sequence

Of the generalized Jacobsthal-Like sequence $\{V_n\}$, like generalized Fibonacci-Like sequences have many interesting properties (Benjamin and Quinn 1999, Lee and Z. Lee 1987, Badshah et al. 2012, Harne et al. 2014, Singh et al. 2010, Soykan et al. 2018, Soykan and Göcen 2022).

Sums of generalized Jacobsthal-Like terms can be given in the following theorems.

Theorem 1. Sum of the first terms of the generalized Jacobsthal-Like sequence $\{V_n\}$ is

$$V_1 + V_2 + V_3 + \cdots + V_n = \sum_{k=1}^{n} V_k = \frac{V_{n+2} - V_2}{2}. \tag{9}$$

This identity becomes

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = \sum_{k=1}^{2n} V_k = \frac{V_{2n+2} - V_2}{2}. \tag{10}$$

Proof. We know that the following relations hold:

$$2V_1 = V_3 - V_2,$$
$$2V_2 = V_4 - V_3,$$
$$2V_3 = V_5 - V_4,$$
$$\vdots$$
$$2V_{n-1} = V_{n+1} - V_n,$$
$$2V_n = V_{n+2} - V_{n+1}.$$

Term wise addition of all above equations, we obtain

$$2(V_1 + V_2 + V_3 + \cdots + V_n) = V_{n+2} - V_2,$$
$$V_1 + V_2 + V_3 + \cdots + V_n = \frac{V_{n+2} - V_2}{2}.$$

Theorem 2. Sum of the first terms of the generalized Jacobsthal-Like sequence $\{V_n\}$ is

$$V_1 + V_2 + V_3 + \cdots V_{2n} = V_{2n+1} - V_1.$$

Proof.

$$V_2 = V_3 - 2V_1,$$
$$V_4 = V_5 - 2V_3,$$
$$V_6 = V_7 - 2V_5,$$
$$\vdots$$
$$V_{2n-2} = V_{2n-1} - 2V_{2n-3},$$
$$V_{2n} = V_{2n+1} - 2V_{2n-1}.$$

Term wise addition of all above equations, we obtain

$$V_2 + V_4 + V_6 + \cdots + V_{2n} = -(V_1 + V_3 + \cdots + V_{2n-1}) + V_{2n+1} - V_1. \tag{12}$$

Adding odd indices to the both sides of the equation,we have

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = V_{2n+1} - V_1.$$

Theorem 3. Sum of the first $2n-1$ terms of the generalized Jacobsthal-Like sequence $\{V_n\}$ is

$$V_0 + V_1 + V_2 + V_3 + \cdots + V_{2n-1} = V_{2n} - V_0. \tag{13}$$

Proof.

$$V_1 = V_2 - 2V_0,$$
$$V_3 = V_4 - 2V_2,$$
$$V_5 = V_6 - 2V_4,$$
$$\vdots$$
$$V_{2n-3} = V_{2n-2} - 2V_{2n-4},$$
$$V_{2n-1} = V_{2n} - 2V_{2n-2}.$$

Term wise addition of all above equations, we obtain

$$V_1 + V_3 + V_5 + \cdots + V_{2n-1} = -(V_0 + V_2 + \cdots + V_{2n-2}) + V_{2n} - V_0,$$
$$V_0 + V_1 + V_2 + V_3 + \cdots + V_{2n-1} = V_{2n} - V_0.$$

We state and prove the following identity for the generalized Jacobsthal-Like sequence $\{V_n\}$

Lemma 4. For every positive integer , we have

$$2V_{2n} - V_{2n+1} = 3 - m. \tag{14}$$

Proof. Combining (10) and (11) and putting $V_1 = 1 + m, V_2 = 5 + m$, we obtain

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = \sum_{k=1}^{2n} V_k = \frac{V_{2n+2} - (5+m)}{2}$$
$$= V_{2n+1} - (1+m),$$
$$V_{2n+2} - (5+m) = 2V_{2n+1} - 2(1+m),$$
$$V_{2n+2} - 2V_{2n+1} = 3 - m,$$
$$V_{2n+1} + 2V_{2n} - 2V_{2n+1} = 3 - m,$$
$$2V_{2n} - V_{2n+1} = 3 - m.$$

Theorem 5. Sum of the first $(n+1)$ terms of the generalized Jacobsthal-Like sequence $\{V_n\}$ with odd and even indices are

$$V_1 + V_3 + V_5 + \cdots + V_{2n+1} = \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3}, \tag{15}$$

and

$$V_0 + V_2 + V_4 + \cdots + V_{2n} = \frac{V_{2n+2} + (n+1)(3-m) - 2}{3} \tag{16}$$

respectively.

