Universal Journal of Mathematics and Applications, 7 (3) (2024) 102-110
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1443527

LP-Kenmotsu Manifolds Admitting Bach Almost Solitons

Rajendra Prasad[1], Abhinav Verma[1], Vindhyachal Singh Yadav[1], Abdul Haseeb[2*] and Mohd Bilal[3]

[1]*Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226007, India*
[2]*Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, Kingdom of Saudi Arabia*
[3]*Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al Qura University, Makkah 21955, Saudi Arabia*
[*]*Corresponding author*

Article Info

Keywords: *Bach almost solitons, LP-Kenmotsu manifolds, Perfect fluid, Weyl tensor*
2010 AMS: *53C25, 53C44, 53C50*
Received: *27 February 2024*
Accepted: *12 May 2024*
Available online: *25 August 2024*

Abstract

For a Lorentzian para-Kenmotsu manifold of dimension m (briefly, $(LPK)_m$) admitting Bach almost soliton (g,ζ,λ), we explored the characteristics of the norm of Ricci operator. Besides, we gave the necessary condition for $(LPK)_m$ $(m\geq 4)$ admitting Bach almost soliton to be an η-Einstein manifold. Afterwards, we proved that Bach almost solitons are always steady when a Lorentzian para-Kenmotsu manifold of dimension three has Bach almost soliton.

1. Introduction

In 1976, the concept of almost paracontact manifolds was proposed by Sato [1]. An almost paracontact structure on a semi-Riemannian manifold $\mathscr{M}$ was established by Kaneyuki and Kozai in [2]. They created almost paracomplex shape on $\mathscr{M}\times R$. According to Kaneyuki et al. [3], the key variation among an almost paracontact manifold is the signature of metric. In 1995, the authors Sinha and Prasad described para-Kenmotsu as well as special para-Kenmotsu manifolds and found significant properties of para-Kenmotsu manifolds [4]. Afterwards, para-Kenmotsu manifolds drew huge attention and a number of mathematicians brought forward the significant characteristics of such manifolds [5–9].
Semi-Riemannian geometry, used in the relativity theory, was studied in [10]. About four decades ago, Kaigorodov has explored the curvature structure of the spacetime [11]. Raychaudhuri et al. [12] extended the above concepts of the general theory of spacetime. Recently, Haseeb and Rajendra introduced and studied the Lorentzian para-Kenmotsu manifolds [13, 14].
1921 was the year, when Bach initiated Bach tensor [15] to explore conformal geometry. He proved that the Bach tensor is a rank 2 trace-free tensor and is conformally invariant in dimension 4. So, in lieu of Hilbert-Einstein functional, the functional is taken in the following way

$$\mathscr{W}(g)=\int_{\mathscr{M}}\|\mathscr{C}\|_g^2 d\nu_g,$$

where, $\mathscr{M}$ is a manifold of dimension-four and $\mathscr{C}$ repersents the Weyl tensor of type $(1,3)$ given by

$$\begin{aligned}\mathscr{C}(U,\mathscr{V})\mathscr{W}=&\mathscr{R}(U,\mathscr{V})\mathscr{W}+\frac{1}{m-2}[\mathscr{S}(U,\mathscr{W})\mathscr{V}-\mathscr{S}(\mathscr{V},\mathscr{W})U+g(U,\mathscr{W})\mathscr{Q}\mathscr{V}-g(\mathscr{V},\mathscr{W})\mathscr{Q}U]\\&-\frac{r}{(m-1)(m-2)}[g(U,\mathscr{W})\mathscr{V}-g(\mathscr{V},\mathscr{W})U],\end{aligned}\tag{1.1}$$

here, $\mathscr{R}$ represents the Riemannian curvature tensor, $\mathscr{Q}$ is the Ricci operator and $\mathscr{S}$ denotes the Ricci tensor, such that, $g(\mathscr{Q}U,\mathscr{V})=\mathscr{S}(U,\mathscr{V})$, $\forall$ differentiable vector fields $U,\mathscr{V},\mathscr{W}$. Bach tensor of type $(0,2)$ on a semi-Riemannian manifold $(\mathscr{M}^m,g)$ of dimension $m(\geq 3)$ is given by

$$\begin{aligned}\mathscr{B}(U,\mathscr{V})=&\frac{1}{(m-3)}\sum_{i\in\{1,\dots,m\}}\sum_{j\in\{1,\dots,m\}}\varepsilon_i\varepsilon_j(\nabla_{\mathscr{E}_i}\nabla_{\mathscr{E}_j}\mathscr{C}')(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V})\\&+\frac{1}{(m-2)}\sum_{i\in\{1,\dots,m\}}\sum_{j\in\{1,\dots,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V}),\end{aligned}\tag{1.2}$$

Email addresses and ORCID numbers: **rp.lucknow@rediffmail.com, 0000-0002-7502-0239 (R. Prasad), vabhinav831@gmail.com, 0009-0004-7998-5224 (A. Verma), vs.yadav4@gmail.com, 0009-0009-2810-2723 (V. S. Yadav), haseeb@jazanu.edu.sa, malikhaseeb80@gmail.com, 0000-0002-1175-6423 (A. Haseeb), mohd7bilal@gmail.com, 0000-0002-6789-2678 (M. Bilal)**
Cite as: R. Prasad, A. Verma, V. S. Yadav, A. Haseeb, M. Bilal, *LP-Kenmotsu manifolds admitting bach almost solitons*, Univers. J. Math. Appl., 7(3) (2024), 102-110.

here, $g(\mathscr{E}_i,\mathscr{E}_i)=\varepsilon_i$, $g(\mathscr{C}(U,\mathscr{V})\mathscr{W},\mathscr{Y})=\mathscr{C}'(U,\mathscr{V},\mathscr{W},\mathscr{Y})$ and $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ is a local orthonormal frame at each point p of $T_p\mathscr{M}$. Relation (1.1), together with contracting Bianchi second identity, we obtain

$$div\mathscr{C}=\frac{(m-3)}{(m-2)}C_0, \tag{1.3}$$

where, C_0 is Cotton tensor [16] given by

$$C_0(U,\mathscr{V})\mathscr{W}=-(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W})+(\nabla_U\mathscr{S})(\mathscr{V},\mathscr{W})+\frac{1}{2(m-1)}[(\mathscr{V}r)g(U,\mathscr{W})-(Ur)g(\mathscr{V},\mathscr{W})]. \tag{1.4}$$

In view of equation (1.3), together with equation (1.2), the Bach tensor takes the form,

$$\mathscr{B}(U,\mathscr{V})=\frac{1}{(m-2)}[\sum_{i\in\{1,\ldots,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}+\sum_{i\in\{1,\ldots,m\}}\sum_{j\in\{1,\ldots,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V})], \tag{1.5}$$

$\forall$ differentiable vector fields $U,\mathscr{V}$. For dimension three, the Weyl tensor vanishes. Therefore, Bach tensor given in equation (1.5) reduces to

$$\mathscr{B}(U,\mathscr{V})=\sum_{i\in\{1,2,3\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}. \tag{1.6}$$

For further study, the references [17–24] may be seen.
In 2012, Das and Kar [25] studied different characteristics of Bach flow on product manifolds and analysed their outcomes with the Ricci flow. Bach flow is suggested in [26] to specify the Harava-Lifschitz gravity in general relativity. In 2011, Bahuaud and Helliwell in [27] studied the presence of Bach flow for short time. Cao and Chen, in the year 2013, explored Bach flat Ricci solitons [28]. Subsequently, Ho [29] worked comprehensively on the solitons of Bach flow. He also studied the Bach flows on Lie group of dimension 4. In 2020, Helliwell specified Bach flow of dimension 4 on locally homogeneous product manifolds [30]. In recent times, Ghosh [31] investigated the Bach almost solitons (g,ζ,λ) in semi-Riemannian geometry and is given by

$$(\pounds_{\mathscr{X}}g+2\mathscr{B}-2\lambda g)(U,\mathscr{V})=0, \tag{1.7}$$

here, $\pounds_{\mathscr{X}}$ is the Lie derivative operator along $\mathscr{X}$; $\mathscr{X}$ is a potential vector field and $\lambda\in C^\infty(\mathscr{M}^m)$. The Bach almost solitons (g,ζ,λ) is said to be expanding, steady and shrinking according to $\lambda<0$, $\lambda=0$ and $\lambda>0$, respectively.
This article is organized in the following manner: Section 1 contains introduction, based on development of almost paracontact manifold and other concepts. Preliminaries are given in Section 2, based on $(LPK)_m$. Section 3 contains the work on (g,ζ,λ) in $(LPK)_m$. In Section 4, we examine $(LPK)_m$ of dimension 3, which admits Bach almost solitons.

2. Preliminaries

An m-dimensional smooth manifold $\mathscr{M}^m$ is called Lorentzian almost paracontact manifold, if it is equipped with a (1,1)-tensor field ϕ, a contravariant vector field ζ, a 1-form η and a Lorentzian metric g of type (0, 2). The following relations for an m-dimensional Lorentzian metric manifold hold [32],

$$\phi^2(U)=U+\eta(U)\zeta,\ \ \eta(\zeta)+1=0, \tag{2.1}$$

$$g(U,\zeta)=\eta(U),\ \ g(\phi U,\phi\mathscr{V})=\eta(U)\eta(\mathscr{V})+g(U,\mathscr{V}), \tag{2.2}$$

$\forall\, U,\mathscr{V}$ on $\mathscr{M}^m$, and the structure (ϕ,ζ,η,g) is named the Lorentzian almost paracontact structure. An $\mathscr{M}^m$ endowed with (ϕ,ζ,η,g) is known as Lorentzian almost paracontact manifold and holding the following results:

$$\phi\zeta=0,\ \ \eta(\phi U)=0,\ \ \Omega(U,\mathscr{V})=\Omega(\mathscr{V},U), \tag{2.3}$$

here, $\Omega(U,\mathscr{V})=g(U,\varphi\mathscr{V})$.

Definition 2.1. *A Lorentzian almost paracontact manifold $\mathscr{M}^m$ is known as $(LPK)_m$ if*

$$(\nabla_U\phi)(\mathscr{V})=-\eta(\mathscr{V})\phi U-g(\phi U,\mathscr{V})\zeta,$$

$\forall\, U$ and $\mathscr{V}$ on $\mathscr{M}^m$.