Proof.

Using (13),

$$V_0 + V_1 + V_2 + \cdots + V_{2n} + V_{2n+1} = V_{2n+2} - 2.$$

For $V_0 + V_2 + V_4 + \cdots + V_{2n-2} + V_{2n} = X,$

$$V_1 + V_3 + V_5 + \cdots + V_{2n-1} + V_{2n+1} = Y$$
$$X + Y = V_{2n+2} - 2. \tag{17}$$

Using (14),

$$\sum_{k=0}^{n} (2V_{2k} - V_{2k+1}) = \sum_{k=0}^{n} (3-m),$$
$$2\sum_{k=0}^{n} V_{2k} - \sum_{k=0}^{n} V_{2k+1} = (n+1)(3-m), \tag{18}$$
$$2X - Y = (n+1)(3-m).$$

Using (17) and (18) we get

$$V_0 + V_2 + V_4 + \cdots + V_{2n} = \frac{V_{2n+2} + (n+1)(3-m) - 2}{3},$$
$$V_1 + V_3 + V_5 + \cdots + V_{2n+1} = \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3}.$$

From the above theorem we can calculate the alternating sum of the first n numbers.

Corollary 6. The alternating sum of the first n numbers of the of the generalized Jacobsthal-Like sequence $\{V_n\}$ yazılabilir.

$$V_0 - V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + (-1)^n V_n$$
$$= \frac{(-1)^n V_{n+1} + (n+1)(3-m) + 2}{3}. \tag{19}$$

Proof. If we subtract equation (15) term wise from equation (16), we get alternating sum of the first $2n$+1 numbers:

$$V_0 + V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + V_{2n} - V_{2n+1}$$
$$= \frac{V_{2n+2} + 5n + 1)(3-m) - 2}{3} - \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3}$$
$$= \frac{-V_{2n+2} + 2(n+1)(3-m) + 2}{3}.$$

If we want to calculate the alternating sum of the first n numbers from the above equation, substituting $2n$+1 by n we get the following result

$$V_0 - V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + (-1)^n V_n$$
$$= \frac{(-1)^n V_{n+1} + (n+1)(3-m) + 2}{3}.$$

Now, some identities for the generalized Jacobsthal-Like sequence $\{V_n\}$ are stated and proven below.

Theorem 7. For every integer $n \geq 0$, for each real coefficient m,

$$mV_{n+2} - mV_{n+1} = 2mV_n. \tag{20}$$

Proof.

$$m(V_{n+2} - V_{n+1}) = m(2V_n) = 2mV_n.$$

Theorem 8. For every integer $n \geq 1$, we have

$$V_n^2 = V_n V_{n+1} - 2V_{n-1} V_n. \tag{21}$$

Proof.

$$V_n V_{n+1} - 2V_{n-1} V_n = V_n (V_{n+1} - 2V_{n-1}) = V_n^2.$$

Theorem 9. (Simson formula) For every integer $n \geq 1$ we have

$$V_{n+1} V_{n-1} - V_n^2 = (-1)^{n+1} 2^{n-1} (9 - m^2). \tag{22}$$

Proof.

We shall use mathematical induction over n.

It is easy to see that for $n = 1$,

$$V_2 V_0 - V_1^2 = (-1)^2 2^0 (9 - m^2)$$
$$2(5+m) - (1+m)^2 = (9 - m^2), \text{ which is true.}$$

Assume that the result is true for $n = k$. Then

$$V_{k+1} V_{k-1} - V_k^2 = (-1)^{k+1} 2^{k-1} (9 - m^2). \tag{23}$$

Multiplying by 2 and adding $V_k V_{k+1}$ to each side of equation (23), we get

$$2V_{k+1}V_{k-1} - 2V_k^2 + V_k V_{k+1} = (-1)^{k+1}2^k(9-m^2) + V_k V_{k+1},$$
$$V_{k+1}(2V_{k-1}+V_k) - V_k(2V_k + V_{k+1}) = (-1)^{k+1}2^k(9-m^2),$$
$$V_{k+1}^2 - V_k V_{k+2} = (-1)^{k+1}2^k(9-m^2),$$
$$-(V_k V_{k+2} - V_{k+1}^2) = (-1)^{k+1}2^k(9-m^2),$$
$$V_{k+2}V_k - V_{k+1}^2 = (-1)^{k+2}2^k(9-m^2).$$

Therefore, the result is also true for $n = k+1$.