Further, for $(LPK)_m$, following results hold good:

$$\nabla_U\zeta+U+\eta(U)\zeta=0, \tag{2.4}$$

$$(\nabla_U\eta)(\mathscr{V})+g(U,\mathscr{V})+\eta(U)\eta(\mathscr{V})=0, \tag{2.5}$$

$$\mathscr{R}(U,\mathscr{V})\zeta=\eta(\mathscr{V})U-\eta(U)\mathscr{V}, \tag{2.6}$$

$$\mathscr{R}(\zeta,\mathscr{V})U = g(U,\mathscr{V})\zeta - \eta(U)\mathscr{V}, \tag{2.7}$$

$$\mathscr{R}(\zeta,U)\zeta = U + \eta(U)\zeta, \tag{2.8}$$

$$\mathscr{S}(U,\zeta) = (m-1)\eta(U), \tag{2.9}$$

$$\mathscr{Q}\zeta = (m-1)\zeta, \tag{2.10}$$

$$\mathscr{S}(\phi\mathscr{V},\phi U) = \mathscr{S}(\mathscr{V},U) + (m-1)\eta(\mathscr{V})\eta(U), \tag{2.11}$$

$\forall\ U,\mathscr{V},\mathscr{W}$ on $(LPK)_m$ [33, 34]. In the above results, ∇ represents the covariant differentiation operator w.r.t. g in semi-Riemannian manifolds.

Proposition 2.2. *We assume $\mathscr{M}$ to be an $(LPK)_m$. Subsequently, we have*

$$\mathscr{S}(\phi U,\mathscr{V}) = \mathscr{S}(U,\phi\mathscr{V}), \tag{2.12}$$

$\forall\ U,\ \mathscr{V}$ on $(LPK)_m$.

Proof. Setting ϕU for U in (2.11), we get,

$$\mathscr{S}(\phi^2U,\phi\mathscr{V}) = \mathscr{S}(\phi U,\mathscr{V}) + (m-1)\eta(\phi U)\eta(\mathscr{V}).$$

Using equations (2.1) and (2.3) in the foregoing equation, we yield

$$\mathscr{S}(U+\eta(U)\zeta,\phi\mathscr{V}) = \mathscr{S}(\phi U,\mathscr{V}). \tag{2.13}$$

From equation (2.13), the Proposition 2.2 follows. □

3. Bach Almost Solitons and $(LPK)_m$

Definition 3.1. *A semi-Riemannnian manifold is called Bach perfect fluid if Bach almost tensor is given by*

$$\mathscr{B}(U,\mathscr{V}) = \beta\eta(U)\eta(\mathscr{V}) + \alpha g(U,\mathscr{V}), \quad \forall\mathscr{V},U,$$

where, α and β are scalars.

Let $(LPK)_m$ admit (g,ζ,λ). Then (1.7) holds and thus, we have

$$(\pounds_\zeta g)(U,\mathscr{V}) + 2\mathscr{B}(U,\mathscr{V}) = 2\lambda g(U,\mathscr{V}). \tag{3.1}$$

As we have

$$(\pounds_\zeta g)(U,\mathscr{V}) = g(\nabla_U\zeta,\mathscr{V}) + g(U,\nabla_{\mathscr{V}}\zeta). \tag{3.2}$$

The result (2.4), together with (3.2) yields

$$(\pounds_\zeta g)(U,\mathscr{V}) + 2[g(U,\mathscr{V}) + \eta(U)\eta(\mathscr{V})] = 0. \tag{3.3}$$

Putting the preceding result (3.3) in (3.1), we lead to

$$\mathscr{B}(\mathscr{V},U) = (1+\lambda)g(\mathscr{V},U) + \eta(\mathscr{V})\eta(U). \tag{3.4}$$

Result (3.4) shows the succeeding proposition:

Proposition 3.2. *An $(LPK)_m$ admitting a Bach almost soliton (g,ζ,λ) is Bach perfect fluid.*

Replacing $\mathscr{W}$ by ζ in (1.1), we have

$$\begin{aligned}\mathscr{C}(U,\mathscr{V})\zeta = \mathscr{R}(U,\mathscr{V})\zeta &+ \frac{1}{(m-2)}[\mathscr{S}(U,\zeta)\mathscr{V} - \mathscr{S}(\mathscr{V},\zeta)U + g(U,\zeta)\mathscr{Q}\mathscr{V} - g(\mathscr{V},\zeta)\mathscr{Q}U] \\ &- \frac{r}{(m-1)(m-2)}[g(U,\zeta)\mathscr{V} - g(\mathscr{V},\zeta)U],\end{aligned} \tag{3.5}$$

$\forall$ differentiable vector fields $U,\mathscr{V}$. Operating $\mathscr{Q}$ in (3.5) and using relations (2.2), (2.6), (2.7) and (2.10), we get

$$\mathscr{Q}(\mathscr{C}(U,\mathscr{V})\zeta) = \frac{(r-m+1)}{(m-1)(m-2)}[-\eta(U)\mathscr{Q}\mathscr{V} + \eta(\mathscr{V})\mathscr{Q}U] - \frac{1}{(m-2)}[\eta(\mathscr{V})\mathscr{Q}^2U - \eta(U)\mathscr{Q}^2\mathscr{V}]. \tag{3.6}$$

The inner product of (3.6) with $\mathscr{X}$ leads to

$$g(\mathscr{Q}(\mathscr{C}(U,\mathscr{V})\zeta),\mathscr{X}) = \frac{(r-m+1)}{(m-1)(m-2)}[\eta(\mathscr{V})g(\mathscr{Q}U,\mathscr{X})-\eta(U)g(\mathscr{Q}\mathscr{V},\mathscr{X})] - \frac{1}{(m-2)}[\eta(\mathscr{V})g(\mathscr{Q}^2U,\mathscr{X})-\eta(U)g(\mathscr{Q}^2\mathscr{V},\mathscr{X})]. \tag{3.7}$$

Let $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ be an orthonormal frame at each point p of $T_p\mathscr{M}$. Now, setting $\mathscr{V}=\mathscr{X}=\mathscr{E}_i$ in (3.7) with summation $i=1$ to m and on evaluation, we get

$$\sum_{i\in\{1,..,m\}} \varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\zeta),\mathscr{E}_i) = -\frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U)+\frac{1}{(m-2)}[|\mathscr{Q}|^2-(m-1)^2]\eta(U). \tag{3.8}$$

Setting ζ in place of $\mathscr{W}$ in relation (1.4) gives

$$C_0(U,\mathscr{V})\zeta = g((\nabla_U\mathscr{Q})\mathscr{V},\zeta)-g((\nabla_{\mathscr{V}}\mathscr{Q})U,\zeta)-\frac{1}{2(m-1)}[U(r)\eta(\mathscr{V})-\mathscr{V}(r)\eta(U)]. \tag{3.9}$$

From equation (2.12), we have the relation

$$\phi\mathscr{Q}U=\mathscr{Q}\phi U. \tag{3.10}$$

From the equation (3.10), we also have

$$g((\nabla_U\mathscr{Q})\mathscr{V},\zeta)=g(\mathscr{Q}U,\mathscr{V})-(m-1)g(U,\mathscr{V}). \tag{3.11}$$

Applying above equation (3.11) in (3.9), it gives

$$C_0(U,\mathscr{V})\zeta = -\frac{1}{2(m-1)}[U(r)\eta(\mathscr{V})-\mathscr{V}(r)\eta(U)]. \tag{3.12}$$

After differentiating covariantly the above relation w.r.t. $\mathscr{W}$ and using the relation (2.5), we obtain

$$(\nabla_{\mathscr{W}}C_0)(U,\mathscr{V})\zeta = -(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W})+(\nabla_U\mathscr{S})(\mathscr{V},\mathscr{W}) - \frac{1}{2(m-1)}[g(\nabla_{\mathscr{W}}\mathscr{D}r,U)\eta(\mathscr{V})-g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(U)], \tag{3.13}$$

here $\mathscr{D}$ represents the gradient operator. Let $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ be the orthonormal frame at each point p of $T_p\mathscr{M}$. Replacing $U=\mathscr{W}=\mathscr{E}_i$ with summation over $i=1$ to m in equation (3.13), this gives

$$\sum_{i\in\{1,\ldots,m\}} \varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,\mathscr{V})\zeta = -\frac{1}{2(m-1)}[(div\mathscr{D}r)\eta(\mathscr{V})-g(\nabla_\zeta\mathscr{D}r,\mathscr{V})]-\frac{\mathscr{V}(r)}{2}. \tag{3.14}$$

Now, by rewriting the equation (1.5), we have

$$\mathscr{B}(U,\mathscr{V}) = \frac{1}{(m-2)}[\sum_{i\in\{1,\ldots,m\}} \varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}+\sum_{i\in\{1,\ldots,m\}}\sum_{j\in\{1,\ldots,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V})]. \tag{3.15}$$

After evaluation, the second term of the above equation takes the form

$$\begin{aligned}\sum_{i\in\{1,\ldots,m\}}\sum_{j\in\{1,\ldots,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V}) &= -\sum_{i\in\{1,\ldots,m\}}\sum_{j\in\{1,\ldots,m\}}\varepsilon_i\varepsilon_j g(\mathscr{Q}\mathscr{E}_i,\mathscr{E}_j)g(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V},\mathscr{E}_j),\\ &= -\sum_{i\in\{1,\ldots,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V}),\mathscr{E}_i).\end{aligned}$$

Taking the above equation and equation (3.15) together, we obtain

$$\mathscr{B}(U,\mathscr{V}) = \frac{1}{(m-2)}[\sum_{i\in\{1,\ldots,m\}} \varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}-\sum_{i\in\{1,\ldots,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V}),\mathscr{E}_i)]. \tag{3.16}$$

Replacing $\mathscr{V}$ for ζ in the above relation (3.16), it gives

$$\mathscr{B}(U,\zeta) = \frac{1}{(m-2)}[\sum_{i\in\{1,\ldots,m\}} \varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\zeta-\sum_{i\in\{1,\ldots,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\zeta),\mathscr{E}_i)]. \tag{3.17}$$

Equations (3.8), (3.14) and (3.17) taken together give

$$\mathscr{B}(U,\zeta) = \frac{1}{(m-2)}[-\frac{U(r)}{2}-\frac{1}{2(m-1)}\{(div\mathscr{D}r)\eta(U)-g(\nabla_\zeta\mathscr{D}r,U)\} + \frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U)-\frac{1}{(m-2)}\{|\mathscr{Q}|^2-(m-1)^2\}\eta(U)]. \tag{3.18}$$

Setting $\mathscr{V}$ for ζ in equation (3.4), we get

$$\mathscr{B}(U,\zeta) = \lambda\eta(U). \tag{3.19}$$

Relation (3.18) and (3.19), taken together give

$$\begin{aligned}\lambda\eta(U) = &\frac{1}{(m-2)}[-\frac{U(r)}{2} - \frac{1}{2(m-1)}\{(div\mathscr{D}r)\eta(U) - g(\nabla_\zeta \mathscr{D}r, U)\} \\ &+ \frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U) - \frac{1}{(m-2)}\{|\mathscr{Q}|^2 - (m-1)^2\}\eta(U)].\end{aligned} \tag{3.20}$$

Setting U for ϕU in relation (3.20), we obtain

$$\frac{1}{(m-2)}[-\frac{\phi U(r)}{2} + \frac{1}{2(m-1)} g(\nabla_\zeta \mathscr{D}r, \phi U)] = 0.$$