Hence, $V_{n+1}V_{n-1} - V_n^2 = (-1)^{n+1}2^{n-1}(9-m^2)$, for every $n \geq 1$

Theorem 10 For every positive integer *n*,

$$V_3 + V_6 + V_9 + \cdots + V_{3n} = \begin{cases} \frac{1}{7}(V_{3n+3} - 16), & \text{if } n \text{ is odd} \\ \frac{1}{7}(V_{3n+3} - V_3), & \text{if } n \text{ is even.} \end{cases} \quad (24)$$

Proof. We use the Binet's formula of generalized Jacobsthal-Like,

$$V_3 + V_6 + V_9 + \cdots + V_{3n}$$
$$= \frac{m}{3}(\alpha^3 - \beta^3) + (\alpha^3 + \beta^3) + \frac{m}{3}(\alpha^6 - \beta^6) + (\alpha^6 + \beta^6)$$
$$+ \frac{m}{3}(\alpha^9 - \beta^9) + (\alpha^9 + \beta^9) + \cdots + \frac{m}{3}(\alpha^{3n} - \beta^{3n}) + (\alpha^{3n} + \beta^{3n})$$
$$= \frac{m}{3}[(\alpha^3 + \alpha^6 + \alpha^9 + \cdots + \alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 + \cdots + \beta^{3n})] +$$
$$[(\alpha^3 + \alpha^6 + \cdots + \alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 + \cdots + \beta^{3n})],$$
$$= \frac{m}{3}\left[\left(\frac{\alpha^{3n+3} - \alpha^3}{\alpha^3 - 1}\right) - \left(\frac{\beta^{3n+3} - \beta^3}{\beta^3 - 1}\right)\right] + \left[\left(\frac{\alpha^{3n+3} - \alpha^3}{\alpha^3 - 1}\right) + \left(\frac{\beta^{3n+3} - \beta^3}{\beta^3 - 1}\right)\right]$$
$$= \frac{m}{3}\left[\left(\frac{2^{3n+3} - 8}{7}\right) - \left(\frac{(-1)^{3n+3} + 1}{-2}\right)\right] + \left[\left(\frac{2^{3n+3} - 8}{7}\right) + \left(\frac{(-1)^{3n+3} + 1}{-2}\right)\right]$$
$$= \begin{cases} \frac{m}{3}\left[\left(\frac{2^{3n+3} - 8}{7}\right) + 1\right] + \left[\left(\frac{2^{3n+3} - 8}{7}\right) - 1\right], & \text{if } n \text{ is odd} \\ \frac{m}{3}\left(\frac{2^{3n+3} - 8}{7}\right) + \frac{2^{3n+3} - 8}{7}, & \text{if } n \text{ is even} \end{cases}$$
$$= \begin{cases} \frac{m}{3}\left(\frac{2^{3n+3} - 1}{7}\right) + \frac{2^{3n+3} - 15}{7}, & \text{if } n \text{ is odd} \\ \frac{1}{7}\left[\frac{m}{3}(2^{3n+3} - (-1)^{3n+3} - 9) + (2^{3n+3} + (-1)^{3n+3} - 7)\right], & \text{if } n \text{ is even} \end{cases}$$
$$= \begin{cases} \frac{m}{3}\left(\frac{2^{3n+3} - (-1)^{3n+3}}{7}\right) + \frac{2^{3n+3} + (-1)^{3n+3} - 16}{7}, & \text{if } n \text{ is odd} \\ \frac{1}{7}[V_{3n+3} - (3m+7)], & \text{if } n \text{ is even} \end{cases}$$
$$= \begin{cases} \frac{1}{7}(V_{3n+3} - 16), & \text{if } n \text{ is odd} \\ \frac{1}{7}(V_{3n+3} - V_3), & \text{if } n \text{ is even.} \end{cases}$$

4. Conclusion

In this article, generalized Jacobsthal-Like sequences are defined and their algebraic properties like Binet's formula, generating functions, Simson formula and the summation formula are investigated. Some other summation formulas like sum of even and odd indices and alternating sum of generalized Jacobsthal-Like sequences are presented.

We believe that the generalized Jacobsthal-Like sequences considered in this article can be extended to generalize other sequences like Pell and Narayana and the results given in this article could be useful for further research on this topic.

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