This implies that

$$g(\nabla_\zeta \mathscr{D}r, \phi U) = (m-1)g(\mathscr{D}r, \phi U).$$

This gives

$$\phi\nabla_\zeta \mathscr{D}r = (m-1)\phi\mathscr{D}r. \tag{3.21}$$

Taking covariant differentiation of equation (2.10) w.r.t. U and using the relations (2.3) and (2.4), we get

$$(\nabla_U \mathscr{Q})\zeta = \mathscr{Q}U - (m-1)U. \tag{3.22}$$

Contracting the preceding equation w.r.t. U, we have

$$\sum_{i\in\{1,\ldots,m\}} \varepsilon_i g(\nabla_{\mathscr{E}_i}\mathscr{Q})\zeta, \mathscr{E}_i) = \sum_{i=1}^{m} \varepsilon_i [g(\mathscr{Q}\mathscr{E}_i, \mathscr{E}_i) - (m-1)g(\mathscr{E}_i, \mathscr{E}_i)].$$

or,

$$(div\mathscr{Q})\zeta = r - (m-1)m,$$

or,

$$\zeta(r) = 2[r - m(m-1)], \tag{3.23}$$

which can be written as

$$\pounds_\zeta r = 2r - 2m(m-1).$$

Applying the exterior derivative in the above relation, we have

$$d\pounds_\zeta r = 2dr.$$

Since, d and the Lie derivative commutes, therefore, we have

$$\pounds_\zeta dr = 2dr.$$

Writing the above relation in the form of gradient operator, we have

$$\pounds_\zeta \mathscr{D}r = 2\mathscr{D}r,$$

or,

$$\nabla_\zeta \mathscr{D}r - \nabla_{\mathscr{D}r}\zeta = 2\mathscr{D}r.$$

Using the relation (2.4) in the above relation, we lead to

$$\nabla_\zeta \mathscr{D}r = \mathscr{D}r - \zeta(r)\zeta. \tag{3.24}$$

Applying ϕ in the above relation (3.24) and using the relations in (2.3) and (3.21), we get

$$\phi\mathscr{D}r = 0.$$

This implies

$$\mathscr{D}r = -\zeta(r)\zeta. \tag{3.25}$$

Differentiating (3.25) covariantly w.r.t. $\mathscr{X}$, it yields

$$\nabla_\mathscr{X} \mathscr{D}r = -[g(\nabla_\mathscr{X} \mathscr{D}r, \zeta)\zeta - g(\mathscr{D}r, \mathscr{X})\zeta - g(\mathscr{D}r, \zeta)\mathscr{X} - 2g(\mathscr{D}r, \zeta)\eta(\mathscr{X})\zeta], \tag{3.26}$$

which by contracting over $\mathscr{X}$ gives

$$(div\mathscr{D}r) = (m-3)\zeta(r). \tag{3.27}$$

Relations (3.24) and (3.25) give

$$\nabla_{\zeta}\mathscr{D}r = -2\zeta(r)\zeta. \tag{3.28}$$

Using relations (3.25), (3.27) and (3.28) in (3.20), we obtain

$$\begin{aligned}\lambda\eta(U) = \frac{1}{(m-2)}[\frac{\zeta(r)}{2}\eta(U) - \frac{1}{2(m-1)}\{(m-3)\zeta(r)\eta(U) + 2\zeta(r)\eta(U)\} \\ + \frac{(r+1-m)^2}{(m-1)(m-2)}\eta(U) - \frac{1}{(m-2)}\{|\mathscr{Q}|^2 - (m-1)^2\}\eta(U)].\end{aligned} \tag{3.29}$$

On simplification, relation (3.29) gives

$$\lambda = \frac{1}{(m-2)^2}[\frac{(r+1-m)^2}{(m-1)} + (m-1)^2 - |\mathscr{Q}|^2]. \tag{3.30}$$

In the light of the relation (3.30), succeeding theorem holds:

Theorem 3.3. *The Bach almost solitons (g,ζ,λ) on an $(LPK)_m$ are expanding, steady and shrinking according as*

$$[\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] > |\mathscr{Q}|^2,\ [\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] = |\mathscr{Q}|^2 \ \text{and}\ [\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] < |\mathscr{Q}|^2.$$

Consider a Lorentzian para-Kenmotsu space form of m-dimension. Then by relation (3.23), we have $r = m(m-1)$. Hence,

$$\lambda = \frac{1}{(m-2)^2}[m(m-1)^2 - |\mathscr{Q}|^2].$$

The above relation leads the following corollary:

Corollary 3.4. *The Bach almost solitons (g,ζ,λ) on an LP-Kenmotsu space form of dimension m is expanding, steady and shrinking according as $m(m-1)^2 > |\mathscr{Q}|^2$, $m(m-1)^2 = |\mathscr{Q}|^2$ and $m(m-1)^2 < |\mathscr{Q}|^2$.*

Definition 3.5. *An $(LPK)_m$ is named η-Einstein if its $\mathscr{S}$ satisfies [35]*

$$\mathscr{S}(\mathscr{V},U) = ag(\mathscr{V},U) + b\eta(\mathscr{V})\eta(U),$$

$\forall\ \mathscr{V},U$, where, a and b are scalars.

Now, replacing $U=\zeta$ in relation (3.13), we have

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = -(\nabla_{\mathscr{V}}\mathscr{S})(\zeta,\mathscr{W}) + (\nabla_{\zeta}\mathscr{S})(\mathscr{V},\mathscr{W}) - \frac{1}{2(m-1)}[g(\nabla_{\mathscr{W}}\mathscr{D}r,\zeta)\eta(\mathscr{V}) - g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(\zeta)]. \tag{3.31}$$

Taking the inner product of relation (3.26) with $\mathscr{V}$ and replacing $\mathscr{X}$ by $\mathscr{W}$, we obtain

$$g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V}) = -[g(\nabla_{\mathscr{W}}\mathscr{D}r,\zeta)\eta(\mathscr{V}) - g(\mathscr{D}r,\mathscr{W})\eta(\mathscr{V}) - g(\mathscr{D}r,\zeta)g(\mathscr{V},\mathscr{W}) - 2g(\mathscr{D}r,\zeta)\eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.32}$$

The relations (3.22), (3.31) and (3.32) give

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = g((\nabla_{\zeta}\mathscr{Q})\mathscr{V},\mathscr{W}) - g((\nabla_{\mathscr{V}}\mathscr{Q})\zeta,\mathscr{W}) - \frac{\zeta(r)}{2(m-1)}[g(\mathscr{V},\mathscr{W}) + \eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.33}$$

In an $(LPK)_m$, the following result holds (for perusal, see [36])

$$(\nabla_{\zeta}\mathscr{Q})\mathscr{V} = 2\mathscr{Q}\mathscr{V} - 2(m-1)\mathscr{V}. \tag{3.34}$$

Applying relations (3.22) and (3.34) into (3.33), it yields

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = g(\mathscr{Q}\mathscr{V},\mathscr{W}) - (m-1)g(\mathscr{V},\mathscr{W}) - \frac{\zeta(r)}{2(m-1)}[g(\mathscr{V},\mathscr{W}) + \eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.35}$$

If $(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = 0$ and (3.23), then (3.35) leads to

$$\mathscr{S}(\mathscr{V},\mathscr{W}) = (\frac{r}{m-1} - 1)g(\mathscr{V},\mathscr{W}) + (\frac{r}{m-1} - m)\eta(\mathscr{V})\eta(\mathscr{W}). \tag{3.36}$$

The relation (3.36) leads the following theorem:

Theorem 3.6. *An $(LPK)_m$ $(m \geq 4)$ admitting (g,ζ,λ) is an η-Einstein manifold provided $(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = 0$, $\forall\ \mathscr{V},\mathscr{W}$.*

4. 3-Dimensional Bach Perfect Fluid Lorentzian Para-Kenmotsu Manifold

We consider an $(LPK)_3$ admitting (g,ζ,λ). Curvature tensor of Riemannian manifold in dimension 3 states

$$\mathscr{R}(U,\mathscr{V})\mathscr{W}=-\mathscr{S}(U,\mathscr{W})\mathscr{V}+\mathscr{S}(\mathscr{V},\mathscr{W})U-g(U,\mathscr{W})\mathscr{Q}\mathscr{V}+g(\mathscr{V},\mathscr{W})\mathscr{Q}U-\frac{r}{2}[g(U,\mathscr{W})\mathscr{V}-g(\mathscr{V},\mathscr{W})U], \tag{4.1}$$

$\forall$ differentiable vector fields U, $\mathscr{V}$ and $\mathscr{W}$.
Replacing $U=\mathscr{W}=\zeta$ in (4.1) and using (2.1), (2.8), (2.9) and (2.10), we obtain

$$\mathscr{Q}\mathscr{V}=(\frac{r}{2}-3)\eta(\mathscr{V})\zeta+(\frac{r}{2}-1)\mathscr{V}. \tag{4.2}$$

The preceding result gives

$$\mathscr{Q}\phi=\phi\mathscr{Q}.$$

The equation (4.2), together with (2.4), gives

$$(\nabla_U\mathscr{Q})\zeta=\mathscr{Q}U-2U. \tag{4.3}$$

Equation (3.12), together with (4.3) leads to

$$C_0(U,\mathscr{V})\zeta=\frac{1}{4}[\mathscr{V}(r)\eta(U)-U(r)\eta(\mathscr{V})].$$

The covariant differentiation of above result w.r.t. $\mathscr{W}$ yields

$$(\nabla_{\mathscr{W}}C_0)(U,\mathscr{V})\zeta=-(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W})-(\nabla_U\mathscr{S})(\mathscr{V},\mathscr{W})-\frac{1}{4}[g(\nabla_{\mathscr{W}}\mathscr{D}r,U)\eta(\mathscr{V})-g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(U)].$$

Putting $\mathscr{W}=U=\mathscr{E}_i$ and taking sum over $i=1,2,3$ in above relation, where $\{\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3=\zeta\}$ is orthonormal frame at each point p of $T_p\mathscr{M}$, we have

$$\sum_{i\in\{1,2,3\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_o)(\mathscr{E}_i,\mathscr{V})\zeta=-\frac{\mathscr{V}(r)}{2}-\frac{1}{4}[(div\mathscr{D}r)\eta(\mathscr{V})-g(\nabla_\zeta\mathscr{D}r,\mathscr{V})]. \tag{4.4}$$

Taking $\mathscr{V}=\zeta$ in (1.6), we have

$$\mathscr{B}(U,\zeta)=\sum_{i\in\{1,2,3\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\zeta. \tag{4.5}$$

Equations (3.4), (4.4) and (4.5) taken together give

$$\lambda\eta(U)=-\frac{1}{2}g(\mathscr{D}r,U)-\frac{1}{4}[(div\mathscr{D}r)\eta(U)-g(\nabla_\zeta\mathscr{D}r,U)]. \tag{4.6}$$

Replacing ϕU for U in (4.6), we get

$$\phi\nabla_\zeta\mathscr{D}r=2\phi\mathscr{D}r. \tag{4.7}$$

We have the relation (3.23) and (3.24), for $m=3$, which yields

$$\nabla_\zeta\mathscr{D}r=\mathscr{D}r-2(r-6)\zeta. \tag{4.8}$$

The relations (4.7) and (4.8) provide

$$\mathscr{D}r=-2(r-6)\zeta. \tag{4.9}$$

By the covariant diffentiation of (4.9) w.r.t. $\mathscr{X}$ yields

$$\nabla_{\mathscr{X}}\mathscr{D}r=-2g(\mathscr{D}r,\mathscr{X})\zeta+2(r-6)\mathscr{X}+2(r-6)\eta(\mathscr{X})\zeta. \tag{4.10}$$

By contracting the relation (4.10) over $\mathscr{X}$, we get

$$(div\mathscr{D}r)=0. \tag{4.11}$$

Using relations (4.8), (4.9) and (4.11) in (4.6), it yields

$$\lambda=0. \tag{4.12}$$

With the help of (4.12), the relation (3.4) reduces to

$$\mathscr{B}(U,\mathscr{V})=\eta(U)\eta(\mathscr{V})+g(U,\mathscr{V}).$$

The above results imply the succeeding theorem:

Theorem 4.1. *Let $(LPK)_3$ admit a (g,ζ,λ), then the manifold is a Bach perfect fluid and (g,ζ,λ) is always steady.*

5. Example

We assume a manifold $\mathscr{M}^3 = \{(u_1,v_1,w_1) \in R^3 : w_1 > 0\}$, here (u_1,v_1,w_1) are the general coordinates in R^3. Consider $\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3$, the vector fields on $\mathscr{M}^3$ given as

$$\hat{\mathscr{E}}_1 = w_1 \frac{\partial}{\partial u_1}, \qquad \hat{\mathscr{E}}_2 = w_1 \frac{\partial}{\partial v_1}, \qquad \hat{\mathscr{E}}_3 = w_1 \frac{\partial}{\partial w_1} = \zeta$$

and are linearly independent at each point of $\mathscr{M}^3$. This implies

$$g(\hat{\mathscr{E}}_i, \hat{\mathscr{E}}_j) = \begin{cases} 0, & 1 \le i \ne j \le 3, \\ -1, & i = j = 1,2, \\ 0, & otherwise. \end{cases}$$

Suppose that η is 1-form on $\mathscr{M}^3$ given by $\eta(U) = g(U, \hat{\mathscr{E}}_3) = g(U,\zeta)$, $\forall$ U $\in \chi(\mathscr{M}^3)$. Again, assume that ϕ is $(1,1)$ tensor field on $\mathscr{M}^3$ given below:

$$\phi\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_2, \qquad \phi\hat{\mathscr{E}}_2 = -\hat{\mathscr{E}}_1, \qquad \phi\hat{\mathscr{E}}_3 = 0.$$

The linear property of g and ϕ give the following relations

$$\eta(\zeta) = g(\zeta,\zeta) = -1, \phi^2 = U + \eta(U)\zeta, \ \ g(U,\zeta) = \eta(U), \ \eta(\phi U) = 0, \ g(\phi U, \phi\mathscr{V}) = \eta(U)\eta(\mathscr{V}) + g(U,\mathscr{V}).$$

Assuming ∇ to be Levi-Civita connection w.r.t. Lorentzian metric g, then

$$[\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_1] = 0, \ [\hat{\mathscr{E}}_3, \hat{\mathscr{E}}_1] = \hat{\mathscr{E}}_1, \ [\hat{\mathscr{E}}_3, \hat{\mathscr{E}}_2] = \hat{\mathscr{E}}_2.$$

Applying Koszul's formula, we can comfortably obtain

$$\nabla_{\hat{\mathscr{E}}_i} \hat{\mathscr{E}}_j = \begin{cases} -\hat{\mathscr{E}}_3, & i = j = 1,2, \\ -\hat{\mathscr{E}}_i, & i = 1,2, j = 3, \\ 0, & otherwise \end{cases} \tag{5.1}$$

Let $U \in \chi(\mathscr{M}^3)$, then the following relations can also be verified

$$\nabla_U \zeta + U + \eta(U)\zeta = 0, \ \ (\nabla_U \phi)\mathscr{V} = -g(\phi U, \mathscr{V})\zeta - \eta(\mathscr{V})\phi(U).$$

For $U, \mathscr{V}, \mathscr{W} \in \chi(\mathscr{M}^3)$.
Equation (5.1) helps to get the following non-vanishing values:

$$\begin{cases} \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2)\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_2, \ \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_3, \ \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2)\hat{\mathscr{E}}_2 = \hat{\mathscr{E}}_1, \\ \mathscr{R}(\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_2 = -\hat{\mathscr{E}}_3, \mathscr{R}(\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_3 = -\hat{\mathscr{E}}_2. \end{cases}$$

The above results help to verify

$$\mathscr{R}(U,\mathscr{V})\mathscr{W} = -g(U,\mathscr{W})\mathscr{V} + g(\mathscr{V},\mathscr{W})U. \tag{5.2}$$

Hence, $\mathscr{M}^3$ is a Lorentzian para-Kenmotsu manifold of constant curvature. By contracting (5.2) over W, we obtain

$$\mathscr{S}(U,\mathscr{V}) = 2g(\mathscr{V},\mathscr{W}).$$

This implies

$$r = 6.$$

Then, (4.6) provides $\lambda = 0$. Hence, in this manifold, the Bach almost solitons are steady.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] I. Sato, *On a structure similar to the almost contact structure*, Tensor (N. S.), **30** (1976), 219-224.
[2] S. Kaneyuki, M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math., **8** (1985), 81-98.
[3] S. Kaneyuki, F. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J., **99** (1985), 173-187.
[4] B.B. Sinha, K.L. Sai Prasad, *A class of almost para contact metric manifolds*, Bull. Calcutta Math. Soc., **87** (1995), 307-312.
[5] T. Q. Binh, L. Tamassy, U. C. De, M. Tarafdar, *Some remarks on almost Kenmotsu manifolds*, Maths. Pannonica, **13** (2002), 31-39.
[6] A. M. Blaga, *Conformal and paracontactly geodesic transformations of almost paracontact metric structures*, Facta Univ. Scr. Math. Inform., **35** (2020), 121-130.
[7] A. M. Blaga, M. Crasmareanu, *Statistical structures in almost paracontact geometry*, Bull. Iranian Math. Soc., **44** (2018), 1407-1413.
[8] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, Illinois J. Math., **55** (2011), 696-718.
[9] S. Dirik, M. Atceken, U. Yildirim, *Anti invariant submanifolds of a normal para contact metric manifolds*, Gulf J. Math., **10** (2021), 38-49.
[10] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983.
[11] V. R. Kaigorodov, *The curvature structure of spacetime*, Pro. Geom., **14** (1983), 177-202.
[12] A. K. Raychaudhuri, S. Banerji, A. Banerjee, *General Relativity, Astrophysics and Cosmology*, Springer-Verlag, 1992.
[13] A. Haseeb, R. Prasad, *Certain results on Lorentzian para-Kenmotsu manifolds*, Bol. Soc. Paran. Mat., **39** (2021), 201-220.
[14] A. Haseeb, R. Prasad, *Some results on Lorentzian para-Kenmotsu manifolds*, Bulletin of the Transilvania University of Brasov, Series III : Mathematics, Informatics, Physics, **13(62)** (2020), 185-198.
[15] R. Bach, *Zur Weylschen relativitatstheorie and der Weylschen Erweiterung des Krummungstensorbegriffs*, Math. Z., **9** (1921), 110-135.
[16] Y. Wang, *Cotton tensors on almost co-Kähler 3-manifolds*, Ann. Polon. Math., **120** (2017), 135-148.
[17] J. Bergman, *Conformal Einstein spaces and Bach tensor generalization in n-dimensions*, Ph. D. Thesis, Linköping University, 2004.
[18] U. C. De, K. De, *On a class of three-dimensional trans-Sasakian manifolds*, Commun. Korean Math. Soc., **27** (2012), 795-808.
[19] U. C. De, G. Ghosh, J. B. Jun, P. Majhi, *Some results on para Sasakian manifolds*, Bull. Transilv. Univ. Brasov, Series III : Mathematics, Informatics, Physics, **11 (60)** (2018), 49-63.
[20] U. C. De, A. Sardar, *Classification of (k,μ)-almost co-Kähler manifolds with vanishing Bach tensor and divergence free cotton tensor*, Commun. Korean Math. Soc., **35** (2020), 1245-1254.
[21] H.I. Yoldas, *Notes on η-Einstein solitons on para-Kenmotsu manifolds*, Math. Method Appl. Sci., **46** (2023), 17632-17640.
[22] H. Fu, J. Peng, *Rigidity theorems for compact Bach-flat manifolds with positive constant scalar curvature*, Hokkaido Math. J., **47** (2018), 581-605.
[23] A. Ghosh, R. Sharma, *Sasakian manifolds with purely transversal Bach tensor*, J. Math. Phys., **58** (2017), 103502, 6 pp.
[24] P. Szekeres, *Conformal tensors*, Proc. R. Soc. Lond. Ser. A-Contain. Pap. Math. Phys., **304** (1968), 113-122.
[25] S. Das, S. Kar, *Bach flows of product manifolds*, Int. J. Geom. Methods Mod. Phys., **9** (2012), 1250039, 18 pp.
[26] I. Bakas, F. Bourliot, D. Lust, M. Petropoulos, *Geometric flows in Horava-Lifshitz gravity*, J. High Energy Phys., **2010** (2010), 1-58.
[27] E. Bahuaud, D. Helliwell, *Short-time existence for some higher-order geometric flows*, Commun. Partial Differ. Equ., **36** (2011), 2189-2207.
[28] H.-D. Cao, Q. Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J., **162** (2013), 1149-1169.
[29] P. T. Ho, *Bach flow*, J. Geom. Phys., **133** (2018), 1-9.
[30] D. Helliwell, *Bach flow on homogeneous products*, SIGMA Symmetry Integrability Geom. Methods Appl., **16** (2020), 35 pp.
[31] A. Ghosh, *On Bach almost solitons*, Beitr. Algebra Geom., **63** (2022), 45-54.
[32] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Nat. Sci., **12** (1989), 151-156.
[33] R. Prasad, A. Haseeb, A. Verma, V. S. Yadav, *A study of φ-Ricci symmetric LP-Kenmotsu manifolds*, Int. J. Maps Math., **7** (2024), 33-44.
[34] Pankaj, S. K. Chaubey, R. Prasad, *Three dimensional Lorentzian para-Kenmotsu manifolds and Yamabe soliton*, Honam Math. J., **43** (2021), 613-626.
[35] D. E. Blair, *Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics*, Springer-Verlag, 509, 1976.
[36] Li, Y., Haseeb, A., Ali, M., *LP-Kenmotsu manifolds admitting η-Ricci solitons and spacetime*, J. Math., **2022**, Article ID 6605127, 10 pages.